

A Volume Formula for Medial Sections of Simplices

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Abstract. Let S_d be a d -dimensional simplex in \mathbb{R}^d , and let H be an affine hyperplane of \mathbb{R}^d . We say that H is a medial hyperplane of S_d if the distance between H and any vertex of S_d is the same constant. The intersection of S_d and a medial hyperplane is called a medial section of S_d . In this paper we give a simple formula for the $(d - 1)$ -volume of any medial section of S_d in terms of the lengths of the edges of S_d . This extends the result of Yetter [5] from the three-dimensional case to arbitrary dimension. We also show that a generalization of the obtained formula measures the volume of the intersection of some analogously chosen “medial” affine subspace of \mathbb{R}^d and the simplex.

1. Introduction

In a recent paper by Yetter [5] a simple formula is given for the area of a parallelogram whose vertices are the midpoints of four edges of a tetrahedron. There the area is expressed in terms of the lengths of the edges of the tetrahedron. Note that the parallelogram considered there can be always obtained as the intersection of the tetrahedron and a plane parallel to two opposite edges of the tetrahedron and lying at equal distance from them.

In this paper we consider a natural generalization of this problem. Let S_d be a d -dimensional simplex, $S_d \subset \mathbb{R}^d$, and let H be an affine hyperplane of \mathbb{R}^d given so that the distance between H and any vertex of S_d is the same constant. We say that such a hyperplane H is a *medial hyperplane* of S_d . Let us define the *medial section* of S_d with respect to H as the $(d - 1)$ -dimensional convex body $S_d \cap H$. Then the analogous question is whether there exists a “simple” formula for the $(d - 1)$ -dimensional volume of the medial section $S_d \cap H$ in terms of the lengths of the edges of S_d .

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The answer to this question is affirmative and it turns out that the square of the volume of a medial section is a polynomial of the squares of the edge lengths of the simplex S_d . For the sake of nicer formularization, instead of the volume we express the square of the volume in the following.

Theorem 1. *Let S_d be a simplex in \mathbb{R}^d with vertices v_1, \dots, v_{d+1} , and let H be a medial hyperplane of S_d separating $\{v_1, \dots, v_m\}$ from $\{v_{m+1}, \dots, v_{d+1}\}$. Then*

$$\text{Vol}_{d-1}^2(S_d \cap H) = \frac{1}{(-8)^{d-1}((m-1)!)^2((d-m)!)^2} \det M_{d+3}^{(m)}, \tag{1}$$

where $M_{d+3}^{(m)}$ is a symmetric $(d+3) \times (d+3)$ matrix with block form

$$M_{d+3}^{(m)} = \begin{pmatrix} O_2 & I_{d+1}^{(m)} \\ I_{d+1}^{(m)T} & A_{d+1} \end{pmatrix}$$

in which O_2 is a 2×2 zero-matrix, $I_{d+1}^{(m)}$ is a $2 \times (d+1)$ matrix of form

$$I_{d+1}^{(m)} = \begin{pmatrix} 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 1 \end{pmatrix} \tag{2}$$

$\underbrace{\hspace{10em}}_m \qquad \underbrace{\hspace{10em}}_{d+1-m}$

with m consecutive 1's and $(d+1-m)$ consecutive 0's in the first row, and m consecutive 0's and $(d+1-m)$ consecutive 1's in the second row, and $A_{d+1} = \{a_{i,j}^2\}$ is a $(d+1) \times (d+1)$ matrix with entries $a_{i,j}^2$, where $a_{i,j}$ measures the distance between v_i and v_j .

Note that for $d = 3$ and $m = 2$ formula (1) of Theorem 1 gives the square of the area of a medial parallelogram P of the tetrahedron S_3 . However, one needs to write out the determinant in (1) and to do some algebraic simplification to get

$$\text{Area}^2(P) = \frac{1}{64}(a_{1,2}^2 a_{3,4}^2 - (a_{1,4}^2 - a_{4,2}^2 + a_{2,3}^2 - a_{3,1}^2)^2),$$

which is the formula of Yetter [5].

Recall the well-known formula

$$\text{Vol}^2(S_d) = \frac{(-1)^{d+1}}{2^d(d!)^2} \det M_{d+2}, \tag{3}$$

where M_{d+2} is the $(d+2) \times (d+2)$ matrix with block form

$$M_{d+2} = \begin{pmatrix} 0 & 1_{d+1} \\ 1_{d+1}^T & A_{d+1} \end{pmatrix}$$

(see [1] and [4]), in which 1_{d+1} is the row matrix of length $d+1$ with all entries equal to 1, and A_{d+1} is the same matrix as in Theorem 1. The determinant in (3) is the so-called Cayley–Menger determinant. The volume formula (3) has various applications in geometry, see [2] and [3] for some examples.

Since (1) and (3) are similar formulas for squares of certain volumes, and both contain analogous determinant factors, one may suspect that there is a common generalization

of these formulas that still expresses the square of the l -volume of the intersection of S_d and an analogously chosen “medial” l -dimensional affine subspace of \mathbb{R}^d , for any $0 \leq l \leq d$. In the following we describe such a generalization which includes both (1) and (3) as special cases.

Let $\mathcal{P} = \{V_1, V_2, \dots, V_k\}$ be a k -partition of the vertex set $\text{vert}(S_d)$, for which

$$V_i = \{v_{n_{i-1}+1}, v_{n_{i-1}+2}, \dots, v_{n_i}\}$$

holds for every $1 \leq i \leq k$, with $0 = n_0 < n_1 < \dots < n_k = d + 1$. Let $m_i = n_i - n_{i-1}$. Clearly, each V_i has cardinality m_i . This means that

$$\mathcal{P} = \{\{v_1, \dots, v_{m_1}\}, \{v_{m_1+1}, \dots, v_{m_1+m_2}\}, \dots, \{v_{d+2-m_k}, \dots, v_{d+1}\}\}.$$

That is, \mathcal{P} is a collection of k disjoint nonempty subsets of $\text{vert}(P)$ each containing consecutively labeled vertices. Let $P_i = \text{aff}(V_i)$ for every $1 \leq i \leq k$, where $\text{aff}()$ stands for the affine hull. Let $H' = \frac{1}{k} \sum_{i=1}^k P_i$. In other words, H' is the Minkowski sum (or set theoretical sum) of the P_i 's rescaled by a homothety with factor $1/k$ (for an exact definition of Minkowski sums, see Section 2.1). Since each P_i is an affine subspace of \mathbb{R}^d of dimension $m_i - 1$, therefore H' is an affine subspace of dimension $\sum_{i=1}^k (m_i - 1) = d + 1 - k$. The following holds.

Theorem 2. *Let \mathcal{P} be a k -partition of the vertices of a d -dimensional simplex S_d as described previously ($1 \leq k \leq d + 1$). Using the notation above, we have*

$$\text{Vol}_{d+1-k}^2(S_d \cap H') = \frac{(-1)^{d+1}}{(2k^2)^{d+1-k} \prod_{i=1}^k (m_i - 1)!^2} \det M_{\mathcal{P}}, \tag{4}$$

where $M_{\mathcal{P}}$ is a symmetric $(d + k + 1) \times (d + k + 1)$ matrix with block form

$$M_{\mathcal{P}} = \begin{pmatrix} O_k & I_{\mathcal{P}} \\ I_{\mathcal{P}}^T & A_{d+1} \end{pmatrix}$$

in which O_k is a $k \times k$ zero-matrix, $I_{\mathcal{P}} = \{g_{i,j}\}$ is a $k \times (d + 1)$ zero-one matrix with entries

$$g_{i,j} = \begin{cases} 1, & \text{if } n_{i-1} + 1 \leq j \leq n_i, \\ 0, & \text{otherwise,} \end{cases}$$

and A_{d+1} is the same $(d + 1) \times (d + 1)$ matrix as described in Theorem 1.

Note that in Theorem 2 the entries of $I_{\mathcal{P}}$ are defined so that we have $g_{i,j} = 1$ if and only if $v_j \in V_i$. Therefore $I_{\mathcal{P}}$ is the characteristic matrix of the k -partition \mathcal{P} of $\text{vert}(S_d)$. It is also clear that $I_{\mathcal{P}}$ is a generalization of the matrix $I_{d+1}^{(m)}$ of Theorem 1. For example, for $k = 3$ we have

$$I_{\mathcal{P}} = \begin{pmatrix} 1 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 1 \end{pmatrix},$$

$\underbrace{\hspace{1.5cm}}_{m_1} \quad \underbrace{\hspace{1.5cm}}_{m_2} \quad \underbrace{\hspace{1.5cm}}_{m_3}$

which is analogous to (2).

Observe that H' of Theorem 2 is a generalization of H of Theorem 1 since the latter can be written as

$$H = \frac{1}{2}(\text{aff}(v_1, v_2, \dots, v_m) + \text{aff}(v_{m+1}, \dots, v_{d+1})).$$

Therefore one may call H' a *medial* $(d + 1 - k)$ -dimensional affine subspace of S_d , and $S_d \cap H'$ may be called a *medial* $(d + 1 - k)$ -section of S_d .

We remark that Theorem 1 is a special case of Theorem 2 with $k = 2$ and $m_1 = m$, while (3) is a special case of Theorem 2 with $k = 1$. However, we prove only Theorem 1 in detail, and for the proof of Theorem 2 just a sketch is given, since the latter goes the same way as the proof of Theorem 1. Only the more extensive notation needed there would make a detailed proof of Theorem 2 more lengthy and complicated.

2. Proofs

2.1. Notation

If K is an l -dimensional convex subset in \mathbb{R}^d for some l , $0 \leq l \leq d$, then we denote its l -dimensional volume by $\text{Vol}_l(K)$. For $l = d$ we use the notation $\text{Vol}(K)$.

If $U, V \subseteq \mathbb{R}^d$, then let $U + V = \{u + v \in \mathbb{R}^d \mid u \in U, v \in V\}$, that is, $U + V$ is the Minkowski sum of U and V . For $v \in \mathbb{R}^d$ we simply write $U + v$ instead of $U + \{v\}$. For $U \subseteq \mathbb{R}^d$ and $\alpha \in \mathbb{R}$, let αU be the set $\{\alpha u \mid u \in U\}$. If $a, b \in \mathbb{R}^d$, then both $[a, b]$ and ab denote the segment with endpoints a and b . We denote by o the origin of \mathbb{R}^d .

Throughout the paper $\text{conv}(\cdot)$ and $\text{aff}(\cdot)$ stand for the convex hull and the affine hull, respectively. For a convex polyhedron P we use the notation $\text{vert}(P)$ for its vertex set. For $u \in \mathbb{R}^d$ we denote its Euclidean norm by $\|u\|$. If U is a linear subspace of \mathbb{R}^d , then U^\perp is the orthogonal complement of U in \mathbb{R}^d .

We denote by $1_l, -1_l$ and 0_l the row matrices of length l with all entries equal to 1, -1 and 0, respectively. If M is a matrix, then we use the notation M^T for the matrix transposed to M .

2.2. Proof of Theorem 1

Throughout this part we assume that the conditions of Theorem 1 hold. Our goal is to prove (1). Let $C_1 = \text{conv}(v_1, \dots, v_m)$ and $C_2 = \text{conv}(v_{m+1}, \dots, v_{d+1})$. Then C_1 is an $(m - 1)$ -dimensional simplex and C_2 is a $(d - m)$ -dimensional simplex. Let $C'_1 = C_1 - v_m$ and $C'_2 = C_2 - v_{d+1}$. Since $S_d = \text{conv}(C_1 \cup C_2)$ and $H = \frac{1}{2}(\text{aff}(C_1) + \text{aff}(C_2))$, therefore $S_d \cap H = \frac{1}{2}(C_1 + C_2)$. This means that $S_d \cap H$ is a translate of $\frac{1}{2}(C'_1 + C'_2)$.

On the other hand, $D_1 = \sum_{i=1}^{m-1} [o, v_i - v_m]$ and $D_2 = \sum_{i=m+1}^d [o, v_i - v_{d+1}]$ are paralleletoes, and their Minkowski sum $D = D_1 + D_2$ is also a paralleletope. Let

$$v'_i = \begin{cases} v_i - v_m, & \text{if } 1 \leq i \leq m - 1, \\ v_{i+1} - v_{d+1}, & \text{if } m \leq i \leq d - 1. \end{cases} \quad (5)$$

Let

$$C' = \text{conv}(o, \{v'_i \mid 1 \leq i \leq d-1\}).$$

Then C' is a $(d-1)$ -dimensional simplex.

We have

$$\begin{aligned} \text{Vol}_{m-1}(D_1) &= (m-1)! \text{Vol}_{m-1}(C'_1), \\ \text{Vol}_{d-m}(D_2) &= (d-m)! \text{Vol}_{d-m}(C'_2), \\ \text{Vol}_{d-1}(D) &= (d-1)! \text{Vol}_{d-1}(C'), \end{aligned}$$

and

$$\begin{aligned} \text{Vol}_{d-1}(D) &= \text{Vol}_{m-1}(D_1) \alpha \text{Vol}_{d-m}(D_2), \\ \text{Vol}_{d-1}(C'_1 + C'_2) &= \text{Vol}_{m-1}(C'_1) \alpha \text{Vol}_{d-m}(C'_2), \end{aligned}$$

for some constant $\alpha > 0$, since the affine hulls of D_1 and D_2 are the same as the affine hulls of C'_1 and C'_2 , respectively, so $\alpha \text{Vol}_{d-m}(D_2)$ is the $(d-m)$ -volume of the parallel projection of D_2 to $\text{aff}(D_1)^\perp$ and $\alpha \text{Vol}_{d-m}(C'_2)$ is the $(d-m)$ -volume of the parallel projection of C'_2 to $\text{aff}(C_1)^\perp$, if α is chosen appropriately.

From these equalities follows

$$\begin{aligned} \text{Vol}_{d-1}(S_d \cap H) &= \frac{1}{2^{d-1}} \text{Vol}_{d-1}(C'_1 + C'_2) = \frac{\text{Vol}_{m-1}(C'_1) \text{Vol}_{d-m}(C'_2) \text{Vol}_{d-1}(D)}{2^{d-1} \text{Vol}_{m-1}(D_1) \text{Vol}_{d-m}(D_2)} \\ &= \frac{(d-1)! \text{Vol}_{d-1}(C')}{2^{d-1} (m-1)! (d-m)!}. \end{aligned}$$

Applying (3) for the factor $\text{Vol}_{d-1}(C')$ in the previous expression we get

$$\text{Vol}_{d-1}^2(S_d \cap H) = \frac{(-1)^d \det N_{d+1}}{8^{d-1} ((m-1)!)^2 ((d-m)!)^2}, \quad (6)$$

where N_{d+1} is a $(d+1) \times (d+1)$ matrix with block form

$$N_{d+1} = \begin{pmatrix} 0 & 1_d \\ 1_d^T & B_d \end{pmatrix}$$

with $B_d = \{b_{i,j}^2\}_{i,j=1}^d$, where $b_{i,j}$ is the distance between v'_{i-1} and v'_{j-1} , for $1 \leq i, j \leq d$ (with the notation $v'_0 = o$). It is clear that to complete the proof of Theorem 1 we only need to show that

$$\det M_{d+3}^{(m)} = -\det N_{d+1}.$$

To prove this, we need the following.

Lemma 3. *Let T be a tetrahedron in \mathbb{R}^3 with vertices p_1, p_2, p_3 and p_4 . Let h be the length of the segment connecting the midpoints of the two opposite edges $p_1 p_2$ and $p_3 p_4$ of T . If $s_{i,j}$ denotes the distance between the vertices p_i and p_j of T , $1 \leq i, j \leq 4$, then we have*

$$h^2 = \frac{1}{4}(s_{1,3}^2 + s_{3,2}^2 + s_{2,4}^2 + s_{4,1}^2 - s_{1,2}^2 - s_{3,4}^2). \quad (7)$$

Proof. Let c be the baricenter of T , that is, $c = \frac{1}{4}(p_1 + p_2 + p_3 + p_4)$. Let T' be the centrally reflected image of T with center c . Denote by p'_i that vertex of T' which is the centrally reflected image of p_i , for any $1 \leq i \leq 4$. Since c is clearly the midpoint of any segment connecting the midpoints of two opposite edges of T' , therefore for any two opposite edges $p_i p_j$ and $p_k p_l$ of T it holds that the segment $p'_i p'_j$ intersects $p_k p_l$ in a point which is the midpoint of both segments, and consequently $p'_i p_k p'_j p_l$ is a parallelogram. This means that

$$Q = \text{conv}(p_1, p_2, p_3, p_4, p'_1, p'_2, p'_3, p'_4)$$

is a parallelepiped, and $\text{vert}(Q) = \text{vert}(T) \cup \text{vert}(T')$. Let e_2, e_3 and e_4 be the lengths of the edges $p_1 p'_2, p_1 p'_3$ and $p_1 p'_4$ of Q , respectively. We recall the well-known fact that the sum of the squares of the four sides of any parallelogram equals the sum of the squares of the two diagonals of that parallelogram. Using this fact for the three faces $p_1 p'_2 p_4 p'_3, p_1 p'_2 p_3 p'_4$ and $p_1 p'_3 p_2 p'_4$ of Q we get

$$\begin{aligned} 2(e_2^2 + e_3^2) &= s_{3,2}^2 + s_{4,4}^2, \\ 2(e_2^2 + e_4^2) &= s_{1,3}^2 + s_{2,4}^2, \\ 2(e_3^2 + e_4^2) &= s_{1,2}^2 + s_{3,4}^2. \end{aligned}$$

Summing up the first two equations, and subtracting the third one from the sum, we obtain

$$4e_2^2 = s_{1,3}^2 + s_{3,2}^2 + s_{2,4}^2 + s_{4,1}^2 - s_{1,2}^2 - s_{3,4}^2. \quad (8)$$

Finally, observe that $h = e_2$. Thus, after dividing both sides of (8) by 4, and replacing e_2 with h , we get (7). \square

We now continue the proof of Theorem 1. By (5) we have the following block form for B_d :

$$B_d = \begin{pmatrix} 0 & w_1 & w_2 \\ w_1^T & X & Y \\ w_2^T & Y^T & Z \end{pmatrix},$$

where w_1 and w_2 are row matrices of lengths $m - 1$ and $d - m$, respectively, of form

$$\begin{aligned} w_1 &= (a_{1,m}^2, a_{2,m}^2, \dots, a_{m-1,m}^2), \\ w_2 &= (a_{m+1,d+1}^2, a_{m+2,d+1}^2, \dots, a_{d,d+1}^2), \end{aligned}$$

and $X = \{x_{i,j}\}, Y = \{y_{i,j}\}$ and $Z = \{z_{i,j}\}$ are matrices of dimensions $(m - 1) \times (m - 1), (m - 1) \times (d - m)$ and $(d - m) \times (d - m)$, respectively, with entries

$$\begin{aligned} x_{i,j} &= a_{i,j}^2 && \text{for } 1 \leq i, j \leq m - 1, \\ y_{i,j} &= b_{i+1,m+j}^2 && \text{for } 1 \leq i \leq m - 1 \text{ and } 1 \leq j \leq d - m, \\ z_{i,j} &= a_{m+i,m+j}^2 && \text{for } 1 \leq i, j \leq d - m. \end{aligned}$$

Since

$$y_{i,j} = b_{i+1,m+j}^2 = 4 \left\| \frac{v_i + v_{d+1}}{2} - \frac{v_{m+j} + v_m}{2} \right\|^2,$$

thus by Lemma 3 we have

$$y_{i,j} = a_{i,m+j}^2 + a_{m+j,d+1}^2 + a_{m,d+1}^2 + a_{i,m}^2 - a_{i,d+1}^2 - a_{m,m+j}^2. \tag{9}$$

It is evident that $\det N_{d+1} = \det \widehat{M}$, where \widehat{M} is a $(d+3) \times (d+3)$ matrix with block form

$$\widehat{M} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0_{m-1} & 0_{d-m} \\ 0 & 1 & 1 & 0 & 0_{m-1} & w_2 \\ 1 & 0 & 0 & 1 & 1_{m-1} & 1_{d-m} \\ 0 & 0 & 1 & 0 & w_1 & w_2 \\ 0_{m-1}^T & 0_{m-1}^T & 1_{m-1}^T & w_1^T & X & Y \\ w_2^T & 0_{d-m}^T & 1_{d-m}^T & w_2^T & Y^T & Z \end{pmatrix}. \tag{10}$$

We will use row and column operations to alter \widehat{M} until we get $M_{d+3}^{(m)}$. Namely, we will multiply some rows and columns by -1 , we will add some multiples of rows to other rows and some multiples of columns to other columns, and furthermore, we will reposition some rows and columns. Clearly, the determinant of such a newly obtained matrix will be $\pm \det \widehat{M}$, and the sign will depend on the parity of the total change of indices of the rows and columns repositioned, and on the number of rows and columns multiplied by -1 .

In the following we give a list of 16 row and column operations which transform \widehat{M} into $M_{d+3}^{(m)}$. At each step, denote by R_i and C_i the i th row and column of the actual matrix, respectively. By the actual matrix we mean that matrix which is obtained from \widehat{M} as the result of those steps already made. Note that this means that in different steps R_i and C_i may stand for different rows and columns for some i . We use arrow notation, e.g. $C_i \rightarrow C_i - C_1$ in Step 1 means that a new i th column is obtained after applying this step by subtracting the 1st column from the i th column of the actual matrix. Here is the list of the row and column operations:

- Step 1. $C_i \rightarrow C_i - C_1$ for $4 \leq i \leq m+3$.
- Step 2. $R_i \rightarrow R_i - R_2$ for $4 \leq i \leq m+3$.
- Step 3. $C_i \rightarrow C_i - C_4$ for $m+4 \leq i \leq d+3$.
- Step 4. $R_i \rightarrow R_i - R_4$ for $m+4 \leq i \leq d+3$.
- Step 5. C_4 is shifted to between C_{m+3} and C_{m+4} .
- Step 6. R_4 is shifted to between R_{m+3} and R_{m+4} .
- Step 7. C_1 is shifted to be the last column.
- Step 8. R_2 is shifted to be the last row.
- Step 9. $C_1 \rightarrow -C_1 + C_2$.
- Step 10. $R_1 \rightarrow -R_1 + R_2$.
- Step 11. $C_i \rightarrow a_{m,i-2}^2 C_1 + C_i$ for $m+3 \leq i \leq d+2$.
- Step 12. $R_i \rightarrow a_{m,i-2}^2 R_1 + R_i$ for $m+3 \leq i \leq d+2$.
- Step 13. $C_i \rightarrow (a_{i-2,d+1}^2 - a_{m,d+1}^2) C_2 + C_i$ for $3 \leq i \leq m+1$.

Step 14. $R_i \rightarrow (a_{i-2,d+1}^2 - a_{m,d+1}^2)R_2 + R_i$ for $3 \leq i \leq m+1$.

Step 15. $C_{d+3} \rightarrow a_{m,d+1}^2 C_1 + C_{d+3}$.

Step 16. $R_{d+3} \rightarrow a_{m,d+1}^2 R_1 + R_{d+3}$.

Now we describe how the matrix \widehat{M} is affected by the row and column operations of the steps above.

After Steps 1 and 2 we get the following matrix \widehat{M}_1 with block form

$$\widehat{M}_1 = \begin{pmatrix} 1 & 0 & 0 & -1 & -1_{m-1} & 0_{d-m} \\ 0 & 1 & 1 & 0 & 0_{m-1} & w_2 \\ 1 & 0 & 0 & 0 & 0_{m-1} & 1_{d-m} \\ 0 & -1 & 0 & 0 & w_1 & 0_{d-m} \\ 0_{m-1}^T & -1_{m-1}^T & 0_{m-1}^T & w_1^T & X & Y_1 \\ w_2^T & 0_{d-m}^T & 1_{d-m}^T & 0_{d-m}^T & Y_1^T & Z \end{pmatrix},$$

in which Y_1 is a submatrix with entries defined similarly to (9) except that the second term of the right side of (9) is dropped out.

After Steps 3 and 4 we get

$$\widehat{M}_2 = \begin{pmatrix} 1 & 0 & 0 & -1 & -1_{m-1} & 1_{d-m} \\ 0 & 1 & 1 & 0 & 0_{m-1} & w_2 \\ 1 & 0 & 0 & 0 & 0_{m-1} & 1_{d-m} \\ 0 & -1 & 0 & 0 & w_1 & 0_{d-m} \\ 0_{m-1}^T & -1_{m-1}^T & 0_{m-1}^T & w_1^T & X & Y_2 \\ w_2^T & 1_{d-m}^T & 1_{d-m}^T & 0_{d-m}^T & Y_2^T & Z \end{pmatrix},$$

in which Y_2 is a submatrix with entries defined similarly to (9) except that the second and the fourth terms of the right side of (9) are dropped out.

After Steps 5 through 8 we get

$$\widehat{M}_3 = \begin{pmatrix} 0 & 0 & -1_{m-1} & -1 & 1_{d-m} & 1 \\ 0 & 0 & 0_{m-1} & 0 & 1_{d-m} & 1 \\ -1_{m-1}^T & 0_{m-1}^T & X & w_1^T & Y_2 & 0_{m-1}^T \\ -1 & 0 & w_1 & 0 & 0_{d-m} & 0 \\ 1_{d-m}^T & 1_{d-m}^T & Y_2^T & 0_{d-m}^T & Z & w_2^T \\ 1 & 1 & 0_{m-1} & 0 & w_2 & 0 \end{pmatrix},$$

in which some rows and columns are repositioned in such a way that the total change of indices of repositioned rows and columns is $(m-1) + (m-1) + (d+2) + (d+1) = 2m + 2d + 1$, which is an odd number. Therefore $\det \widehat{M}_3 = -\det \widehat{M}_2$.

After Steps 9 and 10 we get

$$\widehat{M}_4 = \begin{pmatrix} 0 & 0 & 1_{m-1} & 1 & 0_{d-m} & 0 \\ 0 & 0 & 0_{m-1} & 0 & 1_{d-m} & 1 \\ 1_{m-1}^T & 0_{m-1}^T & X & w_1^T & Y_2 & 0_{m-1}^T \\ 1 & 0 & w_1 & 0 & 0_{d-m} & 0 \\ 0_{d-m}^T & 1_{d-m}^T & Y_2^T & 0_{d-m}^T & Z & w_2^T \\ 0 & 1 & 0_{m-1} & 0 & w_2 & 0 \end{pmatrix}.$$

By Steps 11 through 14, Y_2 is modified so that after Step 14 it has the desired form $Y_2 = \{a_{i,m+j}^2\}$. However, in the last column and in the last row of the actual matrix some undesired terms remain, which are canceled out by Steps 15 and 16. After these last steps we get the matrix $M_{d+3}^{(m)}$. Since the sign of the determinant did not change except Steps 5 through 10, and obviously we have $\det \widehat{M}_3 = \det \widehat{M}_4$, therefore it holds that

$$\det N_{d+1} = \det \widehat{M} = \det \widehat{M}_2 = -\det \widehat{M}_3 = -\det \widehat{M}_4 = -\det M_{d+3}^{(m)}.$$

This completes the proof of Theorem 1. □

2.3. Proof of Theorem 2

We only give an outline of the proof since the more general settings of Theorem 2 would make a detailed proof lengthy and complicated. Apart from this, a proof analogous to the proof of Theorem 1 can be given. We now indicate the main steps of such a proof, pointing out how they are analogous to the corresponding ones of Theorem 1.

For the given k -partition $\mathcal{P} = \{V_1, V_2, \dots, V_k\}$, one can define $C_j = \text{conv}(V_j)$ for $1 \leq j \leq k$, and $C'_j = C_j - v_{n_j}$. Let $C' = \text{conv}(\bigcup_{j=1}^k \text{vert}(C'_j))$. Then C' is a $(d+1-k)$ -dimensional simplex in \mathbb{R}^d . Analogously to the proof of Theorem 1, using corresponding parallelepipeds to each C'_j and to C' , $\text{Vol}_{d+1-k}(S_d \cap H')$ can be expressed as a multiple of $\text{Vol}_{d+1-k}(C')$ with a coefficient depending only on d, k and m_i 's ($1 \leq i \leq k$). Applying (3) for $\text{Vol}_{d+1-k}(C')$, a formula analogous to (6) can be found, by which the proof is reduced to the justification of $\det M_{\mathcal{P}} = (-1)^{k-1} \det N_{d+3-k}$, where N_{d+3-k} is a certain $(d+3-k) \times (d+3-k)$ matrix, having block form

$$N_{d+3-k} = \begin{pmatrix} 0 & 1 & 1_{d+1-k} \\ 1 & 0 & z \\ 1_{d+1-k}^T & z^T & U \end{pmatrix},$$

where z is a row matrix of block form $z = (w_1, w_2, \dots, w_k)$, with the notation

$$w_i = (a_{n_i-m_i+1, n_i}^2, a_{n_i-m_i+2, n_i}^2, \dots, a_{n_i-1, n_i}^2), \quad 1 \leq i \leq k.$$

Analogously to (10), N_{d+3-k} can be expanded with $2k-2$ new rows and columns so that its determinant remains unchanged. The expanded matrix can be written in the following

block form:

$$\hat{M}_{\mathcal{P}} = \begin{pmatrix} I_{k-1} & O_{k-1} & O_{k-1, d+3-k} \\ O_{k-1} & I_{k-1} & V \\ V^T & O_{d+3-k, k-1} & N_{d+3-k} \end{pmatrix},$$

where V is the $(k-1) \times (d+3-k)$ -dimensional matrix whose i th row has block form

$$(0_{n_i-i+2}, w_{i+1}, 0_{d+4+i-k-n_{i+1}}), \quad 1 \leq i \leq k-1.$$

We also used the notation I_r and $O_{r,s}$ for unit square matrices and zero matrices, respectively, whose dimensions are shown in their indices. Next, the entries of the submatrix U of N_{d+3-k} can be calculated by Lemma 3, having forms similar to (9). Finally, one can use row and column operations to get $M_{\mathcal{P}}$ from $\hat{M}_{\mathcal{P}}$. By counting the parity of the total change of indices during the row and column operations one can establish $\det N_{d+3-k} = (-1)^{k-1} \det M_{\mathcal{P}}$. \square

3. Concluding Remarks

It is worth mentioning that a medial section $S_d \cap H^l$ of Theorem 2 may belong to classes of objects of really different shapes, depending on the choice of d , k and m_i 's ($1 \leq i \leq k$). As extremal cases, for $k=1$ and $k=d+1$ we get S_d itself and its baricenter as a single point, respectively. Also, certain segments, triangles and parallelograms can be obtained as medial sections. In general, every medial section can be obtained as the Minkowski sum of certain simplices whose vertices form a partition of the vertices of some d -dimensional simplex. We mention two nonplanar examples: for $d=4$, $k=2$, $m_1=3$ and $m_2=2$, the medial section is a three-dimensional (possibly skew) cylinder with a triangular base, while for $d=5$, $k=2$ and $m_1=m_2=3$ we get a four-dimensional convex polytope with six facets of (possibly skew) cylinders with triangular bases.

The question arises whether there is a volume formula for other, not necessarily medial sections of S_d , obtained as the intersection of S_d and some affine subspace of \mathbb{R}^d . The answer is positive if the affine subspace is parallel to a medial affine subspace of S_d and it intersects the interior of S_d . In fact, then the affine subspace can be written in a form $\hat{H} = \sum_{i=1}^k \alpha_i \text{aff}(V_i)$ for some positive coefficients α_i , $1 \leq i \leq k$, with $\sum_{i=1}^k \alpha_i = 1$. Thus

$$\text{Vol}_{d+1-k}(S_d \cap \hat{H}) = \left(\prod_{i=1}^k \alpha_i^{m_i-1} \right) \text{Vol}_{d+1-k}(S_d \cap H^l),$$

and the explicit formula can be found by substituting (4) into the right side. However, it is not clear at all if there is a formula of similar kind for the volumes of intersections of S_d and generally positioned affine subspaces. We pose this question in the following.

Problem. Let S_d be a d -dimensional simplex with vertices $\{v_1, v_2, \dots, v_{d+1}\}$ in \mathbb{R}^d , and let \tilde{H} be an arbitrary l -dimensional affine subspace of \mathbb{R}^d for some l , $1 \leq l \leq d-1$,

given so that \tilde{H} intersects the interior of S_d . Let \tilde{H}^* be the l -dimensional affine subspace of \mathbb{R}^{d+1} defined as

$$\tilde{H}^* = \left\{ (\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \in \mathbb{R}^{d+1} \mid \sum_{i=1}^{d+1} \alpha_i v_i \in \tilde{H}, \sum_{i=1}^{d+1} \alpha_i = 1 \right\},$$

which encodes the position of \tilde{H} relative to S_d in an affine invariant way.

Is there an algebraic formula expressing $\text{Vol}_l(S_d \cap \tilde{H})$ in terms of the lengths of the edges of S_d and some parameters of \tilde{H}^* ?

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