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METRIC DENSITY AND QUASIMÖBIUS MAPPINGS

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Abstract: We study the notion of μ-density of metric spaces which was introduced by V. Aseev and D. Trotsenko. Interrelation between μ-density and homogeneous density is established. We also characterize μ-dense spaces as "arcwise" connected metric spaces in which "arcs" are the quasimöbius images of the middle-third Cantor set. Finally, we characterize quasiconformal self-mappings of $\dot{\mathbb{R}}^n$ in terms of μ-density.

Keywords: metric density, quasiconformal mapping, quasimöbius mapping

1. Introduction

The theory of quasimöbius mappings, originating with the articles by Aseev, Tukia, and Väisälä, includes the theory of quasiconformal mappings as locally quasimöbius embeddings of domains in $\dot{\mathbb{R}}^n$; see [1, 2.6] and [2, 2.6]. The notion of metric density of [1] plays a special role in the theory of quasimöbius mappings. Namely, the distortion function of each quasimöbius mapping given on such a set can be approximated by a power function [1, Theorem 3.2]. Moreover, it was shown in [3, Theorem 4] that every space possessing the above property has to be μ-dense. See also [4, 3.8] and [5, 2.6] for related concepts.

In this paper we study some properties of μ-dense sets in connection with quasiconformal and quasimöbius mappings as well as homogeneously dense sets. Interrelation between μ-density and homogeneous density is established in Lemma 3.1. We give the exact coefficient of metric density of the middle-third Cantor set in Example 3.9. Theorem 4.4 characterizes μ-dense sets as "arcwise" connected metric spaces in which "arcs" are the quasimöbius images of the middle-third Cantor set. Finally, in Theorem 5.4 we characterize quasiconformality in terms of the coefficient of metric density.

2. Notation and Basic Concepts

2.1. Most of the notations are adopted from [6]. All spaces in this paper are metric and contain no isolated points. They are usually denoted by X or Y. The distance between two points a, b is written as $|a-b|$. The one-point extension of a space X is the union $\dot{X} = X \cup \{\infty\}$ where $\infty \notin X$. If $E \subset X$ is closed and bounded, then $\dot{X} \setminus E$ is said to be a neighborhood of ∞. This defines a Hausdorff topology on $\dot{X}$. If every closed bounded set in X is compact, then $\dot{X}$ is the one-point compactification of X. If $A \subset X$, then $\overline{A}$ is the closure of A, $\mathscr{C}A$ is the complement $X \setminus A$, ∂A is the boundary of A, and $\dot{A}$ is the subspace $A \cup \{\infty\}$ of $\dot{X}$. We let $d(A,B)$ denote the distance between two sets A and B, and let $d(A)$ denote the diameter of A. The open ball $\{x : |x-x_0| < r\}$ is written as $B(x_0, r)$. $\mathbb{R}^n$ stands for the euclidean space and $|a|$ stands for the euclidean norm of a point $a \in \mathbb{R}^n$.

2.2. Let a, b, c, d be distinct points in $\dot{X}$. If the points are in X, their cross ratio $\tau = |a,b,c,d|$ is defined as

$$\tau = |a,b,c,d| = \frac{|a-b||c-d|}{|a-c||b-d|}. \tag{2.3}$$

Otherwise we omit the factor containing ∞. For example,

$$|a,b,c,\infty| = \frac{|a-b|}{|a-c|}.$$

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2.4. Suppose that $f : A \to \dot{Y}$, $A \subset \dot{X}$, is an embedding and $\tau = |a, b, c, d|$ is a cross ratio of points in A. We let τ' denote the image cross ratio $|f(a), f(b), f(c), f(d)|$. We say that f is quasimöbius or QM if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that $\tau' \le \eta(\tau)$ whenever $\tau = |a, b, c, d|$ is a cross ratio of a quadruple of distinct points in A. We also say that f is η-QM. Recall that f is called η-quasisymmetric or η-QS if

$$\frac{|f(a) - f(b)|}{|f(a) - f(c)|} \le \eta\left(\frac{|a - b|}{|a - c|}\right) \tag{2.5}$$

for every triple $a, b, c \in A$ with $a \ne c$; see [4].

Following [1], consider a homeomorphism

$$\eta_\alpha(t) = \begin{cases} t^\alpha, & t \ge 1, \\ t^{\frac{1}{\alpha}}, & 0 \le t < 1, \end{cases} \tag{2.6}$$

where $\alpha \ge 1$. Then every ω-QM (ω-QS) embedding with $\omega(t) = M\eta_\alpha(t)$ is called (M, α)-QM ((M, α)-QS), where $M > 0$.

2.7. Let $A \subset X$. Given a triple of distinct points $x, a, b \in \overline{A}$ with $|a - x| < |b - x|$ we let $G_x(a, b)$ denote the set $\{y \in X : |a - x| < |y - x| < |b - x|\}$ and call it an annulus. Let $\mathscr{H}(A)$ be the collection of all $G_x(a, b)$ with $G_x(a, b) \cap A = \varnothing$. We define the modulus of $G_x(a, b)$ to be

$$\Lambda(G_x(a, b)) = \log \frac{|b - x|}{|a - x|}. \tag{2.8}$$

Given $A \subset X$, let

$$\Lambda_0(A) = \sup_{G \in \mathscr{H}(A)} \Lambda(G). \tag{2.9}$$

If $\mathscr{H}(A) = \varnothing$, we put $\Lambda_0(A) = 0$. We say that $G_x(a, b)$ separates the subsets C and D of X if $|c - x| \le |a - x|$ and $|b - x| \le |d - x|$ for all $c \in C$ and $d \in D$, respectively.

3. μ-Dense Sets and Homogeneously Dense Sets

In this section we establish a relation between the concepts of μ-density and homogeneous density. The latter was introduced by P. Tukia and J. Väisälä [4]. Using this relation we then characterize μ-density of a set A in terms of an annulus with largest modulus separating the components of A (Corollaries 3.4 and 3.5).

Definition [4, 3.8]. A space X is said to be *homogeneously dense* (or HD) if there are numbers λ_1, λ_2 such that $0 < \lambda_1 \le \lambda_2 < 1$ and for each pair of points a, b in X there is a point $x \in X$ satisfying the condition $\lambda_1|b - a| \le |x - a| \le \lambda_2|b - a|$. This X is also said to be (λ_1, λ_2)-HD.

Following [1], a sequence $\{x_i, i \in \mathbb{Z}\}$ of points of X, distinct from $a, b \in \dot{X}$, is called a chain joining the points a and b in X if $x_i \to a$ as $i \to -\infty$ and $x_i \to b$ as $i \to +\infty$. If there exists a real number μ, $1 < \mu < \infty$, such that $|\log(|a, x_i, x_{i+1}, b|)| \le \log \mu$ for all $i \in \mathbb{Z}$, then the chain $\{x_i\}$ is said to be a μ-chain.

Definition [1, 3.1]. A space X is called μ-*dense* ($\mu > 1$) if for each pair of points a, b in X there is a μ-chain $\{x_i\}$ joining the points a and b in X.

We next prove that there is a close connection between the concepts of *homogeneous density* and μ-*density.*

Lemma 3.1. *Let X be a metric space. If X is (λ_1, λ_2)-HD then X is μ-dense with*

$$\mu = \left(\frac{1 + \lambda_2}{\lambda_1(1 - \lambda_2)}\right)^2.$$

Conversely, if X is μ-dense then X is $\left(\frac{1}{6\mu}, \frac{1}{4}\right)$-HD.

PROOF. Suppose that X is (λ_1, λ_2)-HD and let $a, b \in X$. Then by assumption there exists $x_1 \in X$ such that

$$\lambda_1|a-b| \le |a-x_1| \le \lambda_2|a-b|.$$

Similarly, for each $n = 2, 3, \ldots$, there exists $x_n \in X$ such that

$$\lambda_1|a-x_{n-1}| \le |a-x_n| \le \lambda_2|a-x_{n-1}|.$$

Then

$$|a-x_n| \le \lambda_2|a-x_{n-1}| \le \cdots \le \lambda_2^n|a-b|$$

and since $\lambda_2 < 1$, it follows that $x_n \to a$ as $n \to +\infty$. We now have

$$|a, x_n, x_{n+1}, b| = \frac{|a-x_n|}{|a-x_{n+1}|} \cdot \frac{|x_{n+1}-b|}{|x_n-b|} \le \frac{1}{\lambda_1} \cdot \frac{|a-b|+|x_{n+1}-a|}{|a-b|-|x_n-a|}$$

$$\le \frac{1}{\lambda_1} \cdot \frac{(1+\lambda_2^{n+1})|a-b|}{(1-\lambda_2^n)|a-b|} \le \frac{1+\lambda_2}{\lambda_1(1-\lambda_2)}$$

and

$$|a, x_n, x_{n+1}, b| \ge \frac{1}{\lambda_2} \cdot \frac{|a-b|-|x_{n+1}-a|}{|a-b|+|x_n-a|} \ge \frac{1}{\lambda_2} \cdot \frac{(1-\lambda_2^{n+1})|a-b|}{(1+\lambda_2^n)|a-b|} \ge \frac{1-\lambda_2}{\lambda_2}.$$

Hence,

$$|\log(|a, x_n, x_{n+1}, b|)| \le \max\left\{\log\frac{1+\lambda_2}{\lambda_1(1-\lambda_2)}, \log\frac{\lambda_2}{1-\lambda_2}\right\} = \log\frac{1+\lambda_2}{\lambda_1(1-\lambda_2)} < \log\left(\frac{1+\lambda_2}{\lambda_1(1-\lambda_2)}\right)^2.$$

Similarly, for each $n = 0, 1, 2, \ldots$, there exists $y_n \in X$ such that $y_n \to b$ as $n \to +\infty$ and

$$|\log(|a, y_n, y_{n+1}, b|)| \le \log\left(\frac{1+\lambda_2}{\lambda_1(1-\lambda_2)}\right)^2.$$

Also

$$|a, x_1, y_0, b| \ge \frac{|a-x_1|}{|a-b|(1+\lambda_2)} \cdot \frac{|y_0-b|}{|a-b|(1+\lambda_2)} \ge \left(\frac{\lambda_1}{1+\lambda_2}\right)^2$$

and

$$|a, x_1, y_0, b| \le \frac{|a-x_1|}{|a-b|(1-\lambda_2)} \cdot \frac{|y_0-b|}{|a-b|(1-\lambda_2)} \le \left(\frac{\lambda_2}{1-\lambda_2}\right)^2.$$

Hence, the chain $\{z_k,\ k \in \mathbb{Z}\}$,

$$z_k = \begin{cases} x_{-k} & \text{for } k = -1, -2, \ldots, \\ y_k & \text{for } k = 0, 1, 2, \ldots, \end{cases}$$

is a μ-chain joining a and b in X with

$$\mu = \left(\frac{1+\lambda_2}{\lambda_1(1-\lambda_2)}\right)^2.$$

This proves the first assertion.

Suppose next that X is μ-dense. Let $a, b \in X$ and let $\{x_i\}$ be a μ-chain joining a and b in X. Then for each $i \in \mathbb{Z}$

$$\frac{1}{\mu} \le \frac{|a-x_i|\,|x_{i+1}-b|}{|a-x_{i+1}|\,|x_i-b|} \le \mu.$$

Put

$$G = \left\{ x \in X : \frac{1}{6\mu}|a-b| \le |a-x| \le \frac{1}{4}|a-b| \right\}.$$

Clearly any $x \in G$ would complete the proof. So it is enough to show that $G \neq \varnothing$. Assume that $G = \varnothing$. Then there exists $i_0 \in \mathbb{Z}$ such that

$$|a - x_{i_0}| < \frac{|a-b|}{6\mu} \quad \text{and} \quad |a - x_{i_0+1}| > \frac{|a-b|}{4}.$$

However, we also have

$$\frac{1}{\mu} \le \frac{|a-x_{i_0}|}{|b-x_{i_0}|} \cdot \frac{|x_{i_0+1}-b|}{|x_{i_0+1}-a|} < \frac{\frac{1}{6\mu}|a-b|}{(1-\frac{1}{6\mu})|a-b|} \cdot \frac{|a-b|+|a-x_{i_0+1}|}{|a-x_{i_0+1}|}$$

which implies

$$\frac{|a-b|+|a-x_{i_0+1}|}{|a-x_{i_0+1}|} > \frac{6\mu-1}{\mu} > 5.$$

This yields $|a - x_{i_0+1}| < \frac{|a-b|}{4}$ and thus gives us the sought contradiction. $\square$

In the following two lemmas we establish a connection between the homogeneous density of a set A and the quantity $\Lambda_0(A)$ in (2.9).

Lemma 3.2. *If $A \subset X$ is (λ_1, λ_2)-HD then $\Lambda_0(A) \le \log \frac{1}{\lambda_1}$.*

Proof. There is nothing to prove if $\mathscr{H}(A) = \varnothing$. Otherwise let G be an arbitrary annulus in $\mathscr{H}(A)$. Then

$$G = \{x \in X : |c-a| < |c-x| < |c-b|\}$$

for some $a, b, c \in \overline{A}$. By assumption, there exists $d \in A$ such that

$$\lambda_1 |c-b| \le |c-d| \le \lambda_2 |c-b|.$$

Since $G \cap A = \varnothing$, we have $|c-a| \ge |c-d| \ge \lambda_1 |c-b|$. Hence,

$$\Lambda(G) = \log \frac{|c-b|}{|c-a|} \le \log \frac{|c-b|}{\lambda_1 |c-b|} = \log \frac{1}{\lambda_1}$$

which implies $\Lambda_0(A) \le \log \frac{1}{\lambda_1}$. $\square$

Lemma 3.3. *Let $A \subset X$. If $\Lambda_0(A) \le k$ then A is $\left(\frac{1}{2e^{2k}}, \frac{1}{2}\right)$-HD.*

Proof. Take $c, b \in A$. Put

$$G = \left\{ x \in X : \frac{|c-b|}{2e^{2k}} < |c-x| < \frac{|c-b|}{2} \right\}.$$

Note that $\Lambda(G) = 2k$ and hence $G \cap A \neq \varnothing$. Then for every $x \in G \cap A$

$$\frac{1}{2e^{2k}}|c-b| \le |c-x| \le \frac{1}{2}|c-b|. \quad \square$$

The above two lemmas together with Lemma 3.1 give us the following two corollaries that connect μ-density of a set A with the quantity $\Lambda_0(A)$.

Corollary 3.4. *If $A \subset X$ is μ-dense then $\Lambda_0(A) \le \log 6\mu$.* $\square$

Corollary 3.5. *Let $A \subset X$. If $\Lambda_0(A) \le k$, then A is μ-dense with $\mu \le 36e^{4k}$.* □

Next in Corollary 3.7 we establish a connection between the modulus of an annulus separating two points and the connectibility of these points by a μ-chain. But first we need the following

Lemma 3.6. *Suppose that the points $x, y \in A \subset X$ cannot be joined in A by any μ-chain. Then*

$$\sup \Lambda(G_x(a,b)) = +\infty \quad \text{or} \quad \sup \Lambda(G_y(a,b)) = +\infty,$$

where $G_x(a,b)$ and $G_y(a,b)$ are in $\mathscr{H}(A)$ and supremum is taken over all $a, b \in \overline{A}$ such that both $G_x(a,b)$ and $G_y(a,b)$ separate the sets $\{x\}$ and $\{y\}$.

Proof. Suppose that $\sup \Lambda(G_x(a,b)) \le p$ and $\sup \Lambda(G_y(a,b)) \le p$ for some $p < \infty$. Then the sets

$$G_x^i = \left\{ z \in X : \frac{1}{2e^{2pi}}|x-y| < |x-z| < \frac{1}{2e^{2p(i-1)}}|x-y| \right\}$$

and

$$G_y^i = \left\{ z \in X : \frac{1}{2e^{2pi}}|x-y| < |y-z| < \frac{1}{2e^{2p(i-1)}}|x-y| \right\}$$

with $\Lambda(G_x^i) = \Lambda(G_y^i) = 2p$ have nonempty intersections with A for each $i = 1, 2, \dots$. Then any sequence $\{z_k,\ k \in \mathbb{Z}\}$, where $z_k \in G_x^{(-k+1)} \cap A$ for $k = 0, -1, -2, \dots$ and $z_k \in G_y^k \cap A$ for $k = 1, 2, \dots$, will be a μ-chain joining x and y in A with $\mu \le 9e^{4p}$. This gives us the sought contradiction. □

Corollary 3.7. *Let $\mu > 9$. If $x, y \in A \subset X$ cannot be joined in A by a μ-chain then there exists $G_z(a,b) \in \mathscr{H}(A)$ such that $G_z(a,b)$ separates the sets $\{x\}$ and $\{y\}$ and $\Lambda(G_z(a,b)) > \frac{1}{4}\log(\mu/9)$, where $z = x$ or $z = y$.* □

Next we define a measure of density for metric spaces.

Definition 3.8. For a metric space X, the quantity

$$\mu_d(X) = \inf\{\mu : X \text{ is } \mu\text{-dense}\}$$

is called the *coefficient of metric density* of X.

We end this section with an example that gives the coefficient of metric density of the middle-third Cantor set F on the unit interval $[0,1]$.

Example 3.9. If F is the middle-third Cantor set, then $\mu_d(F) = 12.25$.

Proof. According to [7], F is constructed as follows. Let

$$h_1(x) = \frac{1}{3}x \quad \text{and} \quad h_2(x) = \frac{1}{3}x + \frac{2}{3}$$

be similarity transformations of $\mathbb{R}^1$. For $I = [0,1]$ we put $I_1 = h_1(I)$, $I_2 = h_2(I)$, and $F_1 = \partial I_1 \cup \partial I_2$. Similarly, we put $F_2 = \partial I_{11} \cup \partial I_{12} \cup \partial I_{21} \cup \partial I_{22}$, where $I_{11} = h_1(h_1(I))$, $I_{12} = h_1(h_2(I))$, $I_{21} = h_2(h_1(I))$, and $I_{22} = h_2(h_2(I))$. Continuing in this fashion we obtain a sequence of compact sets

$$F_k = \bigcup_{i_1, i_2, \dots, i_k \in \{1,2\}} \partial I_{i_1 i_1 \dots i_k} \quad \text{where} \quad I_{i_1 i_1 \dots i_k} = h_{i_1} \circ h_{i_2} \circ \dots \circ h_{i_k}(I).$$

Then $F_1 \subset F_2 \subset \dots \subset F_k \subset F_{k+1} \subset \dots$ and $F = \overline{F'}$ where $F' = \bigcup_{k=1}^{\infty} F_k$. Note that

$$d(I_{i_1 i_2 \dots i_k}) = (1/3)^k \text{ and } d(I_{i_1 i_2 \dots i_k 1}, I_{i_1 i_2 \dots i_k 2}) = (1/3)^{k+1}.$$

Next we observe the following, omitting details.

Claim 1. *For any distinct points $a, b \in F'$ there exists a sequence $\{x_k,\ k = 0, 1, 2, \dots\}$ of points of F' with $x_0 = b$ such that $x_k \to a$ as $k \to \infty$ and $|a - x_{k+1}| < |a - x_k| \le 3|a - x_{k+1}|$ for all $k = 0, 1, 2, \dots$.*

Claim 2. *For any $a \in I_1 \cap F'$ and $b \in I_2 \cap F'$ there exist $x \in I_1 \cap F'$ and $y \in I_2 \cap F'$ such that $|\log(|a, x, y, b|)| \le \log(12.25)$.*

Moreover, if $a = 2/9$ and $b = 7/9$, then $|\log(|a, x, y, b|)| \ge \log 12.25$ for all $x \in I_1$ and $y \in I_2$, with equality only for $x = 0$ and $y = 1$.

Now using Claims 1 and 2 we easily obtain $\mu_d(F') = 12.25$.

Finally, using the fact that $\mu_d(X) \le \mu$ implies $\mu_d(\overline{X}) \le \mu$ for any set $X \in \dot{\mathbb{R}}^n$ we find $\mu_d(F) = 12.25$. We omit details. $\square$

4. Quasimöbius Mappings and μ-Dense Sets

In this section we study some relations between QM-mappings and μ-dense sets. It was first noticed by P. Tukia and J. Väisälä (see [4]) that every QS-mapping given on a HD-set is (C, α)-QS. Then similar results for QM-mappings, proved by V. Aseev and D. Trotsenko, led to the introduction of μ-dense sets.

Theorem 4.1 [1, 3.2]. *Every ω-QM mapping given on a μ-dense set is (M, α)-QM where M and α depend only on ω and μ.*

The fact that the condition of μ-density in Theorem 4.1 is also necessary was partly shown in [1, 4.1]. The complete solution, given in [3, Theorem 3], has led to another characterization of μ-dense sets:

Theorem 4.2 [3, Theorem 4]. *Let $X \subset \dot{\mathbb{R}}^n$ be a set containing no isolated points. Then X is μ-dense if and only if every ω-QM-mapping $f : X \to \dot{\mathbb{R}}^n$ is (M, α)-QM, where M, α and μ, ω depend only on each other.*

Theorem 4.2 should be compared to the following theorem of Trotsenko and Väisälä.

Theorem 4.3 [8, Theorem 6.21]. *The following conditions are quantitatively equivalent for a metric space A:*

(1) *A is M-relatively connected;*

(2) *Every η-quasisymmetric mapping of A is (C, α)-quasisymmetric with (C, α) depending only on η.*

M-relatively connected sets include uniformly perfect sets and they can contain isolated points. See [8] for more discussion on these sets.

Now, suppose that for each pair of distinct points a, b of a set X there is an ω-QM mapping $f : F \to X$ such that $f(0) = a$ and $f(1) = b$ where F is the Cantor set (see Example 3.9 above). Then we can easily see that X is μ-dense with μ depending only on ω. Next we prove the converse of this statement.

Theorem 4.4. *Let X be a complete metric space and let F be the Cantor set. If X is μ-dense, then for all $b_1, b_2 \in X$ with $b_1 \ne b_2$ there exists a (M, n_0)-QM mapping $g : F \to X$ with $g(0) = b_1$ and $g(1) = b_2$ where n_0 and M depend only on μ. Here n_0 is the smallest integer greater than* $\max\{\mu + 1, 4\}$.

Proof. Put $\omega(t) = 5 \cdot 3^{n_0} \eta_{n_0}(t)$ (see 2.6) and $E_0 = \{b_1, b_2\}$. Then by Lemma 3.1 there exist $b_{12}, b_{21} \in X$ such that

$$\frac{|b_1 - b_2|}{6\mu} \le |b_1 - b_{12}| \le \frac{|b_1 - b_2|}{4}$$

and

$$\frac{|b_1 - b_2|}{6\mu} \le |b_{21} - b_2| \le \frac{|b_1 - b_2|}{4}.$$

Put $E_1 = \{b_{11}, b_{12}, b_{21}, b_{22}\}$ where $b_{11} = b_1$ and $b_{22} = b_2$. Next apply Lemma 3.1 to the pairs (b_{11}, b_{12}) and (b_{21}, b_{22}) to get

$$E_2 = \{b_{111}, b_{112}, b_{121}, b_{122}, b_{211}, b_{212}, b_{221}, b_{222}\}$$

where $b_{111} = b_{11},\ b_{122} = b_{12},\ b_{211} = b_{21}$, and $b_{222} = b_{22}$. Observe that

$$\frac{|b_1 - b_2|}{(6\mu)^2} \le |b_{i_1 i_2 1} - b_{i_1 i_2 2}| \le \frac{|b_1 - b_2|}{4^2}$$

for all 2-tuples $i_1 i_2$. Continuing in this fashion we obtain a sequence $\{E_k\}$, $E_k = \{b_{i_1 i_2 \dots i_k i_{k+1}},\ i_j \in \{1,2\}\}$, of subsets of X. Moreover,

$$E_0 \subset E_1 \subset E_2 \subset \dots \subset E_k \subset E_{k+1} \subset \dots$$

and

$$\frac{|b_1 - b_2|}{(6\mu)^k} \le |b_{i_1 i_2 \dots i_k 1} - b_{i_1 i_2 \dots i_k 2}| \le \frac{|b_1 - b_2|}{4^k} \tag{4.5}$$

for all k-tuples $i_1 i_2 \dots i_k$. Put $E' = \bigcup_{k=1}^{\infty} E_k$. We say that E_k is the kth iteration subset of X generated by the pair $\{b_1, b_2\}$.

Now recall from the proof of Example 3.9 that we have $F' = \bigcup_{k=1}^{\infty} F_k$ and $F = \overline{F}'$. For simplicity we adopt the following notation which is self-explanatory:

$$F_1 = \{a_{11}, a_{12}, a_{21}, a_{22}\}, \quad F_2 = \{a_{111}, a_{112}, a_{121}, a_{122}, a_{211}, a_{212}, a_{221}, a_{222}\}, \dots,$$
$$F_k = \{a_{i_1 i_2 \dots i_k i_{k+1}},\ i_j \in \{1,2\}\}.$$

Here $a_{11} = 0$, $a_{12} = 1/3$, $a_{21} = 2/3$, and $a_{22} = 1$. Also $a_{111} = a_{11} = 0$, $a_{112} = 1/9$, $a_{121} = 2/9$, $a_{122} = a_{12} = 1/3$, $a_{211} = a_{21} = 2/3$, $a_{212} = 7/9$, $a_{221} = 8/9$, $a_{222} = a_{22} = 1$. Now for every $k = 1, 2, \dots$ define a mapping $f_k : F_k \to E_k$ by $f_k(a_{i_1 i_2 \dots i_{k+1}}) = b_{i_1 i_2 \dots i_{k+1}}$. This obviously yields a mapping $f : F' \to E'$.

We will first show that f is ω-QS. Clearly, it suffices to prove that f_k is ω-QS for every $k = 1, 2, \dots$. We prove this by induction on k. If $k = 1$, we can easily verify that f_1 is ω-QS. Assume that f_{k-1} is ω-QS. Observe that the assumption implies that f_{k-1} is an ω-QS embedding from F_{k-1} into the kth iteration subset of X generated by any pair of points in X. Let $h_1(x) = \frac{x}{3}$ and $h_2(x) = \frac{x+2}{3}$ be similarity transformations of $\mathbb{R}^1$. Then

$$F_k = h_1(F_{k-1}) \cup h_2(F_{k-1}),$$
$$E_k = f_k(F_k) = (f_k \circ h_1)(F_{k-1}) \cup (f_k \circ h_2)(F_{k-1}).$$

Let x, y, z be a triple of distinct points in F_k. Then by the construction of f_k and the induction hypothesis, $f_{k|h_1(F_{k-1})}$ and $f_{k|h_2(F_{k-1})}$ are ω-QS. Hence it is enough to consider the following two cases:

CASE 1: $x \in h_1(F_{k-1})$ and $y, z \in h_2(F_{k-1})$. Since

$$\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \dots = \frac{1}{3},$$

we have

$$b_{1 i_1 i_2 \dots i_n} \in B(b_1, (1/3)|b_1 - b_2|) \quad \text{and} \quad b_{2 i_1 i_2 \dots i_n} \in B(b_2, (1/3)|b_1 - b_2|)$$

for all n-tuples $i_1 i_2 \dots i_n$, and all $n = 1, 2, \dots$. Hence,

$$f_k(x) \in B\Big(b_1, \frac{1}{3}|b_1 - b_2|\Big) \quad \text{and} \quad f_k(y), f_k(z) \in B\Big(b_2, \frac{1}{3}|b_1 - b_2|\Big)$$

whence we easily see

$$\frac{1}{3} \le \frac{|x - y|}{|x - z|} \le 3 \quad \text{and} \quad \frac{1}{5} \le \frac{|f_k(x) - f_k(y)|}{|f_k(x) - f_k(z)|} \le 5.$$

Hence, f_k is $(5 \cdot 3^{n_0}, n_0)$-QS.

CASE 2: $x, y \in h_1(F_{k-1})$ and $z \in h_2(F_{k-1})$. Then $x = a_{i_1 i_2 \dots i_{k+1}}$ and $y = a_{j_1 j_2 \dots j_{k+1}}$ for some $(k+1)$-tuples $i_1 i_2 \dots i_{k+1}$ and $j_1 j_2 \dots j_{k+1}$, respectively, and $i_1 = j_1 = 1$. Let p be the smallest integer, $2 \le p \le k+1$, for which $i_p \ne j_p$. Then

$$\frac{1}{3^p} \le |x - y| \le \frac{1}{3^{p-1}},$$

where $x = a_{i_1 i_2 \dots i_{(p-1)} i_p \dots i_{(k+1)}}$ and $y = a_{i_1 i_2 \dots i_{(p-1)} j_p \dots j_{(k+1)}}$. Hence,

$$x, y \in [a_{i_1 i_2 \dots i_{(p-1)}} 1,\ a_{i_1 i_2 \dots i_{(p-1)}} 2].$$

Moreover, x and y lie in two different subsegments

$$[a_{i_1 i_2 \dots i_{(p-1)}} 11,\ a_{i_1 i_2 \dots i_{(p-1)}} 12] \quad \text{and} \quad [a_{i_1 i_2 \dots i_{(p-1)}} 21,\ a_{i_1 i_2 \dots i_{(p-1)}} 22]$$

of the segment $[a_{i_1 i_2 \dots i_{(p-1)}} 1,\ a_{i_1 i_2 \dots i_{(p-1)}} 2]$. By symmetry we can assume that

$$x \in [a_{i_1 i_2 \dots i_{(p-1)}} 11,\ a_{i_1 i_2 \dots i_{(p-1)}} 12] \quad \text{and} \quad y \in [a_{i_1 i_2 \dots i_{(p-1)}} 21,\ a_{i_1 i_2 \dots i_{(p-1)}} 22].$$

Then we have

$$f_k(x) \in B\Big(b_{1 i_2 \dots i_{(p-1)}} 1, \frac{1}{3} |b_{1 i_2 \dots i_{(p-1)}} 1 - b_{1 i_2 \dots i_{(p-1)}} 2|\Big)$$

and

$$f_k(y) \in B\Big(b_{1 i_2 \dots i_{(p-1)}} 2, \frac{1}{3} |b_{1 i_2 \dots i_{(p-1)}} 1 - b_{1 i_2 \dots i_{(p-1)}} 2|\Big).$$

Therefore,

$$\frac{|f_k(x) - f_k(y)|}{|f_k(x) - f_k(z)|} \leq \frac{\frac{5}{3} (\frac{1}{4})^{p-1} |b_1 - b_2|}{\frac{1}{3} |b_1 - b_2|} \leq 5(1/3)^{p-1} = 15(1/3)^p$$

and

$$\frac{|f_k(x) - f_k(y)|}{|f_k(x) - f_k(z)|} \geq \frac{\frac{1}{3} (\frac{1}{6\mu})^{p-1} |b_1 - b_2|}{\frac{5}{3} |b_1 - b_2|} \geq \frac{(\frac{1}{3^{n_0}})^{p-1}}{5} = \frac{(\frac{1}{3^{n_0}})^{p-2}}{5 \cdot 3^{n_0}} \geq \frac{1}{5 \cdot 3^{n_0}} (\frac{1}{3^{p-2}})^{n_0}.$$

Since

$$\frac{1}{3^p} \leq \frac{|x - y|}{|x - z|} \leq \frac{1}{3^{p-2}},$$

it follows that

$$\frac{1}{5 \cdot 3^{n_0}} (\frac{|x - y|}{|x - z|})^{n_0} \leq \frac{|f_k(x) - f_k(y)|}{|f_k(x) - f_k(z)|} \leq 15 \cdot \frac{|x - y|}{|x - z|}.$$

Thus, f_k is $(5 \cdot 3^{n_0}, n_0)$-QS.

Combining Cases 1 and 2, we show that $f_k : F_k \to E_k$ is $(5 \cdot 3^{n_0}, n_0)$-QS, implying that so is $f : F' \to E'$. Since X is complete, by [4, 2.25] f can be extended to an $(5 \cdot 3^{n_0}, n_0)$-QS embedding $g : F \to X$. Then by [1, 2.9] g is (M, n_0)-QM, where M depends only on n_0. $\square$

5. Characterization of Quasiconformality in Terms of Metric Density

In this section we (qualitatively) characterize quasiconformal self-mappings of $\dot{\mathbb{R}}^n$ in terms of the coefficients of metric density of subsets of $\dot{\mathbb{R}}^n$. For notations and basic concepts of this section, see [9–11]. Let Γ be a curve family in $\dot{\mathbb{R}}^n$ and let $F(\Gamma)$ be the set of all nonnegative Borel functions $\rho : \mathbb{R}^n \to \dot{\mathbb{R}}^1$ such that

$$\int_{\gamma} \rho \, ds \geq 1$$

for every locally rectifiable curve $\gamma \in \Gamma$. Then the conformal modulus of Γ is defined as

$$M(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{\mathbb{R}^n} \rho^n \, dm.$$

A *ring* is a domain $A \subset \dot{\mathbb{R}}^n$ whose complement is the union of two disjoint connected compact sets $\mathscr{C}_0$ and $\mathscr{C}_1$. We denote such a ring by $R(\mathscr{C}_0, \mathscr{C}_1)$. We say that a ring $R(\mathscr{C}_0, \mathscr{C}_1)$ is degenerate if either $\mathscr{C}_0$ or

$\mathscr{C}_1$ consists of a single point. Let $A = R(\mathscr{C}_0, \mathscr{C}_1)$ and let Γ_A be the family of all curves joining $\mathscr{C}_0$ and $\mathscr{C}_1$ in A. The conformal modulus of the ring A is then

$$\operatorname{mod} A = \operatorname{mod} R(\mathscr{C}_0, \mathscr{C}_1) = \left(\frac{\omega_{n-1}}{M(\Gamma_A)}\right)^{\frac{1}{n-1}} \tag{5.1}$$

where ω_{n-1} is the surface area of the unit sphere in $\dot{\mathbb{R}}^n$.

According to J. Väisälä [9], a homeomorphism $f : D \to D'$, where D and D' are domains in $\dot{\mathbb{R}}^n$, is called K-QC $(1 \le \infty)$ in D iff

$$\frac{1}{K} M(\Gamma) \le M(\Gamma') \le K M(\Gamma)$$

for every path family Γ in D. However, it was later shown that quasiconformality can be characterized by requiring that (5.1) be satisfied for certain classes of path families; in particular, for a family of paths joining the boundary components of rings in D. In Theorem 5.3 below we characterize quasiconformality in terms of the coefficient of metric density of sets in D.

The following lemma will be used in the proof of Theorem 5.3.

Lemma 5.2. *Let $D \subset \dot{\mathbb{R}}^n$ be a domain and let $\{A_m\}$ be a sequence of disjoint, nondegenerate rings with $\overline{A}_m \subset D$. Then $\sup_m \operatorname{mod} A_m = \infty$ iff $\mu_d(D_0) = \infty$ where $D_0 = D \setminus \bigcup_{m=1}^{\infty} A_m$.*

Proof. Suppose that $\sup_m \operatorname{mod} A_m = \infty$ and $A_m = R(C_0^m, C_1^m)$. We first observe that since the A_m's are nondegenerate, the set D_0 contains no isolated points.

Next, we show that for each $\mu > 1$ there exist points $a, b \in D_0$ which cannot be joined in D_0 by any μ-chain. Indeed, we choose m such that

$$\operatorname{mod} A_m > \log \lambda_n(\mu + 1)$$

where λ_n is a positive constant depending only on n; see [12, p. 225]. Let now $a \in D_0 \cap C_0^m$ and $b \in D_0 \cap C_1^m$ be arbitrary points and let $\{x_i\}$ be any μ_1-chain joining a and b in $\dot{\mathbb{R}}^n \setminus A_m$. Then there exists $i \in \mathbb{Z}$ such that $x_i \in C_0^m$ and $x_{i+1} \in C_1^m$. Using [12, Corollary 1], we hence have

$$\log \lambda_n(\mu + 1) < \operatorname{mod} A_m < \operatorname{mod} R_T(|a, x_i, x_{i+1}, b|) \le \log \lambda_n(|a, x_i, x_{i+1}, b| + 1)$$

which implies

$$\mu < |a, x_i, x_{i+1}, b| \le \mu_1.$$

Here $R_T(t)$ is the Teichmüller ring; see [12, p. 225]. Hence, we infer that $\{x_i\}$ is not a μ-chain. In particular, the points a and b cannot be joined in $\dot{\mathbb{R}}^n \setminus A_m$ and so in D_0 by any μ-chain. Thus, $\mu_d(D_0) = \infty$.

Conversely, let $\mu_d(D_0) = \infty$. In view of Möbius invariance we can assume that $\infty \in A_1$. By Corollary 3.5 we have $\sup_{G \in \mathscr{H}(D_0)} \Lambda(G) = \infty$. We will show that for each $G \in \mathscr{H}(D_0)$ there exists k such that Γ_{A_k} is minorized by Γ_G. So we let G be any element of $\mathscr{H}(D_0)$. Then $G = \{x \in \mathbb{R}^n : |b - a| < |x - a| < |c - a|\}$ for some $a, b, c \in \overline{D}_0$ and $G \cap D_0 = \varnothing$ which implies that $G \subset (\bigcup_{m=1}^{\infty} A_m) \cup (\dot{\mathbb{R}}^n \setminus D)$. But since $(\bigcup_{m=1}^{\infty} A_m) \cap (\dot{\mathbb{R}}^n \setminus D) = \varnothing$ and since G is connected, we have $G \subset \bigcup_{m=1}^{\infty} A_m$. By assumption the A_m's are disjoint, so we have $G \subset A_k$ for some k. And finally since $a \in \overline{D}_0$, we have Γ_{A_k} minorized by Γ_G. Then by [9, Theorem 6.4] we have $M(\Gamma_{A_k}) < M(\Gamma_G)$ and hence $\operatorname{mod} A_k > \operatorname{mod} G$. Thus,

$$\infty = \sup_{G \in \mathscr{H}(D_0)} \Lambda(G) \le \sup_m \operatorname{mod} A_m$$

which completes the proof. □

Theorem 5.3. *Let $D, D' \subset \dot{\mathbb{R}}^n$ be domains and suppose that $f : D \to D'$ is a homeomorphism such that $\mu_d(\Sigma) < \infty$ if and only if $\mu_d(f\Sigma) < \infty$ for every subset Σ of D. Then f is QC.*

PROOF. Assume that f is not QC. Then by [13, Lemma 1] there exists a sequence $\{A_m\}$ of disjoint rings A_m with $\overline{A}_m \subset D$ such that

$$\operatorname{mod} A_m < \frac{\kappa}{m^2} \quad \text{and} \quad \operatorname{mod} A'_m > m^2 \text{ for all } \; m = 1, 2, \dots$$

where $A'_m = A'(x_m, r_m) = \{x' : \; l(x_m, r_m) \leq |x' - f(x)| \leq L(x_m, r_m)\}$ is a spherical ring as in [13, Theorem 1], $\kappa = \kappa(n) < \infty$ is a constant from Väisälä's lemma [14, Theorem 3.10], and $A_m = f^{-1}(A'_m)$. In particular, the A_m's are nondegenerate by [9, Theorem 11.10]. Let $D_0 = D \setminus \bigcup_{m=1}^{\infty} A_m$. Then

$$f(D_0) = f(D) \setminus \bigcup_{m=1}^{\infty} f(A_m) = D' \setminus \bigcup_{m=1}^{\infty} A'_m.$$

Hence, by Lemma 5.2, $\mu_d(D_0) < \infty$ while $\mu_d(fD_0) = \infty$ which is a contradiction. Thus, f is QC. $\square$

Theorem 5.4. *A necessary and sufficient condition for a homeomorphism $f : \dot{\mathbb{R}}^n \to \dot{\mathbb{R}}^n$ to be QC is that $\mu_d(f\Sigma) < \infty$ ($\mu_d(f\Sigma) = \infty$) iff $\mu_d(\Sigma) < \infty$ ($\mu_d(\Sigma) = \infty$) for every subset Σ of $\mathbb{R}^n$.*

PROOF. *Necessity* follows from the fact that f is QM (see, for example, [6, 5.3]) and that finiteness of a metric density of a set is preserved under QM-mappings.

Sufficiency follows from Theorem 5.3. $\square$

REMARK. 1. The necessity part in the above proof follows also from [5, Corollary 4.6].

2. The proof of Theorem 5.3 was suggested by V. V. Aseev; see also [15, Theorem 9].

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