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ON SUMS OF OVERLAPPING PRODUCTS OF INDEPENDENT BERNOULLI RANDOM VARIABLES

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Dedicated to A. V. Skorokhod on his 70th birthday

We find the exact distribution of an arbitrary remainder of an infinite sum of overlapping products of a sequence of independent Bernoulli random variables.

Results and Discussion

Let $X_1, X_2, \ldots$ be independent random variables with distribution

$$P\{X_n = 1\} = \frac{1}{\mu + n - 1} = 1 - P\{X_n = 0\}, \qquad n \in \mathbf{N} := \{1, 2, \ldots\}, \tag{1}$$

where $\mu \geq 1$ is a fixed real-valued parameter, and introduce the random variable $N := N_1 = \sum_{n=1}^{\infty} X_n X_{n+1}$ along with the remainders

$$N_l := \sum_{n=l}^{\infty} X_n X_{n+1}, \qquad l \in \mathbf{N},$$

of the infinite sum. The random nonnegative integer N is well defined; in fact by the monotone convergence theorem

$$E(N_l) = \sum_{n=l}^{\infty} \frac{1}{(\mu + n - 1)(\mu + n)} < \infty$$

and so $E(N_l) = 1/l$ in the particular case $\mu = 1$, for every $l \in \mathbf{N}$. The aim of this note is to determine the distribution of N_l for all $l \in \mathbf{N}$.

The problem of computing the distribution of $N = N_1$ was originally posed for the case $\mu = 1$ to the second-named author by Y. S. Chow. When the solution was obtained by the method of generating functions, which states that if $\mu = 1$, then N is a Poisson random variable with mean 1, P. Diaconis [1] kindly informed him that the result was known: Diaconis' own proof for this result was included in unpublished notes of Michel Emery in Strasbourg and in an unpublished dissertation by Lars-Ola Hahlin in Uppsala, and it also follows as the special case $\lambda = 1$ for the first coordinate of an infinite-dimensional convergence theorem in Sec. 3 of the paper by Arratia, Barbour, and Tavare [2]. Considering the distributions

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$$\mathrm{P}\{X_n = 1\} = \frac{\lambda}{\lambda + n - 1} = 1 - \mathrm{P}\{X_n = 0\}, \qquad n \in \mathbf{N}, \tag{2}$$

for some constant $\lambda > 0$ instead of (1), the method in [2] is purely combinatorial, it identifies the Poisson distribution of N with mean λ as the limiting distribution of the number of cycles of size 1 in a random permutation under the Ewens sampling formula. This method does not appear to produce the distribution of N_l for $l > 1$, even for $\lambda = 1$. Our direct proof here does this for all $l \in \mathbf{N}$ and all $\mu \geq 1$ for the distributions in (1), and, in this case, it is of independent interest even for $N = N_1$ when $\mu = 1$. Throughout, all empty sums are understood as zero and all empty products are understood as one.

Theorem 1. *Let $X_1, X_2, \ldots$ be independent random variables with the distributions in (1) for some $\mu \geq 1$. Then, for any $l, n \in \mathbf{N}$ such that $n \geq 1$,*

$$\mathrm{P}\{ X_l X_{l+1} + \ldots + X_n X_{n+1} + X_{n+1} = k \} = \sum_{j=l+k-2}^{n} \frac{(-1)^{j+k+l}}{\prod_{r=l}^{j+1} (\mu+r-1)} \binom{j+2-l}{k} \tag{3}$$

and, hence, for all $l \in \mathbf{N}$,

$$\mathrm{P}\{N_l = k\} = \sum_{j=l+k-2}^{\infty} \frac{(-1)^{j+k+l}}{\prod_{r=l}^{j+1} (\mu+r-1)} \binom{j+2-l}{k} \tag{4}$$

for every nonnegative integer k, and the generating function of N_l is

$$E\left(s^{N_l}\right) = \sum_{j=l-2}^{\infty} \frac{(s-1)^{j+2-l}}{\prod_{r=l}^{j+1} (\mu + r - 1)} = 1 + \frac{s-1}{\mu + l - 1} + \frac{(s-1)^2}{(\mu + l - 1)(\mu + l)} + \ldots \tag{5}$$

for all $s \in [0, 1]$.

Note that the first statement in (3) and formula (6) in the proof below also give the exact distribution of any section $X_l X_{l+1} + \ldots + X_n X_{n+1}$ of the series defining N.

In the special case $\mu = 1$, formulas (3), (4), and (5) take the form

$$\mathrm{P}\{ X_l X_{l+1} + \ldots + X_n X_{n+1} + X_{n+1} = k \} = (l-1)! \sum_{j=l+k-2}^{n} \frac{(-1)^{j+k+l}}{(j+1)!} \binom{j+2-l}{k},$$

$$\mathrm{P}\{ N_l = k \} = (l-1)! \sum_{j=l+k-2}^{\infty} \frac{(-1)^{j+k+l}}{(j+1)!} \binom{j+2-l}{k} \tag{4_1}$$

for every nonnegative integer k and

$$E\left(s^{N_l}\right) = 1 + \frac{s-1}{l} + \frac{(s-1)^2}{l(l+1)} + \frac{(s-1)^3}{l(l+1)(l+2)} + \ldots = \frac{(l-1)!}{(s-1)^{l-1}} \sum_{j=l-1}^{\infty} \frac{(s-1)^j}{j!} \tag{5_1}$$

for all $s \in [0, 1]$. For $l = 1$, it follows from (4_1) in this particular case that

$$\mathrm{P}\{N=k\} = \sum_{j=k-1}^{\infty} \frac{(-1)^{j+1+k}}{(j+1)!}\binom{j+1}{k} = \frac{1}{k!}\sum_{j+1-k=0}^{\infty} \frac{(-1)^{j+1-k}}{(j+1-k)!} = \frac{1}{k!}e^{-1}$$

for all $k = 1, 2, \ldots$, or, equivalently from (5_1), $E(s^N) = e^{s-1}$, $0 \le s \le 1$, the generating function of the Poisson distribution with mean 1. Note the interesting fact, in this connection, that the multiplying factor $\sum_{j=l-1}^{\infty}(s-1)^j/j!$ of the second formula in (5_1) is the remainder of the polynomial approximation of degree $l-2$ of e^{s-1}. All in all, the distributions equivalently given by (4) or (5) may be looked upon as a parametric family ($\mu \ge 1$, $l \in \mathbf{N}$) extending the Poisson distribution with mean 1.

In the converse direction, we conjecture the following: If $X_1, X_2, \ldots$ are independent Bernoulli random variables such that $\mathrm{P}\{X_1X_2 = 1\} > 0$ and the distribution of $N = N_1 = \sum_{n=1}^{\infty} X_nX_{n+1}$ is given by (4) with $l = 1$, for some $\mu = 1$, then $E(X_n) = 1/(\mu + n - 1)$ for the same μ, for each $n \in \mathbf{N}$. As a special case for $\mu = 1$, this would give a joint characterization of the standard (mean 1) Poisson and the Bernoulli distributions in (1) with $\mu = 1$. The following result confirms the conjecture under the extra condition that an extended "scaled" version of the full conclusion of Theorem 1 holds:

Theorem 2. *Let $X_1, X_2, \ldots$ be independent random variables with distribution given by $\mathrm{P}\{X_n = 1\} = p_n = 1 - \mathrm{P}\{X_n = 0\}$ for some $p_n \in (0, 1)$, $n \in \mathbf{N}$, such that the generating function of $N_l = \sum_{n=l}^{\infty} X_nX_{n+1}$ is*

$$f_{l,\lambda}(s) := E\left(s^{N_l}\right) = 1 + \frac{\lambda(s-1)}{\mu+l-1} + \frac{[\lambda(s-1)]^2}{(\mu+l-1)(\mu+l)} + \frac{[\lambda(s-1)]^3}{(\mu+l-1)(\mu+l)(\mu+l+1)} + \ldots,$$

$0 \le s \le 1$, for all $l \in \mathbf{N}$ and some $\lambda > 0$ and $\mu \ge 1$. Then, necessarily, $\lambda = 1$ and $p_n = 1/(\mu + n - 1)$ for every $n \in \mathbf{N}$.

The function $f_{l,\lambda}(\cdot)$ here is a seemingly natural generalization of the generating function in Theorem 1 because, for the pair $(\lambda, \mu) = (1, 1)$, it reduces to $f_{1,\lambda}(s) = e^{\lambda(s-1)}$, $0 \le s \le 1$, the generating function of the Poisson distribution with mean λ. However, Theorem 2 excludes this parameterization by asserting that the only possible λ is 1. The result in [2], stated above, suggests that a version of the conjecture above that if $X_1, X_2, \ldots$ are independent Bernoulli random variables such that $\mathrm{P}\{X_1X_2 = 1\} > 0$ and the distribution of $N = N_1 = \sum_{n=1}^{\infty} X_nX_{n+1}$ is Poisson with mean $\lambda > 0$, then $E(X_n) = \lambda/(\lambda + n - 1)$ for each $n \in \mathbf{N}$. To prove the corresponding weaker version, an analog of Theorem 2, would require the presently unavailable knowledge of the generating functions of N_l for all $l \in \mathbf{N}$ under the distributions in (2), i.e., the corresponding version of Theorem 1. A remark on this and related problems is placed after the proof of Theorem I below.

Finally, we mention another problem that naturally arises and is open even for our present sequence of independent variables $X_1, X_2, \ldots$ satisfying (1) with $\mu = 1$. For a number $k \in \mathbf{N}$, what is the distribution of $S_k := \sum_{n=1}^{\infty} \prod_{j=n}^{n+k} X_j$? Here, $S_1 = N$, of course, and so Theorem 1 answers the question for $k = 1$, but, while various systems of recursive equations may be derived as in the proof of Theorem 1 below, we were unable to identify in any explicit sense the distribution of even the next case, the distribution of $S_2 = \sum_{n=1}^{\infty} X_nX_{n+1}X_{n+2}$.

Proof of Theorem 1. For all admissible values of the integers l, n and k, we introduce

$$p_{l,n}(k) := \mathrm{P}\{X_lX_{l+1} + \ldots + X_nX_{n+1} + X_{n+1} = k\},$$

$$p^*_{l,n}(k) \;=\; \sum_{j=l+k-2}^{n} \frac{(-1)^{j+k+l}}{\prod_{r=l}^{j+1}(\mu+r-1)} \binom{j+2-l}{k}$$

and

$$q_{l,n}(k) \;:=\; \mathrm{P}\{\, X_l X_{l+1} + \ldots + X_n X_{n+1} \;=\; k \,\},$$

and let us agree to understand $p_{l,n}(k)$, $p^*_{l,n}(k)$, and $q_{l,n}(k)$ as zero if k is negative. Conditioning on X_{n+2}, we obtain

$$p_{l,n+1}(k) \;=\; \frac{\mu+n}{\mu+n+1} q_{l,n}(k) + \frac{1}{\mu+n+1} p_{l,n}(k-1)$$

and

$$q_{l,n+1}(k) \;=\; \frac{\mu+n}{\mu+n+1} q_{l,n}(k) + \frac{1}{\mu+n+1} p_{l,n}(k).$$

From the first of these two equations

$$q_{l,n}(k) \;=\; \frac{\mu+n+1}{\mu+n} p_{l,n+1}(k) - \frac{1}{\mu+n} p_{l,n}(k-1), \tag{6}$$

by which the second becomes

$$\frac{\mu+n+2}{\mu+n+1} p_{l,n+2}(k) - \frac{p_{l,n+1}(k-1)}{\mu+n+1} \;=\; \frac{p_{l,n}(k)}{\mu+n+1} + p_{l,n+1}(k) - \frac{p_{l,n}(k-1)}{\mu+n+1},$$

or, equivalently,

$$p_{l,n+2}(k) \;=\; p_{l,n+1}(k) + \frac{\left[\, p_{l,n+1}(k-1) - p_{l,n}(k-1) \,\right] - \left[\, p_{l,n+1}(k) - p_{l,n}(k) \,\right]}{\mu+n+2}. \tag{7}$$

The crux of the argument is to come up with a reasonable conjecture from the recursion in (7) for the form of $p_{l,n}(k)$, which is given by $p^*_{l,n}(k)$ above. Having this, we now proceed to prove the desired identity $p_{l,n}(\cdot) \equiv p^*_{l,n}(\cdot)$ by induction, which for each $m \geq l$ produces $p_{l,m+2}(k)$ from $p_{l,m+1}(\cdot)$ and $p^*_{l,m}(\cdot)$ for all nonnegative integers k.

First, for all $k = 0, 1, 2, \ldots$, we must consider

$$p_{l,l}(k) \;=\; \mathrm{P}\{\, X_l X_{l+1} + X_{l+1} = k \,\}$$

and

$$p_{l,l+1}(k) \;=\; \mathrm{P}\{\, X_l X_{l+1} + X_{l+1} X_{l+2} + X_{l+2} = k \,\}$$

in a direct fashion. Clearly, $p_{l,l}(k) = 0 = p^*_{l,l}(k)$ for all $k > 2$ and $p_{l,l+1}(k) = 0 = p^*_{l,l+1}(k)$ for all $k > 3$. Also,

$$p_{l,l}(0) = \mathrm{P}\{X_{l+1}=0\} = \frac{\mu+l-1}{\mu+l} = 1 - \frac{1}{\mu+l-1} + \frac{1}{(\mu+l-1)(\mu+l)} = p^*_{l,l}(0),$$

$$p_{l,l}(1) = \mathrm{P}\{X_l=0,\ X_{l+1}=1\} = \frac{\mu+l-2}{\mu+l-1}\frac{1}{\mu+l} = \frac{1}{\mu+l-1} - \frac{2}{(\mu+l-1)(\mu+l)} = p^*_{l,l}(1)$$

and

$$p_{l,l}(2) = \mathrm{P}\{X_l=1,\ X_{l+1}=1\} = \frac{1}{\mu+l-1}\frac{1}{\mu+l} = p^*_{l,l}(2)$$

by the formula for the right-hand sides, and one can check similarly that the expressions for

$$p_{l,l+1}(0) = \mathrm{P}\{X_{l+1}=0,\ X_{l+2}=0\} + \mathrm{P}\{X_l=0,\ X_{l+2}=0\},$$

$$p_{l,l+1}(1) = \mathrm{P}\{X_{l+1}=0,\ X_{l+2}=1\} + \mathrm{P}\{X_l=1,\ X_{l+1}=1,\ X_{l+2}=0\},$$

$$p_{l,l+1}(2) = \mathrm{P}\{X_l=0,\ X_{l+1}=1,\ X_{l+2}=1\},$$

and

$$p_{l,l+1}(3) = \mathrm{P}\{X_l=1,\ X_{l+1}=1,\ X_{l+2}=1\}$$

also agree with $p^*_{l,l+1}(0)$, $p^*_{l,l+1}(1)$, $p^*_{l,l+1}(2)$, and $p^*_{l,l+1}(3)$, respectively. Thus, we have $p_{l,n}(\cdot) = p^*_{l,n}(\cdot)$ for $n = l, l+1$.

We now assume that $p_{l,m}(k) = p^*_{l,m}(k)$ and $p_{l,m+1}(k) = p^*_{l,m+1}(k)$ for all $k = 0, 1, 2, \ldots$ for some integer $m \ge l$. Then, by (7) and this induction hypothesis,

$$\begin{aligned}
p_{l,m+2}(k) &= p^*_{l,m+1}(k) + \frac{\left[p^*_{l,m+1}(k-1) - p^*_{l,m}(k-1)\right] - \left[p^*_{l,m+1}(k) - p^*_{l,m}(k)\right]}{\mu+m+2} \\
&= p^*_{l,m+1}(k) + \frac{1}{\mu+m+2}\frac{(-1)^{m+k+l}}{\prod_{r=l}^{m+2}(\mu+r-1)}\left[\binom{m+3-l}{k-1} + \binom{m+3-l}{k}\right] \\
&= p^*_{l,m+1}(k) + \frac{(-1)^{m+2+k+l}}{\prod_{r=l}^{m+3}(\mu+r-1)}\binom{m+4-l}{k} \\
&= \sum_{j=l+k-2}^{m+2}\frac{(-1)^{j+k+l}}{\prod_{r=l}^{j+1}(\mu+r-1)}\binom{j+2-l}{k} = p^*_{l,m+2}(k)
\end{aligned}$$

for all $k = 0, 1, 2, \ldots$. This proves the first statement in (3).

Since X_{n+1} converges in probability to zero as $n \to \infty$, the second statement in (4) follows directly from the first one. Finally, from (4),

$$E\left(S^{N_l}\right) = \sum_{k=0}^{\infty} s^k\,\mathrm{P}\{N_l = k\} = \sum_{k=0}^{\infty} s^k \sum_{j=l+k-2}^{\infty}\frac{(-1)^{j+k+l}}{\prod_{r=l}^{j+1}(\mu+r-1)}\binom{j+2-l}{k}$$

$$= \sum_{j=l-2}^{\infty} \sum_{k=0}^{j+2-l} s^k \frac{(-1)^{j+k+l}}{\prod_{r=l}^{j+1} (\mu+r-1)} \binom{j+2-l}{k}$$

$$= \sum_{j=l-2}^{\infty} \frac{(-1)^{j+l}}{\prod_{r=l}^{j+1} (\mu+r-1)} \sum_{k=0}^{j+2-l} \binom{j+2-l}{k} (-s)^k$$

$$= \sum_{j=l-2}^{\infty} \frac{(-1)^{j+2-l}}{\prod_{r=l}^{j+1} (\mu+r-1)} (1-s)^{j+2-l} = \sum_{j=l-2}^{\infty} \frac{1}{\prod_{r=l}^{j+1} (\mu+r-1)} (s-1)^{j+2-l}$$

for all $s \in [0, 1)$, which proves the third statement in (5).

Remark. For any probabilities

$$p_n = \mathrm{P}\{X_n = 1\} = 1 - \mathrm{P}\{X_n = 0\} \in [0, 1], \qquad n \in \mathbf{N},$$

the first part of the proof gives the general recursion

$$p_{l,\,n+2}(k) = p_{l,\,n+1}(k) + \left[p_{n+3} p_{l,\,n+1}(k-1) - p_{n+2}(1-p_{n+3}) p_{l,\,n+1}(k-1)\right]$$

$$- \left[p_{n+3} p_{l,\,n+1}(k) - p_{n+2}(1-p_{n+3}) p_{l,\,n+1}(k)\right]$$

for all $n \geq l$ and $k = 0, 1, 2, \ldots$, as an extension of (7). So, we see that Theorem 1 is about an "easy" case where the common value $p_{n+3} = p_{n+2}(1 - p_{n+3})$ can be factored out from the two differences, which happens if and only if $p_{n+3} = p_{n+2}/(1 + p_{n+2})$ for every $n \geq l$ and the starting values of p_l and p_{l+1} make it possible to piece the induction together. It would be of interest to know whether in a "difficult" case where $p_{n+3} \neq p_{n+2}/(1 + p_{n+2})$ for some or all $n \geq l$ it is still possible to derive a closed solution of the recursive formula. The most prominent concrete example of this would be when $p_n = \lambda/(\mu + n - 1)^{\alpha}$, $n \in \mathbf{N}$, for some parameters $\alpha, \lambda > 0$ and $\mu \geq \lambda^{1/\alpha}$, when

$$p_{l,\,n+2}(k) = p_{l,\,n+1}(k) + \frac{\lambda}{(\mu+n+2)^{\alpha}} \left[p_{l,\,n+1}(k-1) - \frac{(\mu+n+2)^{\alpha} - \lambda}{(\mu+n+1)^{\alpha}} p_{l,\,n}(k-1) \right]$$

$$- \frac{\lambda}{(\mu+n+2)^{\alpha}} \left[p_{l,\,n+1}(k) - \frac{(\mu+n+2)^{\alpha} - \lambda}{(\mu+n+1)^{\alpha}} p_{l,\,n}(k) \right]$$

for $n \geq l$ and $k = 0, 1, 2, \ldots$, as a special generalization of (7). This recursion is what one ought to solve in order to obtain an extension of (3). Even for $\alpha = 1$, the ensuing results would generalize those in Theorem 1, i.e., the case $\alpha = 1 = \lambda$, or for a class of distributions containing the family in (2) for $\mu = \lambda$.

Proof of Theorem 2. For integers $m \geq l \geq 1$, we set $N_{l,\,m} := \sum_{n=l}^{m} X_n X_{n+1} \geq 0$. Since $N_{l,\,m} \uparrow N_l$ almost surely as $m \to \infty$, by the monotone convergence theorem we have $E(N_l) = \lim_{m\to\infty} E(N_{l,m})$ and $E(N_l^2) = \lim_{m\to\infty} E(N_{l,\,m}^2)$. Since, with prime denoting left-hand-side derivative,

$$E(N_l) = f'_{l,\lambda}(1) = \frac{\lambda}{\mu+l-1}$$

and

$$E\left(N_l^2\right) = f''_{l,\lambda}(1) + f'_{l,\lambda}(1) = \frac{2\lambda^2}{(\mu+l-1)(\mu+l)} + \frac{\lambda}{\mu+l-1}$$

for all $l \in \mathbf{N}$, the equations

$$\frac{\lambda}{\mu+l-1} = E(N_l) = \lim_{m\to\infty} E\left(N_{l,m}\right) = \sum_{n=l}^{\infty} p_n p_{n+1}$$

and

$$E\left(N_l^2\right) = \lim_{m\to\infty} E\left(N_{l,m}^2\right) = E\left(\left[\sum_{n=l}^{\infty} X_n X_{n+1}\right]^2\right)$$

$$= \sum_{n=l}^{\infty} E\left(X_n^2 X_{n+1}^2\right) + 2\sum_{n=l}^{\infty} E\left(X_n X_{n+1}^2 X_{n+2}\right) + 2\sum_{n=l}^{\infty}\sum_{j=n+2}^{\infty} E\left(X_n X_{n+1} X_j X_{j+1}\right)$$

$$= \sum_{n=l}^{\infty} p_n p_{n+1} + 2\sum_{n=l}^{\infty} p_n p_{n+1} p_{n+2} + 2\sum_{n=l}^{\infty}\sum_{j=n+2}^{\infty} p_n p_{n+1} p_j p_{j+1}$$

imply

$$p_l p_{l+1} = \sum_{n=l}^{\infty} p_n p_{n+1} - \sum_{n=l+1}^{\infty} p_n p_{n+1} = \frac{\lambda}{\mu+l-1} - \frac{\lambda}{\mu+l} = \frac{\lambda}{(\mu+l-1)(\mu+l)}$$

and

$$2\sum_{n=l}^{\infty} p_n p_{n+1} p_{n+2} + 2\sum_{n=l}^{\infty} p_n p_{n+1} \frac{\lambda}{\mu+n+1} = \frac{2\lambda^2}{(\mu+l-1)(\mu+l)}$$

for every $l \in \mathbf{N}$. The latter equations in turn imply

$$p_l p_{l+1} p_{l+2} + p_l p_{l+1} \frac{\lambda}{\mu+l-1} = \frac{\lambda^2}{(\mu+l-1)(\mu+l)} - \frac{\lambda^2}{(\mu+l+1)(\mu+l)} = \frac{2\lambda^2}{(\mu+l-1)(\mu+l)(\mu+l+1)},$$

which, combined with the former equations, yields

$$p_l = \frac{1}{p_{l+1} p_{l+2}}\left[\frac{2\lambda^2}{(\mu+l-1)(\mu+l)(\mu+l+1)} - p_l p_{l+1} \frac{\lambda}{\mu+l+1}\right]$$

$$= \frac{(\mu+l)(\mu+l+1)}{\lambda} \frac{\lambda^2}{(\mu+l-1)(\mu+l)(\mu+l+1)} = \frac{\lambda}{\mu+l-1}$$

for all $l \in \mathbf{N}$. Finally, confronting this with the first set of equations, we get $\lambda^2 = (\mu + l - 1)(\mu + l)\, p_l p_{l+1} = \lambda$. Hence, $\lambda = 1$ necessarily, and so $p_l = 1/(\mu + l - 1)$ for all $l \in \mathbf{N}$.

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