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# Effect of heat transfer on compressible boundary layer flow over a circular cylinder

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**Summary.** The effect of heat transfer on the steady laminar compressible boundary layer flow past a horizontal circular cylinder has been studied. The resulting coupled nonlinear partial differential equations have been solved numerically using a very efficient finite-difference scheme. Specific results are given for the boundary layer separation parameter and the skin friction and heat transfer coefficients.

## 1 Introduction

The analysis of heat transfer through a laminar boundary layer in the flow of a viscous fluid over a body of arbitrary shape and arbitrary specified surface temperature constitutes a very important problem in the field of heat transfer. The prediction of heat transfer under such conditions encompasses a wide range of technological applications, such as the calculation of heat transfer at the front portions of a projectile, aircraft or other body moving through the atmosphere, cooling problems in turbine blades, etc.

The problem of heat transfer from a horizontal circular cylinder placed in a laminar viscous and incompressible fluid has been the subject of many theoretical and experimental investigations because of its numerous engineering applications. Although this problem has been successfully studied in the past, to our best knowledge only little work has been conducted to investigate the effect of heat transfer on forced convection boundary layer flow past a circular cylinder in a viscous compressible fluid. Brown [1] was the first to investigate the effect of heat transfer on the growth of the boundary layer in the impulsive motion of a cylinder in a viscous compressible fluid.

An attempt is made in the present paper to investigate the effect of heat transfer on steady laminar boundary layer flow of a viscous compressible fluid past a horizontal circular cylinder. Owing to the external complexity of the fully compressible boundary layer equations a mathematical model is adopted which is possible to justify it to a certain extent of physical ground. In short, the assumptions are made that the effects of compressibility are confined in the boundary layer and the main stream remains incompressible. This could be realized in practice by releasing a stream of small Mach number past a very hot body. The fluid considered is a model fluid in that the viscosity ($\mu$) is proportional to the absolute temperature ($T$) and the Prandtl number ($\sigma$) is unity. This is the simplest and in easy ways the most useful and revealing fluid (see Stewartson [2]).

## 2 Basic equations

The equations describing the steady flow in the compressible, laminar two-dimensional boundary layer flow are (see Stewartson [2])

$$\frac{\partial}{\partial x}(\varrho u) + \frac{\partial}{\partial y}(\varrho v) = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\varrho}\frac{\partial p}{\partial x} + \frac{1}{\varrho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{2}$$

$$\varrho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) - u\frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{3}$$

together with

$$p = \varrho RT \text{ and } \mu = \mu_0(T/T_0) \tag{4}$$

where $(x, y)$ are Cartesian coordinates with $x$- and $y$-axes along and normal to the surface of the cylinder, respectively, $(u, v)$ are the velocity components along $x$- and $y$-axes, $p$ is the pressure, $\varrho$ is the density, $k$ is the thermal conductivity, $C_p$ is the specific heat at constant pressure, $R$ is the gas constant, and the suffix $o$ refers to some standard state, say, $x = 0$.

The boundary conditions are

$$\begin{aligned} &u = v = 0, \quad T = T_w \quad \text{at} \quad y = 0 \\ &u = U_1, \quad T = T_1 \quad \text{at} \quad y = \infty \end{aligned} \tag{5}$$

where $T_w$ is the constant wall temperature and the suffix unity denotes conditions in the main stream. The main stream velocity $U_1$ may be taken as the velocity in the irrotational motion of an incompressible fluid. Thus, if $a$ is the radius of the cylinder, then

$$U_1(x) = U_\infty \sin(x/a) \tag{6}$$

where $x$ measures the distance from the forward stagnation point of the cylinder and $U_\infty$ is the constant velocity in the incompressible flow. Equations (1)–(3) are further reduced to an almost incompressible form by introducing Stewartson's transformation [2], [3]

$$Y = \frac{a_1}{a_0\sqrt{v_0}}\int_0^y \frac{\varrho}{\varrho_0}, \qquad \varrho u = \varrho_0\sqrt{v_0}\frac{\partial \psi}{\partial y} \tag{7}$$

where $\psi$ is the streamfunction, and $a_1$ and $a_0$ are the velocities of sound in the main stream and at some standard states, respectively. Thus, we have

$$\begin{aligned} &u = \left(\frac{a_1}{a_0}\right)\frac{\partial \psi}{\partial Y}, \qquad v = -\left(\frac{\varrho_0}{\varrho}\right)\sqrt{v_0}\left(\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial Y}\frac{\partial Y}{\partial x}\right) \\ &\mu\frac{\partial u}{\partial y} = \frac{\mu_0 p}{p_0}\frac{a_1^2}{a_0^2\sqrt{\mu_0}}\frac{\partial^2 \psi}{\partial Y^2}, \qquad \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) = \frac{\varrho}{\varrho_0}\frac{a_1^3}{a_0^3}\frac{\partial^3 \psi}{\partial Y^3} \\ &k\frac{\partial T}{\partial y} = \mu_0 C_p\frac{p}{p_0}\frac{a_1}{a_0\sqrt{\mu_0}}\frac{\partial T}{\partial Y} \end{aligned} \tag{8}$$

with

$$\frac{p}{p_0} = \left(\frac{a_1}{a_0}\right)^{2\gamma/(\gamma-1)} \tag{9}$$

where $\gamma$ is the ratio of specific heats.

Using the properties of Eulerian equations of motion of the inviscid outer flow, the momentum equation (2) becomes, after some manipulations,

$$\left(\frac{a_1}{a_0}\right)^{(3\gamma-1)/(\gamma-1)} \left(\frac{\partial\psi}{\partial Y}\frac{\partial^2\psi}{\partial Y\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial Y^2}\right) = \frac{\partial^3\psi}{\partial Y^3} + \left(\frac{a_1}{a_0}\right)^{(5\gamma-3)/\gamma-1)} \frac{T}{T_1} U_1 \frac{dU_1}{dx}. \tag{10}$$

Further, if Eq. (2) is multiplied by $u$ and added to the energy equation (3) the latter becomes

$$\varrho\left(u\frac{\partial S}{\partial x} + v\frac{\partial S}{\partial y}\right) = \frac{\partial}{\partial y}\left(k\frac{\partial S}{\partial y}\right) \tag{11}$$

where the function $S$ is related to the absolute temperature $T$ by

$$\left(1 + \frac{\gamma-1}{2}M_1^2\right)S = \frac{T}{T_1} - \frac{\gamma-1}{2}M_1^2\left(1 - \frac{u^2}{U_1^2}\right) - 1 \tag{12}$$

and $M_1$ is the main stream Mach number. Under the transformation (8), Eq. (11) reduces to

$$\left(\frac{a_0}{a_1}\right)^{(3\gamma-1)/(\gamma-1)} \left(\frac{\partial\psi}{\partial Y}\frac{\partial S}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial S}{\partial Y}\right) = \frac{\partial^2 S}{\partial Y^2}. \tag{13}$$

Since the main purpose of this work is to investigate the effect of heat transfer on the boundary layer, it is sufficient to consider a flow in which the Mach number $M_1$ is small ($M_1 \ll 1$). Consideration of Bernoulli's equation gives

$$\frac{a_1^2}{a_0^2} = 1 + O(M_1^2), \tag{14}$$

and so it is sufficient to replace the factor $a_0/a_1$ in (10) and (13) by unity. Thus, the flow is one in which both viscous and compressibility effects are confined to the boundary layer. Also, relation (12) defining the temperature function $S$ becomes

$$\frac{T}{T_1} = 1 + S \tag{15}$$

where $T_1$ is now the constant temperature of the main stream.

The equations describing the flow and heat transfer are therefore, from (10) and (13),

$$\frac{\partial\psi}{\partial Y}\frac{\partial^2\psi}{\partial Y\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial Y^2} = \frac{\partial^3\psi}{\partial Y^3} + U_1\frac{dU_1}{dx}(1+S) \tag{16}$$

$$\frac{\partial\psi}{\partial Y}\frac{\partial S}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial S}{\partial Y} = \frac{\partial^2 S}{\partial Y^2} \tag{17}$$

subject to the boundary conditions

$$\psi = \frac{\partial\psi}{\partial Y} = 0, \quad S = \frac{T_w}{T_1} - 1 = S_w \quad \text{at} \quad Y = 0$$

$$\frac{\partial S}{\partial Y} \to U_1(x), \quad S \to 0 \quad \text{as} \quad Y = \infty . \tag{18}$$

To obtain the solution of Eqs. (16) and (17), we follow Merkin [4] and introduce the nondimensional variables

$$\xi = \frac{x}{a}, \quad \eta = Y\left(\frac{\sqrt{v_0}}{R}\right) \mathrm{Re}^{1/2},$$
$$\psi = \sqrt{v_0}\, \mathrm{Re}^{1/2}\, \xi f(\xi, \eta), \quad S(x, Y) = S(\xi, \eta) \tag{19}$$

where Re $(= U_\infty a/v_0)$ is the Reynolds number of the incompressible flow. Equations (16) and (17) then become

$$f''' + ff'' - f'^2 + \frac{\sin \xi \cos \xi}{\xi}(1 + S) = \xi\left(f' \frac{\partial f'}{\partial \xi} - f'' \frac{\partial f}{\partial \xi}\right) \tag{20}$$

$$S'' + fS' = \xi\left(f' \frac{\partial S}{\partial \xi} - S' \frac{\partial f}{\partial \xi}\right) \tag{21}$$

subject to the boundary conditions

$$f(\xi, 0) = f'(\xi, 0) = 0, \quad S(\xi, 0) = S_w$$
$$f'(\xi, \infty) = \xi^{-1} \sin \xi, \quad S(\xi, \infty) = 0 \tag{22}$$

where primes denote differentiation with respect to $\eta$.

Once we know the solution for $f(\xi, \eta)$ and $S(\xi, \eta)$, we can calculate the skin friction and the rate of heat transfer at the surface of the cylinder from the following relations:

$$\tau_w = \left(\mu \frac{\partial u}{\partial y}\right)_{y=0} = \frac{\mu_0 U_\infty}{a} \mathrm{Re}^{1/2} \xi f''(\xi, 0) \tag{23}$$

$$q_w = -\left(k \frac{\partial T}{\partial y}\right)_{y=0} = -\frac{\varrho_0 C_p T_1 U_\infty}{\mathrm{Re}^{1/2}} S'(\xi, 0). \tag{24}$$

Thus, the coefficients of the skin-friction $C_f$ and the rate of heat transfer $C_h$ are given by

$$C_f\, \mathrm{Re}^{1/2} = \xi f''(\xi, 0), \quad C_h\, \mathrm{Re}^{1/2} = -S'(\xi, 0) \tag{25}$$

where $C_f$ and $C_h$ are defined as

$$C_f = \frac{\tau_w}{\varrho_0 U_\infty{}^2}, \quad C_h = \frac{q_w}{\varrho_0 C_p T_1 U_\infty}. \tag{26}$$

## 3 Results and discussions

Equations (20) and (21) along with the boundary conditions (22) are solved numerically using the Keller-box method [5] for different values of the surface temperature parameter $S_w = 1.0$, 5.0, 10.0, 15.0 and 20.0 and of the curvature parameter $\xi = 0.0$, 0.4, 0.8, 1.2 and 1.6. Before we enter into the Keller-box scheme the initial profiles of the functions $f$ and $S$ are obtained from the

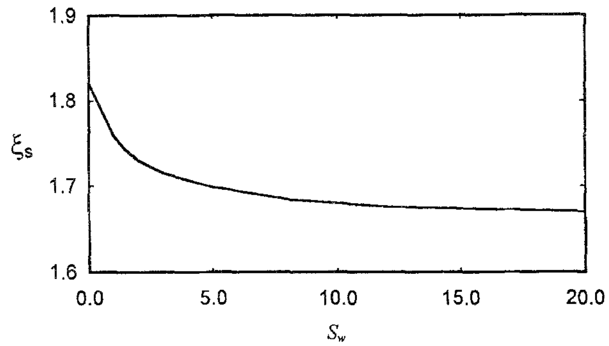


**Fig. 1.** Boundary layer separation parameter $\xi_s$ as a function of $S_w$

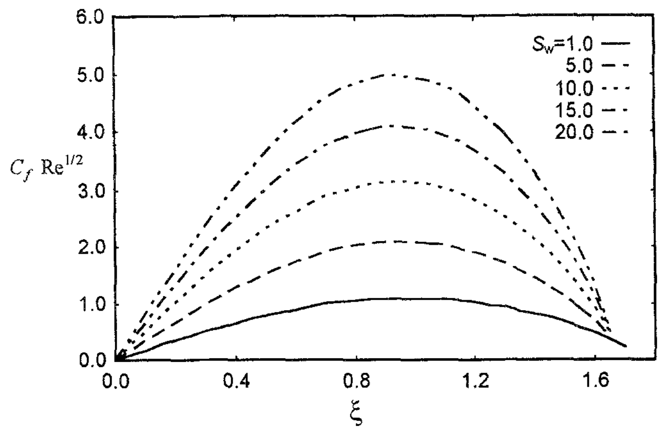


**Fig. 2.** Variation of the skin friction coefficient with $\xi$ for different values of $S_w$

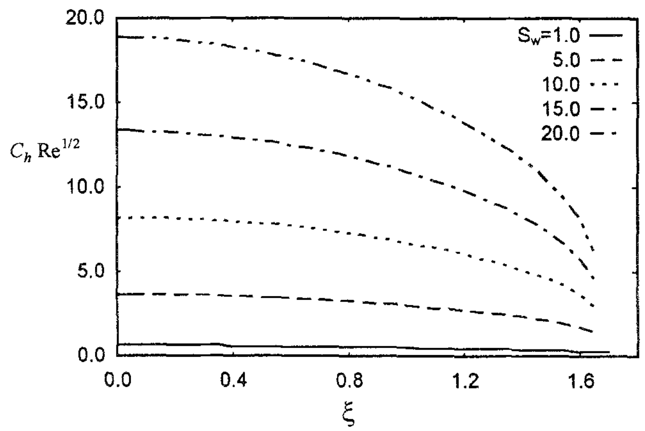


**Fig. 3.** Variation of the rate of heat transfer coefficient with $\xi$ for different values of $S_w$

solution of the following ordinary differential equations valid at the stagnation point ($\xi = 0$) of the cylinder:

$$f''' + ff'' - f'^2 + S + 1 = 0$$
$$S'' + fS' = 0 \tag{27}$$

with the boundary conditions

$$f(0) = f'(0) = 0, \quad S(0) = S_w$$
$$f'(\infty) = 1, \quad S(\infty) = 0. \tag{28}$$

Numerical solutions of Eqs. (27) and (28) are obtained using the Runge-Kutta-Butcher method together with Nachtsheim-Swigert iteration scheme for values of the parameter $S_w$ mentioned above.

Figure 1 shows the variation of the boundary layer separation parameter $\xi_s$ with the surface temperature parameter $S_w$. We first notice that if $S_w = 0.0$, *i.e.* $T_w = T_1$, then $\xi_s = 1.823$ ($\approx 104.45^0$), which is exactly the value reported by Stewartson [2] in Table 4.2. Then, it is seen from Fig. 1 that as $S_w$ increases the point of separation $\xi_s$ decreases to an asymptotic value $\xi_s = 1.678$ ($\approx 96.15^0$). Therefore, the effect of the heat transfer parameter $S_w$ is to move the position of $\xi_s$ upstream to the forward stagnation point of the cylinder.

The skin friction and rate of heat transfer coefficients given by (26) are shown in Figs. 2 and 3. It can easily be seen that these coefficients increase due to the increase of $S_w$. They are higher at the forward stagnation point ($\xi_s = 0.0$) and, as expected, break down at the separation point $\xi_s$ for every value of $S_w$.

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