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Note / Mitteilung

A Note on Dynamic Surface Displacements in an Elastic Half-Space

By

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With 3 Figures

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Many studies have been done on elastic transients generated by surface and buried point sources in an elastic half-space [1]—[9]. In these works, the source time-dependence was that of a Heaviside step or a delta function. Various types of singularities were found to occur in the surface displacements. Partly to investigate the role of the source time-dependence in this connection, the present Note examines the surface displacements due to a surface, vertical, point force with a ramp time-dependence.

Cylindrical coordinates r, φ, z are used, the origin being at the source, with the z axis directed downwards. Following the development given by HASKELL [10] for harmonic waves, the Laplace transformed displacements due to a downward surface vertical point force of unit magnitude can be shown to be, for $z < 0$,

$$\frac{4\pi\varrho p^2}{\bar{h}(p)}\,\bar{u}_r = \int_0^\infty \Big[g(k,p)\,e^{-(2d+z)\eta_2} - 4\eta_1\eta_2(k^2+\eta_2{}^2)\,e^{-d\eta_1-(d+z)\eta_2}$$
$$- 4k^2(k^2+\eta_2{}^2)\,e^{-d\eta_2-(d+z)\eta_1} + g(k,p)\,e^{-(2d+z)\eta_1} + D(k,p)\,e^{z\eta_2}$$
$$- D(k,p)\,e^{z\eta_1}\Big]\frac{k^2 J_1(kr)}{D(k,p)}\,dk, \tag{1}$$

$$\frac{4\pi\varrho p^2}{\bar{h}(p)}\,\bar{u}_z = \int_0^\infty \Big[-4k^2\eta_1\eta_2(k^2+\eta_2{}^2)\,e^{-d\eta_1-(d+z)\eta_2}$$
$$- 4k^2\eta_1\eta_2(k^2+\eta_2{}^2)\,e^{-d\eta_2-(d+z)\eta_1} + k^2 g(k,p)\,e^{-(2d+z)\eta_2}$$
$$+ \eta_1\eta_2 g(k,p)\,e^{-(2d+z)\eta_1} - k^2 D(k,p)\,e^{\eta_2 z}$$
$$+ \eta_1\eta_2 D(k,p)\,e^{z\eta_1}\Big]\frac{kJ_0(kr)}{\eta_2 D(k,p)}\,dk \tag{2}$$

where

$$g(k, p) = (k^2 + \eta_2^2)^2 + 4k^2 \eta_1 \eta_2$$

$$D(k, p) = (k^2 + \eta_2^2)^2 - 4k^2 \eta_1 \eta_2$$

$$\eta_1 = \left(k^2 + \frac{p^2}{c_d^2}\right)^{1/2}, \qquad \eta_2 = \left(k^2 + \frac{p^2}{c_s^2}\right)^{1/2} \tag{3}$$

$$c_s^2 = \frac{\mu}{\varrho}, \qquad c_d^2 = \frac{(\lambda + 2\mu)}{\varrho}$$

λ and μ being the Lame constants, d the source depth, ϱ the density, the bar denoting the Laplace transform, parameter p, the J's denoting Bessel functions of the first kind, and $h(t)$ being the time-dependence of the applied force. As is seen from (3), branch points arise on the imaginary p axis. The branch cuts are taken to run parallel to the negative real axis, the resulting branches having the property that $R\eta_j > 0$, $j = 1, 2$, for $Rp > 0$, R denoting the real part.

On taking the source time-dependence to be a unit ramp, i.e.,

$$h(t) = \begin{cases} 0, & t < 0 \\ \dfrac{t}{q}, & 0 < t < q \\ 1, & t > q \end{cases}$$

and on restricting attention to the surface displacements due to a surface force (set $z = -d$, $d = 0$), (1) and (2) give

$$2\pi\varrho c_s^2 \bar{u}_r = \frac{1}{qp^2}(1 - e^{-pq}) \int_0^\infty [(k^2 + \eta_2^2) - 2\eta_1\eta_2] \frac{k^2 J_1(kr)}{D(k, p)}\, dk, \tag{4}$$

$$2\pi\varrho c_s^4 \bar{u}_z = \frac{1}{q}(1 - e^{-pq}) \int_0^\infty \frac{k\eta_1 J_0(kr)}{D(k, p)}\, dk. \tag{5}$$

The procedure given by PEKERIS [1] and amplified by AGGARWAL and ABLOW [8] will now be followed. On making the change of variable $c_s k = px$, and restricting attention to $\lambda = \mu$,[1] (4), for example, becomes, on replacing the Bessel functions by their Hankel function equivalents,

$$2\pi\varrho c_s^3 q \bar{u}_r = \frac{1}{2p}(1 - e^{-pq}) \int_0^\infty x^2 \left[H_1^{(1)}\left(\frac{p}{c_s} xr\right) + H_1^{(2)}\left(\frac{p}{c_s} xr\right)\right] f(x)\, dx \tag{6}$$

where

$$m_1(x) f(x) = (1 + 2x^2) - 2\sqrt{x^2 + 1}\sqrt{x^2 + \frac{1}{3}},$$

$$m_1(x) = (1 + 2x^2)^2 - 4x^2\sqrt{x^2 + 1}\sqrt{x^2 + \frac{1}{3}}.$$

[1] With this restriction, the roots of the Rayleigh surface wave equation are real, resulting in somewhat simpler algebra later.

The integrals in (6) can be further reduced by using contour integration techniques. When $H_1^{(1)}$ arises, a closed contour consisting of the positive real and imaginary axes and a large quarter circle in the first quadrant is considered. For $H_1^{(2)}$, the contour consists of the positive real axis, the negative imaginary axis, and a large quarter circle in the fourth quadrant. One gets

$$2\pi\varrho q c_s^3 \bar{u}_r = -\frac{2}{\pi} I \frac{(1-e^{-pq})}{p} \int_0^\infty \nu^2 K_1\left(\frac{p}{c_s} r\nu\right) f(i\nu)\, d\nu - \frac{\gamma}{4p}(1-e^{-pq}) K_1\left(\frac{p}{c_s} r\gamma\right) \tag{7}$$

where

$$\gamma = \frac{1}{2}\sqrt{3+\sqrt{3}}$$

K_1 is a modified Bessel function of the first kind and I denotes imaginary part. The last term in (7) stems from identations over simple poles at $x = \pm i\gamma$ (corresponding to the roots of the Rayleigh equation lying on the branches in question). The advantage of the current form is that the inverse Laplace transforms, denoted by L^{-1} in the sequel, can be found in tables. In fact from ERDELYI et al [11]

$$L^{-1}\left[\frac{1}{p} K_1(bp)\right] = \frac{1}{b}\sqrt{t^2-b^2}\, H(t-b) \tag{8}$$

where H denotes the Heaviside unit step function.

Using (8), (7) gives, on noting

$$I f(i\nu) = \begin{cases} 0, & \nu < \frac{1}{\sqrt{3}} \\ \frac{m_3(\nu)}{m_2(\nu)}, & \frac{1}{\sqrt{3}} < \nu < 1 \\ 0, & \nu > 1 \end{cases}$$

$$m_3(\nu) = 6(2\nu^2-1)\sqrt{\nu^2-\frac{1}{3}}\sqrt{1-\nu^2}$$

$$m_2(\nu) = 3 - 24\nu^2 + 56\nu^4 - 32\nu^6$$

and temporarily ignoring the delay factor e^{-pq}, which can be injected later by a suitable time delay,

$$\pi^2 \varrho Q c_s^2 r u_r = \begin{cases} 0, & \tau < \frac{1}{\sqrt{3}} \\ \int_{1/\sqrt{3}}^{\tau} m_4(\nu)\sqrt{\tau^2-\nu^2}\, d\nu, & \frac{1}{\sqrt{3}} < \tau < 1 \\ \int_{1/\sqrt{3}}^{\tau} m_4(\nu)\sqrt{\tau^2-\nu^2}\, d\nu - \frac{1}{8}\sqrt{\tau^2-\gamma^2}\, H(\tau-\gamma), & \tau > 1 \end{cases} \tag{9}$$

where

$$m_4(\nu) = \frac{\nu m_3(\nu)}{m_2(\nu)},$$

$$\tau = \frac{c_s}{r}\,t, \qquad Q = \frac{q c_s}{r}.$$

Letting

$$\nu^2 = \frac{1}{3} + \omega^2 \sin^2\theta, \qquad \omega^2 = \tau^2 - \frac{1}{3}$$

and performing a partial fraction decomposition, (9) yields for $Q = .1$, on taking the delay factor e^{-pq} into account,

$$\pi^2 \varrho Q c_s^2 r u_r = \begin{cases} 0, & \tau < \frac{1}{\sqrt{3}} \\ \zeta_1(\tau), & \frac{1}{\sqrt{3}} < \tau < \frac{1}{\sqrt{3}} + .1 \\ \zeta_1(\tau) - \zeta_1(\tau - .1), & \frac{1}{\sqrt{3}} + .1 < \tau < 1 \\ \zeta_2(\tau) - \zeta_1(\tau - .1), & 1 < \tau < \gamma \\ \zeta_3(\tau) - \zeta_1(\tau - .1), & \gamma < \tau < 1.1 \\ \zeta_3(\tau) - \zeta_2(\tau - .1), & 1.1 < \tau < \gamma + .1 \\ \zeta_3(\tau) - \zeta_3(\tau - .1), & \tau > \gamma + .1 \end{cases} \tag{10}$$

where

$$\begin{aligned} \zeta_1(\tau) = -\frac{\sqrt{2/3}}{64}\big\{&24 m(\xi) + 3(12\omega^2 - 7)K(\xi) - 9(12\omega^2 + 1)\,\Gamma(8\xi^2, \xi) \\ &+ \big[12(3 - 2\sqrt{3})\,\omega^2 + 3 + \sqrt{3}\big]\,\Gamma\big[-(12\sqrt{3} - 20)\,\xi^2, \xi\big] \\ &+ \big[12(3 + 2\sqrt{3})\,\omega^2 + 3 - \sqrt{3}\big]\,\Gamma\big[(12\sqrt{3} + 20)\,\xi^2, \xi\big]\big\}, \end{aligned}$$

$$\begin{aligned} \zeta_2(\tau) = -\frac{\sqrt{2/3}}{64\,\xi}\big\{&24 m(\bar\xi) + 3\bar\xi^2 K(\bar\xi) - 9(8 + \bar\xi^2)\,\Gamma(8, \bar\xi) \\ &+ \big[(3 - \sqrt{3})\,\bar\xi^2 + 8(3 + 2\sqrt{3})\big]\,\Gamma\big[4(3\sqrt{3} + 5), \bar\xi\big] \\ &+ \big[(3 + \sqrt{3})\,\bar\xi^2 + 8(3 - 2\sqrt{3})\big]\,\Gamma\big[-4(3\sqrt{3} - 5), \bar\xi\big]\big\}, \end{aligned}$$

$$\zeta_3(\tau) = \zeta_2(\tau) - \frac{\pi}{8}\sqrt{\tau^2 - \gamma^2},$$

$$\xi^2 = \frac{3}{2}\,\omega^2, \qquad \bar\xi^2 = \frac{2}{(3\tau^2 - 1)},$$

$$K(\xi) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \xi^2 \sin^2\theta}}$$

is the complete elliptic integral of the first kind

$$m(\xi) = \int_0^{\pi/2} \sqrt{1 - \xi^2 \sin^2 \theta}\, d\theta$$

is the complete elliptic integral of the second kind, and

$$\Gamma(n, \xi) = \int_0^{\pi/2} \frac{d\theta}{(1 + n \sin^2 \theta) \sqrt{1 - \xi^2 \sin^2 \theta}}$$

is the complete elliptic integral of the third kind.

Similarly, u_z for $Q = .1$ is given by

$$32\pi^2 \varrho Q c_s^2 r u_z = \begin{cases} 0, & \tau < \frac{1}{\sqrt{3}} \\ \psi_1(\tau), & \frac{1}{\sqrt{3}} < \tau < \frac{1}{\sqrt{3}} + .1 \\ \psi_1(\tau) - \psi_1(\tau - .1), & \frac{1}{\sqrt{3}} + .1 < \tau < 1 \\ \psi_2(\tau) - \psi_1(\tau - .1), & 1 < \tau < \gamma \\ \psi_3(\tau) - \psi_1(\tau - .1), & \gamma < \tau < 1.1 \\ \psi_3(\tau) - \psi_2(\tau - .1), & 1.1 < \tau < \gamma + .1 \\ \psi_3(\tau) - \psi_3(\tau - .1), & \tau > \gamma + .1 \end{cases} \tag{11}$$

where

$$\psi_1(\tau) = 6\left(\tau - \frac{1}{\sqrt{3}}\right) - \sqrt{3} \log \frac{2}{\sqrt{3}} \left(\tau + \sqrt{\tau^2 - \frac{1}{4}}\right)$$
$$+ \sqrt{3\sqrt{3} - 5} \log \left[\left(\tau + \sqrt{\tau^2 - \frac{(3 - \sqrt{3})}{4}}\right) \Big/ \left(\frac{1}{\sqrt{3}} + \frac{\sqrt{3\sqrt{3} - 5}}{2\sqrt{3}}\right)\right]$$
$$- \sqrt{3\sqrt{3} + 5} \left(\sin^{-1} \frac{\tau}{\gamma} - \sin^{-1} \frac{\sqrt{2(3 - \sqrt{3})}}{3}\right),$$

$$\psi_2(\tau) = 12\left(\tau - \frac{2}{3}\gamma^2\right) - \sqrt{3} \log \frac{3 + 2\sqrt{3}}{3} + \sqrt{3\sqrt{3} - 5} \log \frac{2\sqrt{3} + \sqrt{3 + 3\sqrt{3}}}{2 + \sqrt{3\sqrt{3} - 5}}$$
$$- \sqrt{3\sqrt{3} + 5} \left[2 \sin^{-1} \frac{\tau}{\gamma} - \sin^{-1} \frac{1}{\gamma} - \sin^{-1} \frac{\sqrt{2(3 - \sqrt{3})}}{3}\right],$$

$$\psi_3(\tau) = 12\left(\tau - \frac{2\gamma^2}{3}\right) - \sqrt{3} \log \frac{2 + \sqrt{3}}{\sqrt{3}} + \sqrt{3\sqrt{3} - 5} \log \frac{2\sqrt{3} + \sqrt{3 + 3\sqrt{3}}}{2 + \sqrt{3\sqrt{3} - 5}}$$
$$- \sqrt{3\sqrt{3} + 5} \left[\pi - \sin^{-1} \frac{\sqrt{2(3 - \sqrt{3})}}{3} - \sin^{-1} \frac{1}{\gamma}\right].$$

It should be noted that the sequence in (10) and (11) could be quite different for different values of Q. It is interesting that closed form solutions are still obtained, in common with the results of Pekeris [1] for a Heaviside step.

The results of numerical computation based on (10) and (11) are shown in Figs. 1, 2, and 3. The letters indicate the arrival of various events, P being the pressure wave arrival, S the shear wave arrival, and R the Rayleigh wave arrival.

Shown in Fig. 1 are the vertical displacements as functions of time for several values of the ramp rise time Q. It should be noted that no singularities arise at the

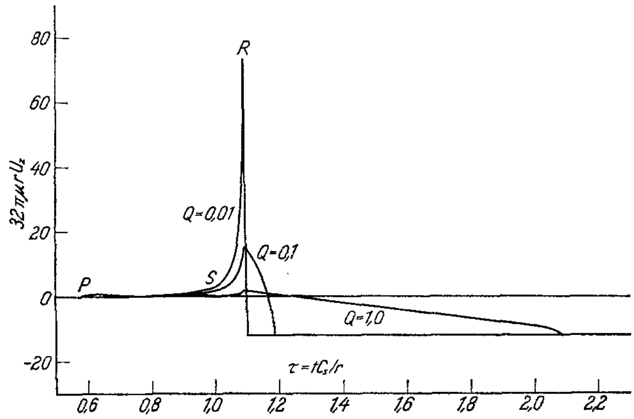

Fig. 1. Vertical displacement versus time for $Q = 0.01$, 0.1, and 1.0

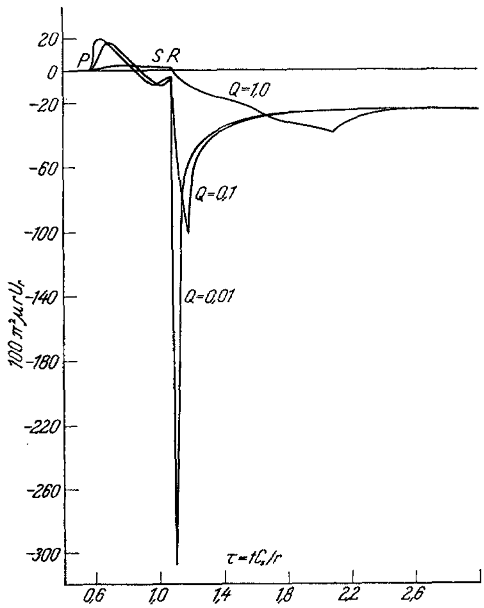

Fig. 2. Horizontal displacement versus time for $Q = 0.01$, 0.1, and 1.0

[2] When $Q = 0.15$, the magnitude of the pulse of the Rayleigh arrival equals that of the static solution in Fig. 1.

Rayleigh arrival in contrast to the corresponding solutions given by PEKERIS [1] for a Heaviside step. As Q approaches zero, the Rayleigh event becomes more and more pronounced. A very interesting item is the maximum amplitude. Shown in Fig. 3 is the amplitude of u_z at the Rayleigh arrival as a function of Q. From Figs. 1 and 3, it can be seen that the maximum magnitude of the pulse occurs at the Rayleigh arrival for $Q < 0.15^2$. However, for $Q > 0.15$, the maximum occurs at the delayed Rayleigh arrival.

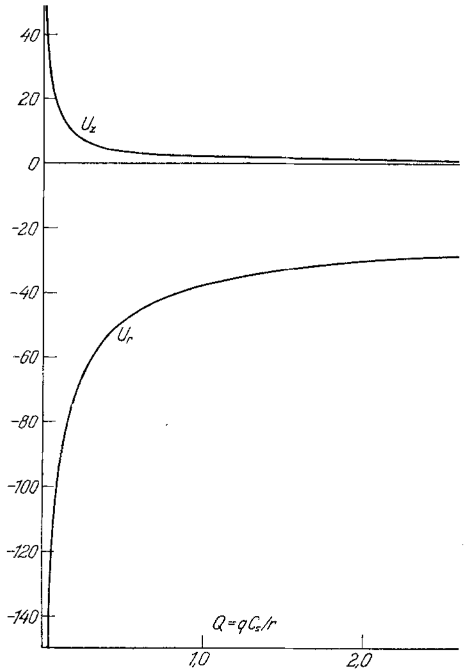

Fig. 3. Displacement amplitudes at the Rayleigh wave arrival as a function of Q

Shown in Fig. 2 are the horizontal displacements as functions of time for several values of Q. Again it is seen that in contrast to Pekeris' results for a Heaviside step, no singularities occur at the Rayleigh arrival. It is interesting that in this case the maximum magnitude always occurs at the Rayleigh arrival. This maximum magnitude is shown in Fig. 3. Fig. 2 also shows that a local maximum occurs between the P and S arrivals, the magnitude decreasing with increasing Q. Finally, it should be noted that the horizontal displacements are not constant after the delayed Rayleigh arrival, in contrast to the vertical displacements.

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