Slow Viscoelastic Flow in Tilted Troughs

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(Received May 17, 1965)

Summary. In viscoelastic flow of an incompressible fluid down a straight trough of arbitrary cross-section, the normal stress effect produces a distortion of the free surface. In slow flow, the shape is given in the lowest order of approximation in terms of the axial velocity for Newtonian flow. Equations governing the second-order perturbation are derived.

Zusammenfassung. Beim viskoelastischen Strömen einer inkompressiblen Flüssigkeit entlang einer geneigten geraden Rinne beliebigen Querschnittes erzeugt der Normalspannungseffekt eine Verzerrung der freien Oberfläche. Deren Form ist bei langsamer Strömung in erster Näherung durch die Axialgeschwindigkeit für NEWTONsche Strömung bestimmt. Die Gleichungen der Näherung zweiter Ordnung werden hergeleitet.

1. Introduction

In the present paper we consider the slow steady motion of incompressible viscoelastic fluids in tilted troughs. The problem is closely related to that of slow viscoelastic flow in tubes, treated by LANGLOIS and RIVLIN [1]. The principal difference is in the presence of a free surface in the trough problem. In Newtonian flow, the free surface remains flat, and the trough problem is equivalent to the problem of flow through a tube whose cross-section consists of that of the trough and its image in the free surface [2], [3]. In viscoelastic flow, however, the normal stress effect produces a distortion of the plane free surface of Newtonian flow. Our main object is to obtain an approximate expression for the form of this surface.

In slow viscoelastic flow through tubes [1], the flow is Newtonian in the lowest order of approximation, and a first approximation to the non-Newtonian normal stresses can be calculated directly from the Newtonian axial velocity field [4]. It can be anticipated that the same results will apply to flow in troughs. There will then be an unequilibrated normal stress on the Newtonian free surface. We can compute the pressure required to equilibrate this normal stress, and the rise (or fall) of the free surface should then be proportional to this pressure.

The present paper is simply a verification that the preceding argument is correct, to first order. We also obtain a second approximation, mainly

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in order to identify sources of error in the first approximation and to provide a means of estimating the accuracy of it.

The problem is outlined in Section 2, and the Newtonian approximation is briefly described. In Section 3, we introduce dimensionless variables suggested by the Newtonian approximation. We also introduce a non-Newtonian parameter $\varepsilon = T\,U/L$, where U/L is a typical shear-rate in Newtonian flow, and T is a characteristic relaxation time for the fluid. The approximate solution is to be obtained by treating ε as a small parameter. The constitutive equations for the viscoelastic flow are expanded in powers of the parameter ε in Section 4. Section 5 deals with the free surface conditions.

In Section 6 we show how the lowest-order approximation to the shape of the free surface is related to the solution of the Newtonian flow problem. Illustrative examples are given in Section 7.

The more complicated problem of obtaining a second approximation is discussed in Sections 8 to 12. We find that before such an approximation can be obtained, it is necessary to calculate the axial velocity perturbations which are produced by the surface distortion and by the variation of apparent viscosity with shear-rate. It is also necessary to obtain a first approximation to the transverse velocity field. However, the manner in which the physical parameters of the problem enter into the approximation can be exhibited explicitly, without carrying out any of these calculations.

2. Basic Equations. Newtonian Approximation

We consider the steady motion of an isotropic incompressible viscoelastic fluid down a long straight trough of uniform cross-section. The trough axis is tilted at an angle θ to the horizontal. We use a system of Cartesian coordinates x_i' with the x_3' axis parallel to the axis of the trough, the x_1' axis horizontal, and the x_2' axis in an upwardly direction. The unperturbed free surface of the fluid lies in the plane $x_2' = 0$. The level of this plane is determined by the general level of the free surface, whose form is to be determined. This level must be specified in some way, for example, by prescribing the point of intersection of the free surface with the trough wall, or by prescribing the total flux down the trough.

We seek a solution in which the velocity components u_i' are independent of x_3'. The continuity equation is then

$$\partial u_\alpha'/\partial x_\alpha' = 0, \tag{2.1}$$

where Greek subscripts have the range 1, 2. We express the stress components σ_{ij} in the form

$$\sigma_{ij} = -p'\,\delta_{ij} + S_{ij}', \tag{2.2}$$

where the pressure p' is arbitrary and the extra stress S_{ij}' is to be specified by a constitutive equation. The momentum equations are then

$$\varrho\, u_\alpha'\,\partial u_i'/\partial x_\alpha' = \varrho\, g_i - \partial p'/\partial x_i' + \partial S_{i\alpha}'/\partial x_\alpha', \tag{2.3}$$

where the components ϱg_i of the weight of fluid per unit volume are

$$(\varrho g_i) = (0, -\varrho g \cos\theta, \varrho g \sin\theta). \tag{2.4}$$

The conditions $u_i' = 0$ are satisfied at the trough wall. On the free surface, the velocity field satisfies the condition $u_i' n_i = 0$, where the vector n_i is normal to the surface. In addition, the traction $\sigma_{ij} n_j$ must reduce to a uniform normal pressure on this surface. Because the fluid is incompressible, there is no loss of generality in taking this pressure to be zero, i. e., $\sigma_{ij} n_j = 0$. The form of the free surface is assumed to be independent of x_3'.

The flow can be made as slow as desired by decreasing the tilt angle θ. It is known [1], [5] that in viscoelastic flows which are sufficiently slow, in a sense to be made more precise, the extra stress S_{ij}' is given approximately by the Newtonian relation

$$S_{ij}' = \mu\,(\partial u_i'/\partial x_j' + \partial u_j'/\partial x_i'), \tag{2.5}$$

where μ is the apparent viscosity at zero shear-rate. In Newtonian flow, the pressure p' is the hydrostatic pressure, $-x_2'\,\varrho g \cos\theta$, produced by the weight of the fluid. The transverse velocity components u_α' vanish, and the axial velocity u_3' satisfies the equation

$$\mu\,(\partial^2/\partial x_1'^2 + \partial^2/\partial x_2'^2)\,u_3' = -\varrho g \sin\theta, \tag{2.6}$$

which is the axial component of the momentum equation (2.3). The condition $\sigma_{ij} n_j = 0$ is satisfied on the flat free surface $x_2' = 0$ provided that $\partial u_3'/\partial x_2' = 0$ on that surface. With $u_3' = 0$ on the trough wall, the axial velocity is then completely determinate. We note that for flow through a tube for which the plane $x_2' = 0$ is a plane of symmetry, the axial velocity u_3' satisfies $\partial u_3'/\partial x_2' = 0$ on the plane $x_2' = 0$. Thus, the problem of Newtonian flow down a trough is equivalent to a problem of flow through a tube.

3. Dimensionless Equations

Let L be a typical length associated with the cross-section of the trough. From (2.6) we then find that the axial velocity in Newtonian flow is of the order of

$$U = (\varrho g L^2/\mu)\sin\theta. \tag{3.1}$$

It is convenient to introduce the following dimensionless variables:

$$x_i = x_i'/L, \quad u_i = u_i'/U, \quad S_{ij} = (L/\mu U)\,S_{ij}'. \tag{3.2}$$

We also express the pressure p' in the form

$$p' = -x_2'\,\varrho g \cos\theta + (\mu U/L)\,p\,(x_1, x_2), \tag{3.3}$$

where $(\mu U/L)\,p$ is the pressure variation which will arise as a reaction to the normal stress effect.

In the momentum equations (2.3), we use (3.1) to (3.3) and (2.4), to obtain

$$R\,u_\beta\,u_{\alpha,\beta} = -p_{,\alpha} + S_{\alpha\beta,\beta} \quad \text{and} \quad R\,u_\alpha\,u_{3,\alpha} = 1 + S_{3\alpha,\alpha}, \tag{3.4}$$

where R is the REYNOLDS number, $\varrho\, U\, L/\mu$. The traction condition on the free surface is, with (2.2) and (3.1) to (3.3),

$$(p - x_2 \operatorname{ctn} \theta)\, n_i = S_{ij}\, n_j. \tag{3.5}$$

In dimensionless form, the continuity equation is $u_{\alpha,\alpha} = 0$, the kinematic free boundary condition is $u_\alpha n_\alpha = 0$, and the no-slip condition on the trough wall is $u_\alpha = 0$.

In Section 4 we will find that a new parameter T with the dimensions of time arises from the constitutive equation for the extra stress. When we speak of slow motion of a viscoelastic fluid, we mean a motion in which the typical shear-rate U/L is small when measured on a time scale intrinsic to the fluid. That is, the dimensionless shear-rate defined by

$$\varepsilon = T\, U/L \tag{3.6}$$

is small. Because T can vary enormously from one fluid to another, the shear-rates U/L which can be considered small will also vary widely, depending on the particular fluid considered. However, in the present problem we find from (3.1) and (3.6) that

$$\varepsilon = \sin\theta/k, \quad \text{where} \quad k = \mu/\varrho\, g\, L\, T. \tag{3.7}$$

Hence, ε can be made as small as desired by decreasing the tilt angle θ.

Although θ is the basic controllable parameter in the problem, it will be convenient to treat ε as the fundamental parameter, and express all other parameters involving θ in terms of ε. In particular, the REYNOLDS number R is written as

$$R = \varepsilon\, R', \quad \text{where} \quad R' = \varrho\, L^2/\mu\, T. \tag{3.8}$$

We shall seek a solution valid in the limit $\varepsilon \to 0$, with k and R' fixed. The solution will thus become increasingly accurate as the tilt angle θ decreases, for a fluid of fixed properties in a given trough.

As in slow viscoelastic flow through tubes [1], symmetry shows that each of the unknowns of the problem is either even or odd in its dependence on the parameter ε [6]. From the results on flow in tubes we can also anticipate that the transverse velocity components u_α will be of order ε^3 in comparison to the axial velocity. On this basis, we write

$$u_3' = U\, u_3 = U\, u(x_\alpha; \varepsilon^2), \quad u_\alpha' = U\, u_\alpha = U\, \varepsilon^3\, v_\alpha(x_\beta; \varepsilon^2) \tag{3.9}$$

and

$$(\mu\, U/L)\, p = (\mu\, U/L)\, \varepsilon\, P(x_\alpha; \varepsilon^2). \tag{3.10}$$

In addition, the equation of the free surface will be of the form

$$x_2 = \varepsilon^2\, F(x_1; \varepsilon^2). \tag{3.11}$$

Dependence on the parameters k and R', which is not at issue here, has been suppressed from the notation.

4. Constitutive Equations

Because we wish to consider a slow steady motion, it is appropriate to write the constitutive equation for the extra stress $\underset{\sim}{S}' = \|S_{ij}'\|$ in the RIVLIN-ERICKSEN [7] form:

$$\underset{\sim}{S}'/\mu = \sum \alpha_n \mathbf{A}_n' + \sum \sum \alpha_{mn} \mathbf{A}_m' \mathbf{A}_n' +$$
$$+ \sum \sum \sum [\alpha_{mnp} \mathbf{A}_m' \mathbf{A}_n' \mathbf{A}_p' + \alpha_{m;np} \mathbf{A}_m' \, tr\, (\mathbf{A}_n'\, {}_p\mathbf{A}')] + \cdots \quad (4.1)$$

The coefficients μ and α are constants, with $\alpha_1 = 1$. The terms indicated by dots are of degree four or higher in the kinematic matrices $\mathbf{A}_n'$. Symmetry of $\underset{\sim}{S}'$ requires that $\alpha_{mn} = \alpha_{nm}$ and $\alpha_{mnp} = \alpha_{pnm}$. We can also take $\alpha_{m;np} = \alpha_{m;pn}$ with no loss of generality. Reduction of the RIVLIN-ERICKSEN equation for an incompressible fluid to a form equivalent to (4.1) has been discussed elsewhere [8].

With U and L defined as in Section 3, we express $\mathbf{A}_n'$ in terms of a dimensionless kinematic matrix $\mathbf{A}_n$ through the relation

$$\mathbf{A}_n' = (U/L)^n \mathbf{A}_n. \quad (4.2)$$

The components $A_{ij}^{(n)}$ of $\mathbf{A}_n$ are defined [7] by

$$A_{ij}^{(1)} = u_{i,j} + u_{j,i} \quad (4.3)$$

and, for $n \geqq 2$,

$$A_{ij}^{(n)} = u_{i,j}^{(n)} + u_{j,i}^{(n)} + \sum_{p=1}^{n-1} \binom{n}{p} u_{k,i}^{(p)} u_{k,j}^{(n-p)}. \quad (4.4)$$

Here we use the notation

$$u_i^{(1)} = u_i \quad \text{and} \quad u_i^{(n)} = (D/Dt)^{n-1} u_i, \quad (4.5)$$

where the dimensionless material derivative D/Dt is of the form $u_\alpha \, \partial/\partial x_\alpha$ in the present problem.

The coefficients α in (4.1) have dimensions of time to various powers. Let T be a characteristic time defined in terms of these coefficients, for example $T = |\alpha_2|$ if $\alpha_2 \neq 0$. Dimensionless coefficients β are then defined by

$$\alpha_n = \beta_n T^{n-1}, \quad \alpha_{mn} = \beta_{mn} T^{m+n-1}, \quad \alpha_{mnp} = \beta_{mnp} T^{m+n+p-1}. \quad (4.6)$$

By using (4.2) and (4.6) in (4.1), and recalling the definition (3.6) of the parameter ε, we obtain

$$(L/\mu U)\, \mathbf{S}' = \mathbf{S} = \sum \varepsilon^{n-1} \beta_n \mathbf{A}_n + \sum \sum \varepsilon^{m+n-1} \beta_{mn} \mathbf{A}_m \mathbf{A}_n +$$
$$+ \sum \sum \sum \varepsilon^{m+n+p-1} [\beta_{mnp} \mathbf{A}_m \mathbf{A}_n \mathbf{A}_p + \beta_{m;np} \mathbf{A}_m \, tr\, (\mathbf{A}_n \mathbf{A}_p)] + \cdots \quad (4.7)$$

Since $\alpha_1 = 1$, it follows that $\beta_1 = 1$ as well.

In the present problem, the transverse velocity components are weak in comparison to the axial velocity [see (3.9)]. For this reason, each material element in the fluid is almost in a state of steady simple shearing motion. In such a motion, all of the kinematic matrices vanish except $\mathbf{A}_1$ and $\mathbf{A}_2$. Hence, in the present problem we can expect that $\mathbf{S}$ will be given to a high degree of approximation in terms of $\mathbf{A}_1$ and $\mathbf{A}_2$ alone.

With (3.9), the material derivative is of the form $\varepsilon^3 v_\alpha \, \partial/\partial x_\alpha$. Hence, with (4.5) we obtain

$$\left.\begin{aligned} u_3^{(1)} &= u, & u_3^{(n)} &= \varepsilon^{3n-3} (v_\alpha \, \partial/\partial x_\alpha)^{n-1} u, \\ u_\alpha^{(1)} &= \varepsilon^3 v_\alpha, & u_\beta^{(n)} &= \varepsilon^{3n} (v_\alpha \, \partial/\partial x_\alpha)^{n-1} v_\beta. \end{aligned}\right\} \tag{4.8}$$

Inspection of (4.4) then shows that $\mathbf{A}_n = 0\,(\varepsilon^{3n-6})$ for $n \geqslant 2$, with $\mathbf{A}_1 = 0\,(1)$. If we neglect terms of order ε^5, the expression (4.7) for $\mathbf{S}$ does indeed involve only $\mathbf{A}_1$ and $\mathbf{A}_2$.

By means of (4.3) and (4.4), with (4.8), these matrices can be expressed as

$$\mathbf{A}_1 = \mathbf{B}_1 + \varepsilon^3 \mathbf{C} \quad \text{and} \quad \mathbf{A}_2 = \mathbf{B}_2 + 0\,(\varepsilon^3), \tag{4.9}$$

where the non-zero components of the matrices $\mathbf{B}_1$, $\mathbf{B}_2$, and $\mathbf{C}$ are

$$B_{3\alpha}^{(1)} = u_{,\alpha}, \quad B_{\alpha\beta}^{(2)} = 2\, u_{,\alpha} u_{,\beta}, \quad \text{and} \quad C_{\alpha\beta} = v_{\alpha,\beta} + v_{\beta,\alpha}. \tag{4.10}$$

The matrices $\mathbf{B}_1$ and $\mathbf{B}_2$ are the kinematic matrices for the axial steady simple shearing motion. $\mathbf{C}$ is twice the strain-rate matrix for the transverse perturbation.

Because of the special forms of the steady simple shearing matrices $\mathbf{B}_1$ and $\mathbf{B}_2$, every symmetric matrix polynomial in these matrices can be expressed as a linear combination of $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_1{}^2$, and the unit matrix $\mathbf{I}$, with scalar coefficients which are polynomials in the invariant

$$\gamma^2 = tr\,\mathbf{B}_1{}^2/2 = u_{,\alpha} u_{,\alpha}, \tag{4.11}$$

where γ is the absolute shear-rate [9]. In particular, by using (4.9) in (4.7), and recalling that $\mathbf{A}_n = 0\,(\varepsilon^{3n-6})$ for $n \geqslant 2$, we obtain an expression of the form

$$\mathbf{S} = \eta\, \mathbf{B}_1 + \varepsilon\, \nu_2\, \mathbf{B}_2/2 + \varepsilon\, \nu_1 (\mathbf{B}_1{}^2 - \mathbf{B}_2/2) + \varepsilon^3 \mathbf{C} + 0\,(\varepsilon^4). \tag{4.12}$$

The dimensionless apparent viscosity η and the normal stress coefficients ν_1 and ν_2 are given by

$$\nu_1 = \nu_{11} + (\varepsilon\gamma)^2 \nu_{12} + 0\,(\varepsilon^4), \quad \nu_2 = \nu_{21} + (\varepsilon\gamma)^2 \nu_{22} + 0\,(\varepsilon^4), \tag{4.13}$$

and

$$\eta = 1 + (\varepsilon\gamma)^2 \eta_1 + 0\,(\varepsilon^4), \tag{4.14}$$

where

$$\nu_{11} = \beta_{11}, \qquad \nu_{12} = 2\,\beta_{121} + \beta_{1111} + 2\,\beta_{11;11}, \tag{4.15}$$

$$\nu_{21} = 2\,\beta_2 + \beta_{11}, \quad \nu_{22} = 4\,\beta_{22} + 4\,\beta_{112} + 4\,\beta_{2;11} + \beta_{1111} + 2\,\beta_{11;11}, \tag{4.16}$$

and

$$\eta_1 = 2\,\beta_{12} + \beta_{111} + 2\,\beta_{1;11}. \tag{4.17}$$

From (4.12), with (4.10), we obtain

$$S_{3\alpha} = \eta\, u_{,\alpha} + 0\,(\varepsilon^4), \tag{4.18}$$

$$S_{\alpha\beta} = \varepsilon\,[\nu_2\, u_{,\alpha} u_{,\beta} + \varepsilon^2 (v_{\alpha,\beta} + v_{\beta,\alpha}) + 0\,(\varepsilon^4)], \tag{4.19}$$

and an expression for S_{33} which will not be needed. It is worth emphasizing that although the flow we consider is not a viscometric flow [10], the constitutive equations involve only the viscometric functions η, ν_1, and ν_2, to the indicated order of approximation.

5. Free Surface Conditions

By resolving the free surface traction conditions (3.5) into axial, tangential, and normal components, we obtain

$$S_{3\alpha}\, n_\alpha = 0, \quad S_{\alpha\beta}\, t_\alpha\, n_\beta = 0, \tag{5.1}$$

and

$$x_2\, n_\alpha\, n_\alpha = \tan\theta\, [p\, n_\alpha\, n_\alpha - S_{\alpha\beta}\, n_\alpha\, n_\beta], \tag{5.2}$$

where n_α and t_α are respectively normal and tangential to the free surface (3.11):

$$n_2 = t_1 = 1, \quad -n_1 = t_2 = \varepsilon^2 F'(x_1). \tag{5.3}$$

The three relations (5.1) and (5.2) can be simplified and made more explicit by using the expressions (4.18) and (4.19) for **S**. By using (4.18) in (5.1a), and noticing that the apparent viscosity η is $0\,(1)$, we obtain

$$n_\alpha\, u,_\alpha = 0\,(\varepsilon^4). \tag{5.4}$$

With this result, on using (4.19) in (5.1b), we obtain

$$(v_{\alpha,\beta} + v_{\beta,\alpha})\, t_\alpha\, n_\beta = 0\,(\varepsilon^2). \tag{5.5}$$

Finally, by setting x_2 equal to $\varepsilon^2 F(x_1)$ in (5.2), and making use of (3.7), (3.10), (4.19), (5.3), and (5.4), we obtain

$$F = k\,(1 - \varepsilon^2 k^2)^{-1/2}\, [P - 2\,\varepsilon^2\, v_{2,2} + 0\,(\varepsilon^4)]. \tag{5.6}$$

We now express the free surface conditions in terms of quantities evaluated on the unperturbed free surface $x_2 = 0$, by expanding each unknown in powers of x_2 and then setting $x_2 = \varepsilon^2 F$. The kinematic condition $n_\alpha\, v_\alpha = 0$ implies, with (5.3), that $v_2 = 0\,(\varepsilon^2)$ on $x_2 = \varepsilon^2 F$. By using the expansion procedure just outlined, we obtain

$$v_2(x_1, 0) = 0\,(\varepsilon^2). \tag{5.7}$$

Because we shall seek only a first approximation to v_α, (5.7) will be sufficiently accurate for our purpose. A condition on the transverse velocity component v_1 is found by applying the expansion procedure to (5.5), and then taking (5.7) into account:

$$v_{1,2}(x_1, 0) = 0\,(\varepsilon^2). \tag{5.8}$$

The expansion of (5.6) yields

$$\begin{aligned} F/k = (1 + \varepsilon^2 k^2/2)\, [P(x_1, 0) + \varepsilon^2 F(x_1)\, P,_2(x_1, 0) - \\ - 2\,\varepsilon^2\, v_{2,2}(x_1, 0) + 0\,(\varepsilon^4)]. \end{aligned} \tag{5.9}$$

Hence in particular

$$F(x_1) = k\, P(x_1, 0) + 0\,(\varepsilon^2), \tag{5.10}$$

and by using this result on the right-hand side of (5.9) we obtain the more accurate expression

$$\begin{aligned} F/k = P(x_1, 0) + \varepsilon^2\, [(k^2/2)\, P(x_1, 0) + \\ + k\, P(x_1, 0)\, P,_2(x_1, 0) - 2\, v_{2,2}(x_1, 0)] + 0\,(\varepsilon^4). \end{aligned} \tag{5.11}$$

On applying the expansion procedure to (5.4), and using (5.3) and (5.10), we obtain

$$u,_2(x_1, 0) + \varepsilon^2 [k\, P(x_1, 0)\, u,_{22}(x_1, 0) - k\, P,_1(x_1, 0)\, u,_1(x_1, 0)] = 0\,(\varepsilon^4). \quad (5.12)$$

6. Solution: First Approximation

In the momentum equations (3.4), we use (3.8) to (3.10), (4.18), and (4.19), to express the axial component as

$$(\eta\, u,_\alpha),_\alpha = -1 + 0\,(\varepsilon^4) \quad (6.1)$$

and the transverse components as

$$P,_\alpha = (\nu_2\, u,_\alpha\, u,_\beta),_\beta + \varepsilon^2\, \nabla^2 v_\alpha + 0\,(\varepsilon^4), \quad (6.2)$$

where the viscometric functions η and ν_2 are defined in (4.14) and (4.13), respectively. To obtain the solutions of (6.1) and (6.2) to the indicated order of approximation, we expand u, P, and v_α in powers of ε^2 as

$$u = u^{(0)} + \varepsilon^2\, u^{(1)} + 0\,(\varepsilon^4),\ \ P = P^{(0)} + \varepsilon^2\, P^{(1)} + 0\,(\varepsilon^4),\ \ v_\alpha = v_\alpha{}^{(0)} + 0\,(\varepsilon^2). \quad (6.3)$$

By using (6.3) in (6.1), we obtain in particular

$$\nabla^2 u^{(0)} = -1, \quad (6.4)$$

which is the dimensionless form of (2.6), the equation governing the axial velocity in Newtonian flow. The condition $u^{(0)} = 0$ is to be satisfied on the trough wall. With (6.3), the free surface condition (5.12) yields $u,_2^{(0)}(x_1, 0) = 0$. Thus, $u^{(0)}$ satisfies all of the conditions on the axial velocity in Newtonian flow.

When the expansions (6.3) are used in the transverse momentum equations (6.2), and (6.4) is taken into account, it is found that the equation for $P^{(0)}$ can be integrated immediately:

$$P^{(0)} = (\nu_{21}/2)\,(\gamma_0^2 - 2\, u^{(0)}) + C_0. \quad (6.5)$$

Here ν_{21} is defined in (4.16), C_0 is an arbitrary constant, and γ_0 is the absolute shear rate for the Newtonian flow:

$$\gamma_0{}^2 = u,_\alpha^{(0)}\, u,_\alpha^{(0)}. \quad (6.6)$$

The lowest-order approximation to the shape of the free surface is found by using (6.3) and (6.5) in (5.10):

$$F(x_1) = (k\, \nu_{21}/2)\, S(x_1) + k\, C_0 + 0\,(\varepsilon^2). \quad (6.7)$$

The shape factor $S(x_1)$ is

$$S(x_1) = [u,_1^{(0)}(x_1, 0)]^2 - 2\, u^{(0)}(x_1, 0). \quad (6.8)$$

Here we have used the condition $u,_2^{(0)}(x_1, 0) = 0$. The constant C_0 in (6.7) affects the level of the free surface but not its shape. This constant can be determined by specifying the height of a single point on the free surface, or by specifying the total flux of fluid down the trough.

We note that although the distortion of the free surface is a non-Newtonian effect, the lowest-order approximation to the shape is found by solving only the Newtonian flow problem (6.4). Converting (6.7) into dimensional form with the aid of (3.1), (3.2), (3.6), (3.7), (3.11), (4.6), and (4.16), we obtain

$$x_2' = (\varrho\, g\, L^2/\mu)\,(\alpha_2 + \alpha_{11}/2)\,(\sin\theta)^2\, S\,(x_1'/L). \tag{6.9}$$

Here we have omitted the constant and the higher-order terms in (6.7).

7. Examples

In the following sections we consider the problem of obtaining an improved approximation to the shape of the free surface. In the present section, we illustrate the results already obtained, with some simple examples.

We first consider a trough in the form of an elliptical arc of semi-axes L and $h\,L$, whose equation in dimensionless form is

$$x^2 + (y/h)^2 = 1, \quad y < 0, \tag{7.1}$$

where we use the notation $x = x_1$ and $y = x_2$. The solution of (6.4) satisfying $u^{(0)} = 0$ on (7.1) and $u_{,2}^{(0)} = 0$ on $y = 0$ is

$$u^{(0)} = [1 - x^2 - (y/h)^2]/2\,(1 + h^{-2}). \tag{7.2}$$

The shape factor defined in (6.8) is then

$$S(x) = [(2 + h^{-2})\,x^2 - 1 - h^{-2}]/(1 + h^{-2})^2. \tag{7.3}$$

Thus, the free surface takes on a parabolic shape. The largest difference in level is

$$\varDelta S = S(1) - S(0) = (2 + h^{-2})/(1 + h^{-2})^2. \tag{7.4}$$

In the case of a shallow trough ($h \to 0$), we obtain $\varDelta S \cong h^2$. For a circular section ($h = 1$), we obtain $\varDelta S = 3/4$, and in the case of a deep channel ($h \to \infty$), $\varDelta S = 2$.

The solution for a trough bounded by the planes $x_1' = L$ and $x_2' = - x_1'/\sqrt{3}$ is also especially simple. The solution of (6.4) which vanishes on $x = 1$ and $y = \pm\, x/\sqrt{3}$ is

$$u^{(0)} = (1/4)\,(1 - x)\,(x^2 - 3\,y^2), \tag{7.5}$$

and this satisfies the condition $\partial u^{(0)}/\partial y = 0$ on $y = 0$. The shape factor is then

$$S(x) = (1/16)\,x^2\,(9\,x^2 - 4\,x - 4). \tag{7.6}$$

The maximum difference in level in this case is

$$S(1) - S(2/3) = 59/432. \tag{7.7}$$

8. Axial Velocity Perturbation

Before a second approximation to the form of the free surface can be obtained, the perturbations $u^{(1)}$, $P^{(1)}$, and $v_\alpha^{(0)}$ in (6.3) must be calculated. We begin by considering the axial velocity perturbation $u^{(1)}$. By using (4.11), (4.14), and (6.3) in the axial momentum equation (6.1), we obtain

$$\nabla^2 u^{(1)} = - \eta_1 (\gamma_0^2 u_{,\alpha}^{(0)})_{,\alpha}, \tag{8.1}$$

where the constant η_1 is defined in (4.17), and γ_0^2 is defined in (6.6). On the trough wall, $u^{(1)}$ is zero. The condition for $u^{(1)}$ on the unperturbed free surface is obtained by using (6.3) in (5.12):

$$u_{,2}^{(1)}(x_1, 0)/k = - P^{(0)}(x_1, 0)\, u_{,22}^{(0)}(x_1, 0) + P_{,1}^{(0)}(x_1, 0)\, u_{,1}^{(0)}(x_1, 0). \tag{8.2}$$

With (6.5) and (6.8), this condition can be rewritten in the form

$$u_{,2}^{(1)}(x_1, 0)/k = - C_0\, u_{,22}^{(0)}(x_1, 0) - (\nu_{21}/2)\, [S(x_1)\, u_{,22}^{(0)}(x_1, 0) - S'(x_1)\, u_{,1}^{(0)}(x_1, 0)]. \tag{8.3}$$

For physical insight, it is helpful to fragment $u^{(1)}$ into three parts:

$$u^{(1)} = \eta_1 \varphi_1 + k\, C_0\, \varphi_2 + k\, \nu_{21}\, \varphi_3. \tag{8.4}$$

Then (8.1) and (8.3) are satisfied if φ_1 satisfies

$$\nabla^2 \varphi_1 = - (\gamma_0^2 u_{,\alpha}^{(0)})_{,\alpha}, \quad \varphi_{1,2}(x_1, 0) = 0, \tag{8.5}$$

φ_2 satisfies

$$\nabla^2 \varphi_2 = 0, \quad \varphi_{2,2}(x_1, 0) = - u_{,22}^{(0)}(x_1, 0), \tag{8.6}$$

and φ_3 satisfies

$$\nabla^2 \varphi_3 = 0, \quad \varphi_{3,2}(x_1, 0) = - (1/2)\, [S(x_1)\, u_{,22}^{(0)}(x_1, 0) - S'(x_1)\, u_{,1}^{(0)}(x_1, 0)]. \tag{8.7}$$

Each of the functions φ_1, φ_2, and φ_3 is to vanish at the trough wall.

In the decomposition (8.4), φ_1 is the axial velocity perturbation which would arise from the rate-dependence of the apparent viscosity, in flow through a tube bounded by the trough and its reflection in the plane $x_2 = 0$. The term φ_2 is purely a Newtonian effect, which would arise if the Newtonian flat free surface were at $x_2 = \varepsilon^2 k\, C_0$ rather than at $x_2 = 0$. The term φ_3 is due to the distortion of the free surface.

9. Pressure Perturbation

The equation for $P^{(1)}$, found by using (4.11), (4.13), and (6.3) in the transverse momentum equation (6.2), is

$$P_{,\alpha}^{(1)} = \nu_{21}\, (u_{,\alpha}^{(0)} u_{,\beta}^{(1)} + u_{,\alpha}^{(1)} u_{,\beta}^{(0)})_{,\beta} + \nu_{22}\, (\gamma_0^2\, u_{,\alpha}^{(0)} u_{,\beta}^{(0)})_{,\beta} + \nabla^2 v_\alpha^{(0)}, \tag{9.1}$$

where ν_{21} and ν_{22} are defined in (4.16), and γ_0^2 is defined in (6.6). By making use of (6.4), (8.1), and (8.4) to (8.7), from (9.1) we obtain

$$P^{(1)} = \nu_{21}\, \eta_1\, P_1 + \nu_{21}\, k\, C_0\, P_2 + \nu_{21}^2\, k\, P_3 + \nu_{22}\, P_4 + (\nu_{21}\, \eta_1 - \nu_{22})\, P_5 + C_1, \tag{9.2}$$

where C_1 is an arbitrary constant, and the functions P_n are given by

$$P_n = u_{,\alpha}^{(0)}\, \varphi_{n,\alpha} - \varphi_n \ (n = 1, 2, 3), \qquad P_4 = \gamma_0^4/4, \tag{9.3}$$

and

$$P_{5,\alpha} = u_{,\alpha}^{(0)}\, \nabla^2 \varphi_1 + (\nu_{21}\, \eta_1 - \nu_{22})^{-1}\, \nabla^2 v_\alpha^{(0)}. \tag{9.4}$$

The integrability condition on (9.4) will yield an equation for $v_\alpha^{(0)}$.

10. Transverse Velocity

To satisfy the continuity equation $v^{(0)}_{\alpha,\alpha} = 0$, we introduce a stream function ψ:

$$v_1^{(0)} = (\nu_{21}\,\eta_1 - \nu_{22})\,\psi,_2, \quad v_2^{(0)} = -(\nu_{21}\,\eta_1 - \nu_{22})\,\psi,_1. \tag{10.1}$$

The factor $\nu_{21}\,\eta_1 - \nu_{22}$ has been introduced in order to simplify the equation for ψ. By eliminating P_5 from (9.4) by cross-differentiation, we obtain

$$\nabla^2\,\nabla^2\,\psi = -\,\partial(u^{(0)}, \nabla^2\,\varphi_1)/\partial(x_1, x_2). \tag{10.2}$$

The conditions $v_\alpha^{(0)} = 0$ on the trough wall imply that ψ is constant along the wall and that its normal derivative vanishes. We can take $\psi = 0$ on the wall without loss of generality. The free surface condition (5.7) implies that ψ is also constant along the free surface, whence $\psi(x_1, 0) = 0$. The condition (5.8) yields $\psi,_{22}(x_1, 0) = 0$. These are exactly the conditions satisfied by the transverse velocity in flow through a tube whose cross-section is bounded by the trough and its image in the plane $x_2 = 0$.

11. Free Surface: Second Approximation

In the expression (5.11) for the free surface, we use the expansions (6.3) to obtain

$$\begin{aligned} F/k = {} & P^{(0)}(x_1, 0) + \varepsilon^2\,[P^{(1)}(x_1, 0) + (k^2/2)\,P^{(0)}(x_1, 0) + \\ & + k\,P^{(0)}(x_1, 0)\,P_{,2}^{(0)}(x_1, 0) - 2\,v_{2,2}^{(0)}(x_1, 0)] + 0\,(\varepsilon^4). \end{aligned} \tag{11.1}$$

With $P^{(0)}$ given by (6.5), we notice that $P_{,2}^{(0)}(x_1, 0) = 0$ since $u_{,2}^{(0)}(x_1, 0) = 0$. This eliminates one of the terms in (11.1). By using (6.5), (6.8), (9.2), and (10.1), from (11.1) we obtain

$$\begin{aligned} F/k = {} & (1 + \varepsilon^2\,k^2/2)\,[C_0 + (\nu_{21}/2)\,S(x_1)] + \\ & + \varepsilon^2\,\{\nu_{21}\,\eta_1\,P_1(x_1, 0) + \nu_{21}\,k\,C_0\,P_2(x_1, 0) + \nu_{21}^2\,k\,P_3(x_1, 0) + \\ & + \nu_{22}\,P_4(x_1, 0) + (\nu_{21}\,\eta_1 - \nu_{22})\,[P_5(x_1, 0) + 2\,\psi,_{12}(x_1, 0)] + C_1\} + \\ & + 0\,(\varepsilon^4). \end{aligned} \tag{11.2}$$

All of the function S, P_n, and ψ in (11.2) are independent of the physical parameters of the problem, and are determined only by the cross-sectional shape of the trough. In summary, we outline the procedure for obtaining these functions. First, $u^{(0)}$ is obtained by solving (6.4). $S(x_1)$ and $P_4(x_1, 0)$ can then be evaluated immediately, from (6.8) and (9.3) respectively. Second, the problems (8.5) to (8.7) [or (8.1)] must be solved, to obtain φ_1, φ_2, and φ_3. The functions P_1, P_2, and P_3 are then given by (9.3). Finally, ψ is found by solving (10.2). The constants C_0 and C_1 in (11.1) can then be determined by specifying the location of some point on the free surface.

12. Concluding Remarks

The first approximation to the shape of the free surface, given by (6.9), is of an especially simple form. Evaluation of the shape factor $S(x_1)$ requires the solution of only one equation (6.4). However, the second approximation (11.2) involves a variety of effects, and evaluation of the functions appearing in this approximation requires the solution of two more partial differential equations, which are more difficult to solve than (6.4) is. We have carried out all of the details of these solutions in the case of a trough of semi-circular cross-section, but since we did not find this exercise to be particularly edifying, we will not report the results here. Our main conclusions are that the first approximation is particularly simple and that the second approximation is excessively complicated.

Acknowledgement

The work described in this paper was carried out under a grant DA-ARO (D)-31-124-G 484 from the Army Research Office, Durham.

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