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AN APPROXIMATION TO INFINITELY DIVISIBLE LAWS

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1. The approximation

One question Professor Tandori asked at my doctoral defense on February 2, 1972, was about infinite divisibility. Since he was satisfied, my answer probably included that, according to Lévy's formula ([9], p. 84), a distribution on the real line $\mathbf{R}$ is infinitely divisible if and only if its characteristic function $\varphi(t)$, $t \in \mathbf{R}$, is given by

$$\varphi(t) = \exp\left\{ i\theta t - \frac{\sigma^2}{2} t^2 + \int_{-\infty}^{0} \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) dL(x) + \right.$$

$$\left. + \int_{0}^{\infty} \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) dR(x) \right\},$$

where i is the imaginary unit, $\theta \in \mathbf{R}$ and $\sigma \geqq 0$ are constants, the function $L(\cdot)$ is left-continuous and non-decreasing on $(-\infty, 0)$ with $L(-\infty) = 0$ and the function $R(\cdot)$ is right-continuous and non-decreasing on $(0, \infty)$ with $R(\infty) = 0$, such that

$$\int_{-\varepsilon}^{0} x^2 dL(x) + \int_{0}^{\varepsilon} x^2 dR(x) < \infty \quad \text{for every} \quad \varepsilon > 0.$$

Little did I think at the time that I should be able to answer the question somewhat more thoroughly twenty-three years later. I hope he will like a few late details here.

For a given quadruple $\big(\theta, \sigma, L(\cdot), R(\cdot)\big)$ with the described properties, let $F_{\theta,\sigma,L,R}(\cdot)$ denote the corresponding distribution function, so that $\varphi(t) =$

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$= \int_{-\infty}^{\infty} e^{itx} dF_{\theta,\sigma,L,R}(x)$, $t \in \mathbf{R}$. Consider the inverse functions

$$\psi_L(u) := \inf\{x < 0 : L(x) > u\}$$

and

$$\psi_R(u) := \inf\{x < 0 : -R(-x) > u\}, \quad 0 < u < \infty,$$

where the infimum of the empty set is taken to be zero. These are non-decreasing, non-positive, right-continuous functions on the half-line $(0, \infty)$ such that

$$(1.1) \qquad \int_\varepsilon^\infty \psi_L^2(u)\,du + \int_\varepsilon^\infty \psi_R^2(u)\,du < \infty \quad \text{for every} \quad \varepsilon > 0.$$

Let $Y_1, Y_2, \ldots$ be independent exponentially distributed random variables with mean 1, so that $P\{Y_k > x\} = e^{-x}$, $x > 0$, $k \in \mathbf{N}$, and consider the corresponding partial sums $S_n := Y_1 + \cdots + Y_n$, $n \in \mathbf{N}$. Let Z be a standard normal random variable, let $\{S_n^{(L)}\}_{n=1}^\infty$ and $\{S_n^{(R)}\}_{n=1}^\infty$ be distributionally equivalent copies of the sequence $\{S_n\}_{n=1}^\infty$ such that the sequences $\{S_n^{(L)}\}_{n=1}^\infty$, $\{S_n^{(R)}\}_{n=1}^\infty$ and Z are independent. Now consider

$$(1.2) \quad V_n^{(M)} := \sum_{j=1}^n \psi_M\big(S_j^{(M)}\big) - \int_1^{S_n^{(M)}} \psi_M(u)\,du, \quad n \in \mathbf{N}, \quad M = L, R,$$

and the problem of approximating $F_{\theta,\sigma,L,R}(\cdot)$ by the distribution functions

$$F_{\theta,\sigma,L,R}^{n,m}(x) := P\Big\{V_n^{(L)} + \sigma Z - V_m^{(R)} + \theta - \theta_L + \theta_R \leqq x\Big\}, \quad x \in \mathbf{R}, \quad n, m \in \mathbf{N},$$

where

$$(1.3) \qquad \theta_M := \int_0^1 \frac{\psi_M(s)}{1 + \psi_M^2(s)}\,ds - \int_1^\infty \frac{\psi_M^3(s)}{1 + \psi_M^2(s)}\,ds, \quad M = L, R.$$

More precisely, we are interested in seeing how fast the Lévy distances $D_{n,m}(L, R)$ between $F_{\theta,\sigma,L,R}^{n,m}(\cdot)$ and $F_{\theta,\sigma,L,R}(\cdot)$, defined as

$$\inf\Big\{\varepsilon > 0 : F_{\theta,\sigma,L,R}^{n,m}(x - \varepsilon) - \varepsilon \leqq F_{\theta,\sigma,L,R}(x) \leqq F_{\theta,\sigma,L,R}^{n,m}(x + \varepsilon) + \varepsilon \quad \text{for all} \quad x \in \mathbf{R}\Big\},$$

go to zero as $n, m \to \infty$. As it turns out, this depends upon how fast the functions $\psi_L(u)$ and $\psi_R(u)$ approach zero as $u \to \infty$.

For $M = L$ and $M = R$ and any $a > 0$ consider

$$(1.4)\qquad \begin{cases} v_M(a) := \sqrt{\int_a^\infty \psi_M^2(u)\,du}, & \text{so that } \left|\psi_M(a)\right| \downarrow 0 \quad \text{and} \\ v_M(a) \downarrow 0 \quad \text{as} \quad a \uparrow \infty, \end{cases}$$

and for a fixed $1/2 \leqq d_M \leqq 2e^{(e-2)/2}$ choose a finite $a_M^* > 0$ so that

$$(1.5)\qquad \begin{cases} \psi_M(a_M^*) = 0 \quad \text{if } \psi_M(a) = 0 \text{ for some } a > 0 \quad \text{and} \\ v_M(a_M^*) < \dfrac{1}{e^{2/e}} \text{ and } \left|\psi_M(a_M^*)\right| \leqq \dfrac{1}{e^{e^{e/2}/d_M}} \text{ if } \psi_M(a) < 0 \text{ for all } a > 0, \end{cases}$$

and, with log standing for the natural logarithm, for all $a \geqq a_M^*$ define

$$w_M(a) :=$$

$$:= \begin{cases} w_1^{(M)}(a) := \sqrt{\frac{e}{2}}\, v_M(a)\sqrt{\log \frac{1}{v_M(a)}}, & \text{if } \sqrt{\frac{2}{e}\log\frac{1}{v_M(a)}} \leqq \frac{v_M(a)}{|\psi_M(a)|}, \\ w_2^{(M)}(a) := \frac{1}{2}\left|\psi_M(a)\right| \log \frac{1}{v_M(a)|\psi_M(a)|}, & \text{if } \sqrt{\frac{2}{e}\log\frac{1}{v_M(a)}} > \frac{v_M(a)}{|\psi_M(a)|} \geqq 1, \\ w_3^{(M)}(a) := d_M \left|\psi_M(a)\right| \log \frac{1}{|\psi_M(a)|}, & \text{if } \frac{v_M(a)}{|\psi_M(a)|} < 1 < \sqrt{\frac{2}{e}\log\frac{1}{v_M(a)}}, \end{cases}$$

where, since $d_M \leqq 2e^{(e-2)/2}$, the second inequality in the specification of $w_3^{(M)}(a)$ is satisfied because $v_M(a) < \left|\psi_M(a)\right| \leqq e^{-e^{e/2}/d_M} \leqq e^{-e/2}$. While it is understood that $w_M(a) := 0$ if $\psi_M(a) = 0$, since otherwise $2w_2^{(M)}(a) \leqq$ $\leqq \left|\psi_M(a)\right| \log\left(1/\left|\psi_M(a)\right|\right) + v_M(a) \log\left(1/v_M(a)\right)$, it is clear that $w_M(a) \to$ $\to 0$ as $a \to \infty$, $M = L, R$. Finally, setting

$$(1.6)\qquad r_n^{(M)}(a) := \begin{cases} P\{S_n \leqq a\} + 2w_M(a), & \text{if } \psi_M(\cdot) \not\equiv 0 \text{ on } (0,\infty), \\ 0, & \text{if } \psi_M(\cdot) \equiv 0 \text{ on } (0,\infty) \end{cases}$$

for $M = L, R$ and $a \geqq a_M^*$, the main result is the following.

THEOREM. *If* $a_L \geqq a_L^*$ *and* $a_R \geqq a_R^*$, *then* $D_{n,m}(L,R) \leqq r_n^{(L)}(a_L) +$ $+ r_m^{(R)}(a_R)$ *for every* $n, m \in \mathbf{N}$.

It will be also clear from the proof (and will be followed in bracketed phrases) that in the case when $\psi_M(u) < 0$ for all $u > 0$, if $w_M(a) = w_1^{(M)}(a)$ for all $a \geqq \tilde{a}_M$ and $v_M(\bar{a}_M) \leqq e^{-2/e}$ for some $\tilde{a}_M, \bar{a}_M > 0$, then the choice

$a_M^* = \max(\tilde a_M, \bar a_M)$ is permissible, while if $w_M(a) = w_2^{(M)}(a)$ for all $a \geqq \tilde a_M$ and $|\psi_M(\bar a_M)| < 1$, $v_M(\bar a_M) < 1$ and the product $v_M(\bar a_M)|\psi_M(\bar a_M)| \leqq e^{-2}$ for some $\tilde a_M, \bar a_M > 0$, then again we may take $a_M^* = \max(\tilde a_M, \bar a_M)$, $M = L, R$. The constant d_M enters the threshold a_M^* as in (1.5) only if the case $w_M(a) = w_3^{(M)}(a)$ cannot be excluded for $M = L$ or $M = R$.

To use the theorem, one will choose two positive sequences $\{a_n^{(L)} : n \in \mathbf{N}\}$ and $\{a_n^{(R)} : n \in \mathbf{N}\}$ such that $\limsup_{n\to\infty} a_n^{(M)}/n < 1$, $M = L, R$, and obtain $D_{n,m}(L,R) \leqq r_n^{(L)}(a_n^{(L)}) + r_m^{(R)}(a_m^{(R)})$ for all n and m such that $a_n^{(L)} \geqq a_L^*$ and $a_m^{(R)} \geqq a_R^*$. For $a_n \equiv a_n^{(L)}$ or $a_n \equiv a_n^{(R)}$, the limsup condition is to force the gamma probabilities

$$P\{S_n \leqq a_n\} = \int_0^{a_n} \frac{x^{n-1}}{(n-1)!} e^{-x}\, dx$$

go to zero as $n \to \infty$. This convergence is the fastest if $a_n \equiv a$ for some $a \geqq a_L^*$ or $a \geqq a_R^*$, in which case an expansion of the incomplete gamma function ([8], p. 135) yields

$$(1.7) \quad P\{S_n \leqq a\} = \frac{a^n}{n!} e^{-a} \sum_{k=0}^{\infty} \frac{a^k}{(n+1)(n+2)\cdots(n+k)} \leqq \frac{a^n}{n!}, \quad n \in \mathbf{N}.$$

On the other hand, $w_M(a_n) \to 0$ fast for $M = L$ or $M = R$ if $a_n \to \infty$ fast as $n \to \infty$. For the fastest possible sequence $a_n \equiv \tau n$, the elementary Lemma 3.1 in [7] gives

$$(1.8) \qquad P\{S_n \leqq \tau n\} \leqq e^{-(1-\tau)^2 n/2} \quad \text{whenever} \quad 0 < \tau < 1, \quad n \in \mathbf{N}.$$

In a concrete situation a trade-off between the opposing tendencies has to be found.

If the limiting infinitely divisible distribution function $F_{\theta,\sigma,L,R}(\cdot)$ is absolutely continuous with density $f_{\theta,\sigma,L,R}(\cdot)$ for which $K_{\theta,\sigma,L,R} := \sup\{f_{\theta,\sigma,L,R}(x) : x \in \mathbf{R}\} < \infty$, then by the theorem and a well-known inequality connecting the Kolmogorov and Lévy distances, for any two positive sequences $\{a_n^{(L)} : n \in \mathbf{N}\}$ and $\{a_n^{(R)} : n \in \mathbf{N}\}$ as above,

$$(1.9) \qquad \sup_{x\in\mathbf{R}} \left| F_{\theta,\sigma,L,R}^{n,m}(x) - F_{\theta,\sigma,L,R}(x) \right| \leqq$$

$$\leqq \left[1 + K_{\theta,\sigma,L,R}\right]\left[r_n^{(L)}(a_n^{(L)}) + r_m^{(R)}(a_m^{(R)})\right]$$

for all n and m such that $a_n^{(L)} \geqq a_L^*$ and $a_m^{(R)} \geqq a_R^*$.

Another general corollary is for the case when the Lévy measure of the underlying infinitely divisible distribution is finite, i.e. in our terminology, both $L(0-) < \infty$ and $R(0+) > -\infty$. In this case, $\psi_L(\cdot)$ is zero on the half-line $\left[L(0-), \infty\right)$ and $\psi_R(\cdot)$ is zero on the half-line $\left[-R(0+), \infty\right)$, and the theorem and (1.7) together yield

$$D_{n,m}(L,R) \leqq \frac{\left[L(0-)\right]^n}{n!} + \frac{\left[-R(0+)\right]^m}{m!} \quad \text{for all} \quad n, m \in \mathbf{R}. \tag{1.10}$$

A result of the type of the theorem, though somewhat different in nature, was first proved by Hall [10] for the approximation of stable laws. A closer version was derived among other results in [2]. Stable laws are considered among the illustrative examples in Section 3, following the proof. The theorem above improves the main result in [3], where a special integrability condition was assumed on the functions ψ_L and ψ_R, restricting (1.1). The approach here differs from that in [3] in the realization that there is no point insisting on the deterministic centering $\int_1^n \psi_M(u)\,du$ instead of the present $\int_1^{S_n^{(M)}} \psi_M(u)\,du$ in $V_n^{(M)}$ in (1.2), $M = L, R$, and in the associated use of moment generating functions, rather than just moments, resulting in faster rates of approximation and no restriction on $L(\cdot)$ and $R(\cdot)$. As explained in [3], these approximations are made possible by a probabilistic representation of a random variable with a given, arbitrary infinitely divisible distribution, obtained in [1]. The sums $\sum_{j=1}^n \psi_L\left(S_j^{(L)}\right) \leqq 0$ in $V_n^{(L)}$ and $-\sum_{j=1}^m \psi_R\left(S_j^{(R)}\right) \geqq 0$ in $-V_m^{(R)}$ are to be viewed as the asymptotic contributions of fixed numbers, n and m, of the smallest and the largest terms in a sum of independent and identically distributed random variables in the domain of partial attraction of the infinitely divisible law given by the quadruple $\left(\theta, \sigma, L(\cdot), R(\cdot)\right)$. (For a recent discussion of such domains the reader is referred to [4].) Thus $V_n^{(L)}$ and $-V_m^{(R)}$ themselves are centered versions of these asymptotic contributions, presently with random centerings. This is why such approximations were called "extreme-sum approximations" in [3].

2. Proof of the theorem

On the same probability space $(\Omega, \mathcal{A}, P)$ where the random variables $V_{n,m} := V_{n,m}(\theta, \sigma, L, R) := V_n^{(L)} + \sigma Z - V_m^{(R)} + \theta - \theta_L + \theta_R$ are defined, and expressed in terms of the same independent sequences $\left\{S_n^{(L)}\right\}_{n=1}^\infty$, $\left\{S_n^{(R)}\right\}_{n=1}^\infty$ and Z, for a given quadruple (θ, σ, L, R) let $V := V(\theta, \sigma, L, R) := V_L + \sigma Z -$

$V_R + \theta - \theta_L + \theta_R$, where for the independent left-continuous Poisson processes

$$N_M(u) := \sum_{k=1}^{\infty} I\{S_k^{(M)} < u\}, \quad 0 \leqq u < \infty, \quad M = L, R,$$

with unit intensity, where $I\{\cdot\}$ is the indicator function,

$$V_M := \int_{S_1^{(M)}}^{\infty} [u - N_M(u)]\, d\psi_M(u) + \int_1^{S_1^{(M)}} u\, d\psi_M(u) + \psi_M(1), \quad M = L, R.$$

Then by Theorem 3 in [1], the distribution function of the variable $V = V(\theta, \sigma, L, R)$ is the function $F_{\theta,\sigma,L,R}(\cdot)$ to be estimated. Since $D_{n,m}(L,R) \leqq r_n^{(L)}(a_L) + r_m^{(R)}(a_R)$ if $P\{|V - V_{n,m}| > r_n^{(L)}(a_L) + r_m^{(R)}(a_R)\} \leqq r_n^{(L)}(a_L) + r_m^{(R)}(a_R)$, the inequality claimed in the theorem will follow if we show that $P\{|V_M - V_n^{(M)}| > r_n^{(M)}(a)\} \leqq r_n^{(M)}(a)$ holds for all $n \in \mathbf{N}$ and $a \geqq a_M^*$, for both $M = L$ and $M = R$. Dropping the indices in (1.4)–(1.6), i.e. setting $v^2(a) := \int_a^\infty \psi^2(u)\, du$, for some $1/2 \leqq d \leqq 2e^{(e-2)/2}$ choosing $a^* > 0$ so that

$$\text{(2.1)} \quad \begin{cases} \psi(a^*) = 0 \quad \text{if } \psi(a) = 0 \text{ for some } a > 0 \quad \text{and} \\ v(a^*) \leqq \dfrac{1}{e^{2/e}} \quad \text{and} \quad |\psi(a^*)| \leqq \dfrac{1}{e^{e^{e/2}/d}} \quad \text{if } \psi(a) < 0 \text{ for all } a > 0, \end{cases}$$

and for $a \geqq a^*$, with the same convention that $w(a) = 0$ if $\psi(a) = 0$, defining

$$\text{(2.2)} \quad w(a) := \begin{cases} w_1(a) := \sqrt{\frac{e}{2}}\, v(a) \sqrt{\log \frac{1}{v(a)}}, & \text{if } \sqrt{\frac{2}{e} \log \frac{1}{v(a)}} \leqq \frac{v(a)}{|\psi(a)|}, \\ w_2(a) := \frac{1}{2} |\psi(a)| \log \frac{1}{v(a)|\psi(a)|}, & \text{if } \sqrt{\frac{2}{e} \log \frac{1}{v(a)}} > \frac{v(a)}{|\psi(a)|} \geqq 1, \\ w_3(a) := d\, |\psi(a)| \log \frac{1}{|\psi(a)|}, & \text{if } \sqrt{\frac{2}{e} \log \frac{1}{v(a)}} > 1 > \frac{v(a)}{|\psi(a)|}, \end{cases}$$

for a non-decreasing, non-positive, right-continuous function $\psi(\cdot)$ on $(0,\infty)$, for which $v(a) < \infty$ for all $a > 0$, we have to show that for all $n \in \mathbf{N}$ and $a \geqq a^*$,

$$\text{(2.3)} \qquad P\{|\Delta_n| > r_n(a)\} \leqq r_n(a),$$

where

$$r_n(a) := \begin{cases} P\{S_n \leqq a\} + 2w(a), & \text{if } \psi(\cdot) \not\equiv 0, \\ 0, & \text{if } \psi(\cdot) \equiv 0, \end{cases}$$

and, using that the jump-points of the Poisson process $N(u) := \sum_{k=1}^{\infty} I\{S_k < u\}$, $0 \leqq u < \infty$, hit the possible discontinuity points of $\psi(\cdot)$ with probability zero,

$$\Delta_n := \int_{S_1}^{\infty} [u - N(u)]\, d\psi(u) + \int_1^{S_1} u\, d\psi(u) +$$

$$+\psi(1) - \sum_{j=1}^{n} \psi(S_j) + \int_1^{S_n} \psi(u)\, du =$$

$$= \int_{S_n}^{\infty} [u - N(u)]\, d\psi(u) + \sum_{j=1}^{n-1} \left\{ \int_{S_j}^{S_{j+1}} u\, d\psi(u) - j\left[\psi(S_{j+1}) - \psi(S_j)\right] \right\} +$$

$$+ \int_1^{S_1} u\, d\psi(u) + \psi(1) - \sum_{j=1}^{n} \psi(S_j) + \int_1^{S_n} \psi(u)\, du =$$

$$= \int_{S_n}^{\infty} [u - N(u)]\, d\psi(u) + \int_1^{S_n} u\, d\psi(u) + \int_1^{S_n} \psi(u)\, du - n\psi(S_n) + \psi(1) =$$

$$= \int_{S_n}^{\infty} [u - N(u)]\, d\psi(u) + \psi(S_n)[S_n - n]$$

almost surely. (Throughout the usual convention $\int_c^d \cdots d\psi := \int_{(c,d]} \cdots d\psi$ applies for all $0 < c < d < \infty$. The integral on the half-line (S_1, ∞) exists almost surely as an improper Riemann integral by (1.1), i.e. by the fact that $v(a) < \infty$ for all $a > 0$.)

If $\psi(u) = 0$ for all $u > 0$, there is in fact nothing to prove. (And here we have $P\{|\Delta_n| > 0\} = 0$ since $\Delta_n \equiv 0$.) If $\psi(\cdot) \not\equiv 0$ on $(0, \infty)$, two cases are distinguished. The trivial case is when $\psi(a^*) = 0$ and hence $\psi(v) = 0$ for all $v \geqq a^*$. In this case, $P\{|\Delta_n| > r_n(a)\} \leqq P\{S_n \leqq a\} + P\{|\Delta_n| > r_n(a), S_n > a\} = P\{S_n \leqq a\}$ and $w(a) = 0$ for all $a \geqq a^*$, and so (2.3) follows with $r_n(a) = P\{S_n \leqq a\}$.

For the non-trivial case, suppose that $\psi(v) < 0$ for all $v > 0$. Fix $n \in \mathbf{N}$ and $a \geqq a^*$, and put $g_n(x) := x^{n-1}e^{-x}/(n-1)!$, $x > 0$, for the density function of S_n. By the definition of $r_n(a)$ in (2.3) and by Markov's inequality we have

$$\text{(2.4)} \quad P\{|\Delta_n| > r_n(a)\} \leqq P\{S_n \leqq a\} + P\{\Delta_n \geqq 2w(a), S_n > a\} +$$

$$+P\{-\Delta_n \geqq 2w(a), S_n > a\} \leqq$$

$$\leqq P\{S_n \leqq a\} + e^{-2sw(a)} E\left(e^{s\Delta_n} I\{S_n > a\}\right) + e^{-2tw(a)} E\left(e^{-t\Delta_n} I\{S_n > a\}\right) =$$

$$= P\{S_n \leqq a\} + e^{-2sw(a)} \int_a^\infty \exp\left\{ \int_x^\infty \left[e^{s\psi(v)} - 1 - s\psi(v) \right] dv \right\} g_n(x)\, dx +$$

$$+ e^{-2tw(a)} \int_a^\infty \exp\left\{ \int_x^\infty \left[e^{-t\psi(v)} - 1 + t\psi(v) \right] dv \right\} g_n(x)\, dx$$

for every $s > 0$ and every $t > 0$, where the last equation for the restricted moment generating functions follows by a slight modification of the first part of the proof of Theorem 4 in [1]. Actually, the slight modification is just the trivial one to account for the restrictive presence of the indicators. Indeed, that taken for granted and setting

$$\Delta_n^* := \int_{S_n}^\infty [v - N(v)]\, d\psi(v) + \int_1^{S_n} v\, d\psi(v) +$$

$$+ \int_1^n \psi(v)\, dv - (n-1)\psi(S_n) + \psi(1),$$

Theorem 4 in [1] directly gives (replacing the *it* there by u) that for all $u \in \mathbf{R}$,

$$E\left(e^{u\Delta_n^*} I\{S_n > a\}\right) = \int_a^\infty \exp\left\{ \int_x^\infty \left[e^{u\psi(v)} - 1 - \frac{u\psi(v)}{1+\psi^2(v)} \right] dv + u\psi(v) + \right.$$

$$\left. + u \int_{1+x}^n \psi(v)\, dv + u \int_x^{1+x} \frac{\psi(v)}{1+\psi^2(v)}\, dv - u \int_{1+x}^\infty \frac{\psi^3(v)}{1+\psi^2(v)}\, dv \right\} g_n(x)\, dx =$$

$$= \int_a^\infty \exp\left\{ \int_x^\infty \left[e^{u\psi(v)} - 1 - u\psi(v) \right] dv + u\psi(x) + u \int_x^n \psi(v)\, dv \right\} g_n(x)\, dx,$$

where the second equation is by straightforward algebra. Hence for

$$\Delta_n^* - \psi(S_n) - \int_{S_n}^n \psi(v)\, dv = \int_{S_n}^\infty [v - N(v)]\, d\psi(v) + \int_1^{S_n} v\, d\psi(v) +$$

$$+ \int_1^{S_n} \psi(v)\, dv - n\psi(S_n) + \psi(1) =$$

$$= \int_{S_n}^\infty [v - N(v)]\, d\psi(v) + S_n\psi(S_n) - n\psi(S_n) = \Delta_n$$

we clearly obtain

$$E\left(e^{u\Delta_n} I\{S_n > a\}\right) = \int_a^\infty \exp\left\{ \int_x^\infty \left[e^{u\psi(v)} - 1 - u\psi(v) \right] dv \right\} g_n(x)\, dx,$$

$$u \in \mathbf{R},$$

proving (2.4), where the integrals on the right may or may not be finite at this stage.

To estimate the integrands there, we use the inequality that if $c \geqq 0$ is a constant, then $e^u - 1 - u \leqq e^c u^2/2$ for all $-\infty < u \leqq c$. For the first integral in (2.4), we have $-\infty < s\psi(v) < 0$ for all $v \geqq a$, so that

$$e^{s\psi(v)} - 1 - s\psi(v) \leqq \frac{s^2}{2}\psi^2(v), \quad v \geqq a, \quad \text{for every} \quad s > 0.$$

For the second, since the negative function $\psi(\cdot)$ is non-decreasing, we obviously have $0 < -t\psi(v) = t|\psi(v)| \leqq |\psi(v)|/|\psi(a)| \leqq 1$ whenever $0 < t \leqq \leqq 1/|\psi(a)|$ and $v \geqq a$, so that

$$e^{-t\psi(v)} - 1 + t\psi(v) \leqq \frac{e\,t^2}{2}\psi^2(v), \quad v \geqq a, \quad \text{for every} \quad 0 < t \leqq \frac{1}{|\psi(a)|}.$$

Hence, moving down x to a in the integrals in both exponents, from (2.4) we obtain

$$\text{(2.5)} \qquad P\{|\Delta_n| > r_n(a)\} \leqq$$

$$\leqq P\{S_n \leqq a\} + \exp\left\{\frac{s^2}{2}v^2(a) - 2sw(a)\right\} + \exp\left\{\frac{et^2}{2}v^2(a) - 2tw(a)\right\}$$

for all $s > 0$ and $0 < t \leqq 1/|\psi(a)|$.

Using (2.2), for all choices of $a \geqq a^*$ for which $w(a) = w_2(a)$ we have

$$w_2(a) > \frac{1}{2}\sqrt{\frac{e}{2}}\frac{v(a)}{\sqrt{\log\frac{1}{v(a)}}}\log\frac{1}{v^2(a)} = \sqrt{\frac{e}{2}}\,v(a)\sqrt{\log\frac{1}{v(a)}} = w_1(a).$$

Also, for all $a \geqq a^*$ for which $w(a) = w_3(a)$ the choice of a^* in (2.1) forces

$$x := \frac{1}{v(a)} > \frac{1}{|\psi(a)|} \geqq e^{e^{e/2}/d} > e^{e/d} \geqq e^{\frac{e}{2d^2}} \quad \text{since}$$

$$v(a) < |\psi(a)| \quad \text{and} \quad d \geqq \frac{1}{2}.$$

This implies that $\sqrt{\log x} > \sqrt{e/2}/d$ or, what is of course the same, $\sqrt{e/2}/ / (d\sqrt{\log x}) < 1$ and, consequently, $\sqrt{e/2}\,x/(d\log x) < x/\sqrt{\log x}$. So,

$$\frac{1}{d}\sqrt{\frac{e}{2}}\frac{y}{\log y} < \frac{1}{d}\sqrt{\frac{e}{2}}\frac{x}{\log x} \leqq \frac{x}{\sqrt{\log x}}$$

whenever $y := 1/|\psi(a)| < x$, since the function $y/\log y$, $y > 0$, is increasing on the half-line $[e, \infty)$ and $y = 1/|\psi(a)| > e$ by the choice of a^* and the upper bound on d. But by (2.2) the inequality $y = 1/|\psi(a)| < 1/v(a) = x$ is equivalent to $w(a) = w_3(a)$. Thus, if $a \geqq a^*$ and $w(a) = w_3(a)$, then $\sqrt{e/2}\, y/\log y < d\, x/\sqrt{\log x}$ or, what is the same,

$$\sqrt{\frac{e}{2}}\,\frac{1}{|\psi(a)|\log\frac{1}{|\psi(a)|}} < d\,\frac{1}{v(a)\sqrt{\log\frac{1}{v(a)}}}, \quad \text{that is,} \quad w_1(a) < w_3(a).$$

(We see that $w_2(a) > w_1(a)$ whenever $v(a) < 1$, $|\psi(a)| < 1$ and $w(a) = w_2(a)$.) For reference purposes the foregoing may be summarized by saying that whenever $a \geqq a^*$,

(2.6) $\qquad$ if $\quad w(a) = w_j(a), \quad$ then $\quad w_j(a) > w_1(a), \quad j = 2, 3.$

Consider the convex function $f_a(s) := \frac{s^2}{2}\, v^2(a) - 2sw(a)$, $s > 0$. Then $f_a(\cdot)$ is negative on the interval $\big(0, 4w(a)/v^2(a)\big)$ and takes its minimum at $s_* = 2w(a)/v^2(a)$. Hence, choosing $s = s_*$ and using (2.6) twice, the second term of the bound in (2.5) is

$$\exp\{f_a(s_*)\} = \exp\left\{-\frac{2w^2(a)}{v^2(a)}\right\} \leqq \exp\left\{-\frac{2w_1^2(a)}{v^2(a)}\right\} = \exp\left\{-e\log\frac{1}{v(a)}\right\} < $$
$$< v(a) \leqq w_1(a) \leqq w(a).$$

The inequality before the last holds since $v(a) \leqq e^{-2/e}$ for all $a \geqq a^*$ by (2.1).

The convex function $h_a(t) := \frac{et^2}{2}\, v^2(a) - 2tw(a)$, $t > 0$, is also negative on the interval $\big(0, [4w(a)]/[ev^2(a)]\big)$ and takes its minimum at the point $t_* := [2w(a)]/[ev^2(a)]$. However, here we also have to satisfy the constraint $0 < t \leqq 1/|\psi(a)|$. So, choosing $t_\diamond := \min\{1/|\psi(a)|, t_*\}$, the third term of the bound in (2.5) becomes $\exp\{h_a(t_\diamond)\}$. Let $a \geqq a^*$. If $w(a) = w_1(a)$, so that

$$0 < t_* = \frac{2\,w_1(a)}{e\,v^2(a)} = \sqrt{\frac{2}{e}}\,\frac{\sqrt{\log\frac{1}{v(a)}}}{v(a)} \leqq \frac{1}{|\psi(a)|},$$

we have (whenever $v(a) \leqq e^{-2/e}$ as above)

$$\exp\{h_a(t_\diamond)\} = \exp\{h_a(t_*)\} = \exp\left\{\log\frac{1}{v(a)} - 2t_* w_1(a)\right\} =$$
$$= \exp\left\{\log\frac{1}{v(a)} - 2\log\frac{1}{v(a)}\right\} = v(a) \leqq w_1(a) = w(a).$$

If $w(a) = w_j(a)$, $j = 2,3$, then by (2.6) again,

$$t_* = \frac{2\,w_j(a)}{e\,v^2(a)} > \frac{2\,w_1(a)}{e\,v^2(a)} = \sqrt{\frac{2}{e}}\ \frac{\sqrt{\log\frac{1}{v(a)}}}{v(a)} > \frac{1}{|\psi(a)|}, \quad j = 2,3.$$

Hence if $w(a) = w_2(a)$, then (whenever $v(a) < 1$, $|\psi(a)| < 1$ and $v(a)|\psi(a)| \leqq$ $\leqq e^{-2}$)

$$\exp\{h_a(t_\diamond)\} = \exp\{h_a(1/|\psi(a)|)\} =$$

$$= \exp\left\{\frac{e}{2}\,\frac{v^2(a)}{\psi^2(a)} - \frac{2w_2(a)}{|\psi(a)|}\right\} < \frac{1}{v(a)}\exp\left\{-\frac{2w_2(a)}{|\psi(a)|}\right\} =$$

$$= \frac{1}{v(a)}\exp\left\{-\log\frac{1}{v(a)|\psi(a)|}\right\} = |\psi(a)| \leqq w_2(a) = w(a)$$

by the choice of a^*, while if $w(a) = w_3(a)$, then

$$\exp\{h_a(t_\diamond)\} = \exp\{h_a(1/|\psi(a)|)\} =$$

$$= \exp\left\{\frac{e}{2}\,\frac{v^2(a)}{\psi^2(a)} - \frac{2w_3(a)}{|\psi(a)|}\right\} < e^{e/2}\exp\left\{-\frac{2w_3(a)}{|\psi(a)|}\right\} =$$

$$= e^{e/2}\,|\psi(a)|^{2d} \leqq e^{e/2}\,|\psi(a)| \leqq d\,|\psi(a)|\log\frac{1}{|\psi(a)|} = w_3(a) = w(a)$$

since $2d \geqq 1$ and $|\psi(a)| \leqq 1/e^{e^{e/2}/d}$ by the choice of a^*. Therefore, the inequality $\exp\{h_a(t_\diamond)\} < w(a)$ holds for all $a \geqq a^*$.

Thus if $a \geqq a^*$, then the bound in (2.5) is less than $P\{S_n \leqq a\} + 2w(a) =$ $= r_n(a)$. This fact establishes (2.3) in the non-trivial case, and hence the theorem. (The collection of bracketed phrases also establishes the remark concerning the choice of the thresholds.)

3. Examples

The first three examples show, in particular, that all three versions of the rate function provided by the three branches of $w_M(\cdot)$, $M = L, R$, defined between (1.5) and (1.6), may in fact occur. For simplicity of exposition, we deal with spectrally one-sided infinitely divisible distributions, that is, we choose $L(\cdot) \equiv 0$, with the exception of the stable and compound Poisson examples. The last four examples are of interest in their own right, the negative binomial being weird enough to deserve attention in any case. In Examples 2–4, the threshold remark beneath the theorem is used without further notice.

EXAMPLE 1. If $L \equiv 0$ and $\psi_R(u) = -e^{-u}$, $u > 0$, then $w_L(a) = 0$, $v_R(a) = e^{-a}/\sqrt{2}$ and $w_R(a) = w_3^{(R)}(a) = d\,a\,e^{-a}$ for all $a > 0$ for $v_R(a)/|\psi_R(a)| = 1/\sqrt{2} < 1$. Regardless of $\theta \in \mathbf{R}$ and $\sigma \geqq 0$, for the corresponding Lévy distance the theorem and (1.8), with $a_n \equiv (2-\sqrt{3})\,n$, give

$$D_{n,n}(0,R) \leqq \frac{3d(2-\sqrt{3})\,n}{\exp\{(2-\sqrt{3})n\}} \quad \text{for all} \quad n \geqq \frac{\max\left(\frac{e^{e/2}}{d}, \frac{2}{e} - \log\sqrt{2}\right)}{2-\sqrt{3}}$$

and each fixed $1/2 \leqq d \leqq 2e^{(e-2)/2} = 2.86419\ldots$. For $d = 2e^{(e-2)/2}$ this holds for all $n \geqq 6$ and for $d = 1/2$ the inequality is true for all $n \geqq 30$.

EXAMPLE 2. If $L \equiv 0$ and $\psi_R(u) = -\sqrt{u}\,e^{-u/2}$, then $w_L(a) = 0$, $v_R(a) = \sqrt{a+1}\,e^{-a/2}$ and $w_R(a) = w_2^{(R)}(a) = 2^{-1}\sqrt{a}\,e^{-a/2}\log\left(e^a[a^2+a]^{-1/2}\right)$ for all $a \geqq 2/(e-2) = 2.78442\ldots$, say. Regardless of $\theta \in \mathbf{R}$ and $\sigma \geqq 0$, for the corresponding Lévy distance the theorem and (1.8), with $a_n \equiv (3-\sqrt{5})\,n/2$, give

$$D_{n,n}(0,R) \leqq \frac{3\sqrt{(3-\sqrt{5})\,n}}{2^{3/2}\exp\left\{\frac{3-\sqrt{5}}{4}n\right\}} \log \frac{\exp\left\{\frac{3-\sqrt{5}}{2}n\right\}}{\sqrt{\frac{3-\sqrt{5}}{2}n}\,\sqrt{1+\frac{3-\sqrt{5}}{2}n}}$$

$$\text{for all} \quad n \geqq 5 > \frac{3.35}{3-\sqrt{5}}.$$

EXAMPLE 3: Stable laws. Let $F_{\alpha,\beta,\eta,\zeta}(\cdot)$ be the distribution function of a non-normal stable law with exponent $0 < \alpha < 2$, given by its characteristic function

$$\int_{-\infty}^{\infty} e^{itx} dF_{\alpha,\beta,\eta,\zeta}(x) =$$

$$= \begin{cases} \exp\left\{i\zeta t - \eta|t|^\alpha\left[1 - i\beta\,\mathrm{sgn}(t)\tan(\alpha\pi/2)\right]\right\}, & \text{if } \alpha \neq 1, \\ \exp\left\{i\zeta t - \eta|t|^\alpha\left[1 + i\beta\,\mathrm{sgn}(t)\frac{2}{\pi}\log|t|\right]\right\}, & \text{if } \alpha = 1, \end{cases}$$

with skewness, scale and location parameters $-1 \leqq \beta \leqq 1$, $\eta > 0$ and $\zeta \in \mathbf{R}$, where $\mathrm{sgn}(t)$ is the sign function, $t \in \mathbf{R}$. In Lévy's canonical form at the beginning of the paper, this is given by some $\theta = \theta(\alpha,\beta,\eta,\zeta)$, $\sigma = 0$ and $L(\cdot)$ and $R(\cdot)$ functions such that $\psi_M(u) = -c_M u^{-1/\alpha}$, $u > 0$, where $c_M = c_M(\alpha,\beta,\eta,\zeta) \geqq 0$ are some constants, $M = L, R$, such that $c_L(\alpha,1,\eta,\zeta) = 0$ and $c_L(\alpha,\beta,\eta,\zeta) > 0$ for every $-1 \leqq \beta < 1$, while $c_R(\alpha,-1,\eta,\zeta) = 0$ and $c_R(\alpha,\beta,\eta,\zeta) > 0$ for every $-1 < \beta \leqq 1$; cf. [9], [1], [4]. Setting $K_{\alpha,\beta,\eta,\zeta} := \sup\{f_{\alpha,\beta,\eta,\zeta}(x) : x \in \mathbf{R}\} < \infty$ for the corresponding density function

$f_{\alpha,\beta,\eta,\zeta}(\cdot) := F'_{\alpha,\beta,\eta,\zeta}(\cdot)$ and $\vartheta = \vartheta(\alpha,\beta,\eta,\zeta) := \theta - \theta_L + \theta_R$, where θ_L and θ_R are given through (1.3), let $F^{n,m}_{\alpha,\beta,\eta,\zeta}(x) := P\{V_n^{(L)} - V_m^{(R)} + \vartheta \leqq x\}$, $x \in \mathbf{R}$, where, in the present situation $V_n^{(M)}$ of (1.2), for $M = L, R$ and $n \in \mathbf{N}$, is given by

$$V_n^{(M)} = \begin{cases} -c_M \sum_{j=1}^n \big(S_j^{(M)}\big)^{-1/\alpha} + \frac{\alpha c_M}{\alpha-1}\big(S_n^{(M)}\big)^{\frac{\alpha-1}{\alpha}} - \frac{\alpha c_M}{\alpha-1}, & \text{if } \alpha \neq 1, \\ -c_M \sum_{j=1}^n \big(S_j^{(M)}\big)^{-1} + c_M \log S_n^{(M)}, & \text{if } \alpha = 1. \end{cases}$$

Elementary calculation shows that $v_M(a) = \sqrt{\alpha/(2-\alpha)}\, c_M\, a^{-(2-\alpha)/(2\alpha)}$, $a >$ > 0, and

$$w_M(a) = w_1^{(M)}(a) = \frac{\sqrt{e}}{2}\,\frac{c_M}{a^{\frac{1}{\alpha}-\frac{1}{2}}}\sqrt{\log\Big(a\Big[c_M\sqrt{\alpha/(2-\alpha)}\Big]^{-2\alpha/(2-\alpha)}\Big)}$$
$$\text{for all} \quad a \geqq a_M^*$$

if $c_M > 0$, where, putting $\rho := \alpha/(2-\alpha)$, $u_M := \big(2/(\rho e)\big)\log\big(1/(c_M\sqrt{\rho})\big)$ and $v_M := 1/(2\rho^2)$, the threshold a_M^* may be chosen as $a_M^* =$ $= \max\big(\rho^\rho\, c_M^{2\rho}\, e^{4\rho/e},\, a_M^\circ\big)$, where a_M° is the smallest positive number such that $a \geqq u_M + v_M \log a$ for all $a \geqq a_M^\circ$, $M = L, R$. Picking now any $\tau \in (0,1)$ in (1.8) and letting $n_M^* := \max\big(a_M^*/\tau,\, n_M^\circ\big)$, where n_M° is the smallest $n \in \mathbf{N}$ for which $\exp\{-(1-\tau)^2 n/2\} \leqq w_1^{(M)}(\tau n)$, $M = L, R$, the inequality in (1.9) gives that for all $n \geqq n_L^*$ and $m \geqq n_R^*$,

$$\sup_{x\in\mathbf{R}}\big|F^{n,m}_{\alpha,\beta,\eta,\zeta}(x) - F_{\alpha,\beta,\eta,\zeta}(x)\big| \leqq 3\big[1 + K_{\alpha,\beta,\eta,\zeta}\big]\,\big[w_1^{(L)}(\tau n) + w_1^{(R)}(\tau m)\big].$$

Neglecting thresholds and constants, the qualitative meaning of this is that

$$\sup_{x\in\mathbf{R}}\big|F^{n,m}_{\alpha,\beta,\eta,\zeta}(x) - F_{\alpha,\beta,\eta,\zeta}(x)\big| = O\bigg(c_L\,\frac{\sqrt{\log n}}{n^{\frac{1}{\alpha}-\frac{1}{2}}} + c_R\,\frac{\sqrt{\log m}}{m^{\frac{1}{\alpha}-\frac{1}{2}}}\bigg)$$
$$\text{as} \quad n, m \to \infty.$$

Improving Theorem 2.2 in [2], the latter rate has also been established in Remark 1.3 of Janssen and Mason [11] by completely different methods.

EXAMPLE 4: Limiting St. Petersburg distributions. In a classical St. Petersburg game, a player gains 2^k ducats with probability 2^{-k}, $k \in \mathbf{N}$. As determined in [5], the class $\{G_\gamma(\cdot) : 1/2 < \gamma \leqq 1\}$ of all possible non-degenerate subsequential limiting types of distribution functions for the cumulative gains of a player in a sequence of independent St. Petersburg games, under any deterministic centering and norming, is described by the family of infinitely divisible characteristic functions

$$\int_{-\infty}^{\infty} e^{itx} dG_\gamma(x) = \exp\left\{ i\theta_\gamma t + \int_0^\infty \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) dR_\gamma(x) \right\}, \quad t \in \mathbf{R},$$

where, with Log standing for the logarithm to the base 2 and, for any $y \in \mathbf{R}$, with $\lfloor y \rfloor$ denoting the greatest integer not greater than y, having a fractional part $\langle y \rangle = y - \lfloor y \rfloor$,

$$\theta_\gamma := \sum_{k=1}^{\infty} \frac{\gamma^2}{\gamma^2 + 4^k} - \sum_{k=0}^{\infty} \frac{1}{1 + \gamma^2 4^k} - \operatorname{Log} \gamma$$

and

$$R_\gamma(x) = -\gamma\, 2^{-\lfloor \operatorname{Log}(\gamma x) \rfloor}, \quad x > 0,$$

so that $\psi_\gamma(u) := \psi_{R_\gamma}(u) = -2^{-\lfloor \operatorname{Log}(u/\gamma) \rfloor}/\gamma$, $u > 0$. Hence by lengthier but elementary and quite delicate computation through (1.2) and (1.3),

$$W_n^{(\gamma)} := -V_n^{(R_\gamma)} + \theta_\gamma + \theta_{R_\gamma} = \frac{1}{\gamma} \sum_{j=1}^{n} \frac{1}{2^{\lfloor \operatorname{Log}(S_j/\gamma) \rfloor}} - \operatorname{Log} S_n + \delta\left(\frac{S_n}{\gamma} \right),$$

where $\delta(s) = 1 + \langle \operatorname{Log} s \rangle - 2^{\langle \operatorname{Log} s \rangle}$, $s > 0$, and it can be seen in similarly elementary fashion that $0 \leqq \delta(s) \leqq 1 - (1 + \log\log 2)/\log 2 = 0.08607\ldots$ for all $s > 0$. (The function $\delta(\cdot)$ plays a special role in the theory of the St. Petersburg game, described in [6], and the present example has some motivational value at some point there.) Also, since $1/u \leqq |\psi_\gamma(u)| < 2/u$ for all $u > 0$, for the corresponding $v_\gamma^2(a) := \int_a^\infty \psi_\gamma^2(u)\, du$ we obtain $1/\sqrt{a} \leqq v_\gamma(a) < \sqrt{2}/\sqrt{a}$ for every $a > 0$ and $1/2 < \gamma \leqq 1$. Thus we have $v_\gamma(a) < e^{-2/e}$ if $a \geqq a^* :=$ $:= 2e^{4/e} = 8.71168\ldots$ and $\sqrt{a}/2 < v_\gamma(a)/|\psi_\gamma(u)| \leqq \sqrt{2a}$, so

$$w_{R_\gamma}(a) = w_1^{(R_\gamma)}(a) = \sqrt{\frac{e}{2}}\, v_\gamma(a) \sqrt{\log\left[1/v_\gamma(a)\right]} < \sqrt{\frac{e}{2}}\, \frac{\sqrt{\log a}}{\sqrt{a}} =: w_1(a)$$

for all $a > 1$ and all $1/2 < \gamma \leqq 1$. Since the densities $g_\gamma(\cdot) = G_\gamma'(\cdot)$ exist and it can be shown that $\sup_{1/2 < \gamma \leqq 1} \sup\{g_\gamma(x) : x \in \mathbf{R}\} \leqq 1/2$, for any $0 < \tau < 1$ in (1.8), finally (1.9) yields

$$\sup_{\frac{1}{2} < \gamma \leqq 1} \sup_{x \in \mathbf{R}} \left| P\{ W_n^{(\gamma)} \leqq x \} - G_\gamma(x) \right| < C(\tau) \frac{\sqrt{\log n}}{\sqrt{n}} \quad \text{for all} \quad n \geqq n^*(\tau),$$

where the bound is a trivial upper bound for $3w_1(\tau n)$ with $C(\tau) := 9\sqrt{e/(2\tau)}/2$ and $n^*(\tau) := \lceil \max(2e^{4/e}/\tau, n_*(\tau)) \rceil$, where $\lceil x \rceil := \min\{k \in \mathbf{N} : k \geqq x\}$, $x > 0$, and $n_*(\tau) := \min\{k \in \mathbf{N} : \exp\{-(1-\tau)^2 k\} \leqq w_1^2(\tau k)\}$. Here, rounding up, $C(1) = 5.24620$ is unachievable, and we get $C(0.707) = 6.23929$, $n^*(0.707) = 13$; $C(0.8) = 5.86543$, $n^*(0.8) = 53$; $C(0.9) = 5.52998$, $n^*(0.9) = 376$; $C(0.95) = 5.38249$, $n^*(0.95) = 2\,107$; $C(0.99) = 5.27263$, $n^*(0.99) = 86\,177$ and $C(0.999) = 5.24883$, $n^*(0.999) = 13\,297\,850$.

EXAMPLE 5: Compound Poisson laws. Let $N_\lambda, X_1, X_2, \ldots$ be independent random variables such that N_λ has the Poisson distribution on the integers $\{0, 1, 2, \ldots\}$ with mean $\lambda > 0$ and $X_1, X_2, \ldots$ have the same distribution function $G(x) := P\{X \leqq x\}$, $x \in \mathbf{R}$. Then Lévy's canonical form of the characteristic function of the infinitely divisible compound Poisson distribution function $F_{\lambda,G}(x) := P\{\sum_{k=1}^{N_\lambda} X_k \leqq x\}$, $x \in \mathbf{R}$, is given by $\sigma = 0$ and, with $G_-(\cdot)$ denoting the left-continuous version of $G(\cdot)$,

$$\theta = \theta_{\lambda,G} = \lambda \int_{-\infty}^{\infty} \frac{x}{1+x^2}\, dG(x), \quad L(x) = \lambda G_-(x),\ x < 0,$$

$$R(x) = \lambda\big[G(x) - 1\big],\ x > 0.$$

Hence, letting $G_+^{-1}(\cdot)$ denote the right-continuous version of the left-continuous generalized inverse $G^{-1}(s) := \inf\{x \in \mathbf{R} : G(x) \geqq s\}$, $0 < s < 1$, the usual quantile function, pertaining to $G(\cdot)$, we have

$$\psi_L(u) = \begin{cases} G_+^{-1}\left(\frac{u}{\lambda}\right), & \text{if } 0 < u < \lambda G_-(0), \\ 0, & \text{if } u \geqq \lambda G_-(0), \end{cases}$$

and

$$\psi_R(u) = \begin{cases} -G^{-1}\left(1 - \frac{u}{\lambda}\right), & \text{if } 0 < u < \lambda[1 - G(0)], \\ 0, & \text{if } u \geqq \lambda[1 - G(0)]. \end{cases}$$

For the Lévy distance $D_{n,m}(\lambda, G)$ between $F_{\lambda,G}(\cdot)$ and its approximation $F_{\lambda,G}^{n,m}(x) := P\{V_n^{(L)} - V_m^{(R)} + \theta_{\lambda,G} - \theta_L + \theta_R \leqq x\}$, $x \in \mathbf{R}$, given by the present $\psi_L(\cdot)$ and $\psi_R(\cdot)$ through (1.2) and (1.3), by (1.10) we obtain

$$D_{n,m}(\lambda, G) \leqq \frac{\big[\lambda G_-(0)\big]^n}{n!} + \frac{\big[\lambda\{1 - G(0)\}\big]^m}{m!} \quad \text{for all} \quad n, m \in \mathbf{N}.$$

The Poisson law itself, with mean λ, is the special case when $G(x)$, $x \in \mathbf{R}$, degenerates at the point $x = 1$ and $\theta_\lambda := \theta_{\lambda,G} = \lambda/2$ for the corresponding quantity. In this case, $\psi_L(u) = 0$ and $\psi_R(u) = -I\{u < \lambda\}$ for all $u > 0$, and enjoyable calculation shows that $-V_n^{(R)} + \theta_\lambda + \theta_R = \sum_{j=1}^n I\{S_j < \lambda\} + (\lambda - S_n)I\{S_n < \lambda\}$ for every $n \in \mathbf{N}$. If $D_n(\lambda)$ denotes the Lévy distance between the distribution function $F_{n,\lambda}(\cdot)$ of the latter random variable and the Poisson distribution function $F_\lambda(x) := e^{-\lambda}\sum_{k=1}^{\lfloor x \rfloor} \lambda^k/k!$, $x \in \mathbf{R}$, of N_λ, with an empty sum understood as zero as above, then the result reduces to the inequality $D_n(\lambda) \leqq \lambda^n/n!$ for all $n \in \mathbf{N}$. Furthermore, if $D_n^*(\lambda)$ is the Lévy distance between $F_\lambda(\cdot)$ and $F_{n,\lambda}^*(x) := P\{\sum_{j=1}^n I\{S_j < \lambda\} \leqq x\}$, $x \in \mathbf{R}$, then a trivial extra step based on the triangle inequality for a Lévy distance yields $D_n^*(\lambda) \leqq 2\,\lambda^n/n!$ for all $n \in \mathbf{N}$.

EXAMPLE 6: Negative binomial distributions. For a fixed order $\ell \in \mathbf{N}$ and success probability $0 < p < 1$, consider the negative binomial distribution function

$$F_{\ell,p}(x) := P\{V_\ell(p) \leqq x\} := p^\ell \sum_{k=\ell}^{\lfloor x \rfloor} \binom{k-1}{\ell-1} q^{k-\ell}, \quad x \in \mathbf{R},$$

where $q := 1 - p$. As is well known, it is infinitely divisible and it is a routine exercise to show that the Lévy form of the characteristic function is

$$\int_{-\infty}^{\infty} e^{itx} dF_{\ell,p}(x) = \exp\left\{ i\theta_{\ell,p} t + \int_0^\infty \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) dR_{\ell,p}(x)\right\}, \quad t \in \mathbf{R},$$

where

$$\theta_{\ell,p} = \ell + \ell \sum_{m=1}^{\infty} \frac{q^m}{1+m^2} \quad \text{and} \quad R_{\ell,p}(x) = \ell \sum_{m=1}^{\lfloor x \rfloor} \frac{q^m}{m} + \ell \log p, \quad x > 0.$$

So, noting that $-\ell \log p = \ell \sum_{m=1}^{\infty} q^m/m$,

$$\psi_{\ell,p}(u) := \psi_{R_{\ell,p}}(u) = -\sum_{k=1}^{\infty} k\, I\left\{\log\frac{1}{p^\ell} - \ell\sum_{m=1}^{k}\frac{q^m}{m} \leqq u < \log\frac{1}{p^\ell} - \ell\sum_{m=1}^{k-1}\frac{q^m}{m}\right\},$$
$$u > 0,$$

thus $\psi_{\ell,p}(u) = 0$ for all $u \geqq -\ell \log p$. Evaluating (1.2) and (1.3) with this, the result is

$$W_n^{\ell,p} := -V_n^{(R_{\ell,p})} + \theta_{\ell,p} + \theta_{R_{\ell,p}} =$$

$$= \ell + \sum_{j=1}^{n} \sum_{k=1}^{\infty} k\, I\Big\{ \ell \sum_{m=k+1}^{\infty} \frac{q^m}{m} \leqq S_j < \ell \sum_{m=k}^{\infty} \frac{q^m}{m} \Big\} +$$

$$+ \sum_{k=1}^{\infty} k \Big[\ell \sum_{m=k}^{\infty} \frac{q^m}{m} - S_n \Big] I\Big\{ \ell \sum_{m=k+1}^{\infty} \frac{q^m}{m} \leqq S_n < \ell \sum_{m=k}^{\infty} \frac{q^m}{m} \Big\} =: \ell + T_n^{\ell,p} + R_n^{\ell,p}.$$

(Note that $P\{W_n^{\ell,p} = \ell\} = P\{\ell + T_n^{\ell,p} = \ell\} = p^\ell = P\{V_\ell(p) = \ell\}$.) If now $D_n(\ell,p)$ is the Lévy distance between $F_{\ell,p}(\cdot)$ and $P\{W_n^{\ell,p} \leqq \cdot\}$ and $D_n^*(\ell,p)$ is the Lévy distance between $F_{\ell,p}(\cdot)$ and $P\{\ell + T_n^{\ell,p} \leqq \cdot\}$, then (1.10) and an extra step as above yield

$$D_n(\ell,p) \leqq \frac{[-\ell \log p]^n}{n!} \quad \text{and} \quad D_n^*(\ell,p) \leqq 2\frac{[-\ell \log p]^n}{n!} \quad \text{for all} \quad n \in \mathbf{N}.$$

If $\ell = 1$, this is of course a result for the approximation of the geometric distribution function $F_{1,p}(x) = p \sum_{k=1}^{\lfloor x \rfloor} q^{k-1}$, $x \in \mathbf{R}$.

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