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Right inverses for linear, constant coefficient partial differential operators on distributions over open half spaces

By

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Abstract. Results of Hörmander on evolution operators together with a characterization of the present authors [Ann. Inst. Fourier, Grenoble **40,** 619–655 (1990)] are used to prove the following: Let $P \in \mathbb{C}[z_1, \ldots, z_n]$ and denote by P_m its principal part. If $P - P_m$ is dominated by P_m then the following assertions for the partial differential operators $P(D)$ and $P_m(D)$ are equivalent for $N \in S^{n-1}$:

(1) $P(D)$ and/or $P_m(D)$ admit a continuous linear right inverse on $C^\infty(H_+(N))$.
(2) $P(D)$ admits a continuous linear right inverse on $C^\infty(\mathbb{R}^n)$ and a fundamental solution $E \in \mathscr{D}'(\mathbb{R}^n)$ satisfying Supp $E \subset \overline{H_-(N)}$,

where $H_+(N) := \{x \in \mathbb{R}^n : \pm\langle x, N\rangle > 0\}$.

In the early fifties L. Schwartz posed the problem of determining when a linear differential operator $P(D)$ with constant coefficients admits a (continuous linear) right inverse on $\mathscr{E}(\Omega)$ or $\mathscr{D}'(\Omega)$, Ω an open subset of $\mathbb{R}^n$. This problem was solved by the present authors in [6] and for systems over convex open sets by Palamodov [10]. The solution in [6] is formulated in terms of a P-convexity condition for general open sets Ω and for convex sets Ω in terms of a Phragmén-Lindelöf condition on the zero variety $V(P) = \{z \in \mathbb{C}^n : P(z) = 0\}$. Though these conditions are rather involved they could be evaluated further in many examples. For example, if $N \in \mathbb{R}^n$ is non-characteristic for P then $P(D)$ admits a right inverse on $\mathscr{E}(H_+(N))$ and/or $\mathscr{D}'(H_+(N))$, where $H_\pm(N) = \{x \in \mathbb{R}^n : \pm\langle x, N\rangle > 0\}$, if and only if P is hyperbolic with respect to N (see [6], 3.2). If N is characteristic for P then $P(D)$ may or may not have a right inverse on $\mathscr{E}(H_+(N))$. However, no characterization is known in this case. The aim of the present paper is to prove the following result.

Theorem. *Let P be a non-constant complex polynomial in n variables, let P_m denote its principal part and assume that P_m dominates $P - P_m$. Then the following assertions are equivalent for $N \in S^{n-1}$:*

(1) *$P(D)$ admits a right inverse on $\mathscr{E}(H_+(N))$*
(2) *$P_m(D)$ admits a right inverse on $\mathscr{E}(H_+(N))$*
(3) *$P(D)$ admits a right inverse on $\mathscr{E}(\mathbb{R}^n)$ and there exists $E \in \mathscr{D}'(\mathbb{R}^n)$ satisfying $P(D)E = \delta$ and* Supp $E \subset \overline{H_-(N)}$.

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It was shown in [9], that condition (3) is sufficient without any further assumption on P. Hence the main point here is to prove the necessity of (3) under the given hypotheses. In order to achieve this we construct special subharmonic functions (which might be of interest also in a different context) to evaluate the corresponding Phragmén-Lindelöf condition for P_m, which also has a right inverse on $\mathscr{E}(H_+(N))$ by the perturbation result [8], 4.1. Then the theorem follows from results of Hörmander [3] on evolution operators. Note that the condition "P_m dominates $P - P_m$" is necessary and sufficient for perturbations that preserve hyperbolicity (see Hörmander [4], Thm. 12.4.6) and sufficient for preserving the property of being an evolution operator (see Hörmander [4], Thm. 12.8.17).

The theorem above can be extended to general convex open sets in $\mathbb{R}^n$ as Franken [1] shows by a different approach which uses (similarly as in Franken and Meise [2]) a connection between "weak extendability of zero-solutions" and the existence of right inverses.

1. Preliminaries. In this section we fix the notation and recall some facts which will be used subsequently.

Definition 1.1. Let Ω be an open subset of $\mathbb{R}^n$. Then $\mathscr{E}(\Omega)$ denotes the complex vector space of all infinitely differentiable functions on Ω, endowed with the Fréchet-space topology of uniform convergence of all derivatives on all compact subsets of Ω. Also, $\mathscr{D}(\Omega)$ denotes the space of all functions in $\mathscr{E}(\Omega)$ which have compact support in Ω. It is endowed with the standard (LF)-space topology. Its dual space $\mathscr{D}'(\Omega)$ is the space of all distributions on Ω.

1.2. Polynomials and differential operators. By $\mathbb{C}[z_1, \ldots, z_n]$ we denote the ring of all complex polynomials in n variables, which will be also regarded as functions on $\mathbb{C}^n$. For $P \in \mathbb{C}[z_1, \ldots, z_n]$,

$$P(z) = \sum_{|\alpha| \leqq m} a_\alpha z^\alpha,$$

with $\sum\limits_{|\alpha|=m} |a_\alpha| \neq 0$, we call $P_m(z) := \sum\limits_{|\alpha|=m} a_\alpha z^\alpha$ the principal part of P. Note that P_m is a homogeneous polynomial of degree m. A vector $N \in \mathbb{R}^n \setminus \{0\}$ is called non-characteristic for P if $P_m(N) \neq 0$, otherwise N is called characteristic.

For $P \in \mathbb{C}[z_1, \ldots, z_n]$ and an open set Ω in $\mathbb{R}^n$ we define the linear partial differential operator

$$P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega), \quad P(D)f := \sum_{|\alpha| \leqq m} a_\alpha i^{-|\alpha|} f^{(\alpha)}.$$

Then $P(D)$ is a continuous endomorphism of $\mathscr{D}'(\Omega)$ and its restriction to $\mathscr{E}(\Omega)$ is a continuous endomorphism of $\mathscr{E}(\Omega)$.

A distribution E in $\mathscr{D}'(\mathbb{R}^n)$ is called a fundamental solution for $P(D)$ if $P(D)E = \delta$, where δ denotes the point evaluation at zero.

1.3. Right inverses. We will say that $P(D)$ admits a right inverse on $\mathscr{E}(\Omega)$ (resp. on $\mathscr{D}'(\Omega)$) if there exists a continuous linear map $R : \mathscr{E}(\Omega) \to \mathscr{E}(\Omega)$ (resp. $R : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$) so that

$P(D) \circ R = \mathrm{id}_{\mathscr{E}(\Omega)}$ (resp. $= \mathrm{id}_{\mathscr{D}'(\Omega)}$). By [6], 2.7, $P(D)$ admits a right inverse on $\mathscr{D}'(\Omega)$ if and only if $P(D)$ admits a right inverse on $\mathscr{E}(\Omega)$. Note that in [6], 3.8, this property is characterized by various conditions of different type.

2. Auxiliary subharmonic functions. In this section we construct the subharmonic functions which will be used in the next section to prove one of our main results.

Definition 2.1. For $a > 1$ define $\omega_a : \mathbb{R} \to \mathbb{R}$ by

$$\omega_a(t) := \begin{cases} \dfrac{|t|}{\log a} & \text{if } |t| \leqq a \\ \dfrac{a}{\log a}\left(1 + \log \dfrac{|t|}{a}\right) & \text{if } |t| > a\,. \end{cases}$$

Remark 2.2. $\omega_a|_{[0,\infty[}$ is continuously differentiable, strictly increasing and concave. Since $\omega_a(0) = 0$, it follows that ω_a is subadditive on $\mathbb{R}$. Furthermore $t \mapsto \omega_a(e^t)$ is convex. Hence $z \mapsto \omega_a(|z|)$ is subharmonic. Obviously, $\int\limits_{-\infty}^{\infty} \dfrac{\omega_a(t)}{1+t^2}\,dt < \infty$ for each $a \geqq 2$. Therefore we can consider the harmonic extension of ω_a, defined as follows.

Definition 2.3. For $a \geqq 2$ define $u_a : \mathbb{C} \to \mathbb{R}$ the harmonic extension of ω_a by $u_a|_{\mathbb{R}} = \omega_a$ and

$$u_a(x+iy) = \frac{|y|}{\pi} \int\limits_{-\infty}^{+\infty} \frac{\omega_a(t)}{(t-x)^2 + y^2}\,dt,\ x \in \mathbb{R},\ y \in \mathbb{R} \setminus \{0\}.$$

Lemma 2.4. *For $a \geqq 3$ the functions u_a have the following properties:*

(1) *u_a is continuous on $\mathbb{C}$ and harmonic on $\mathbb{C} \setminus \mathbb{R}$,*
(2) *$u_a(z) \geqq \omega_a(|z|)$ for all $z \in \mathbb{C}$,*
(3) *$u_a(x+iy) \leqq \omega_a(x) + u_a(iy),\ x \in \mathbb{R},\ y \in \mathbb{R} \setminus \{0\}$,*
(4) *$u_a(iy) \leqq \dfrac{y}{\pi}\left[2 + \dfrac{4}{\log a} + \dfrac{\log 2}{\log a}\right]$, for $y \geqq 1$,*
(5) *$\lim\limits_{a \to \infty} u_a(x+iy) = \dfrac{2}{\pi}|y|$ for $x \in \mathbb{R}$, $y \in \mathbb{R} \setminus \{0\}$,*
(6) *$\dfrac{\partial u_a}{\partial y}(x+i) \geqq 0$ for $x \in \mathbb{R}$,*
(7) *$\left|\dfrac{\partial u_a}{\partial y}(x+iy)\right| \leqq \dfrac{3}{\pi}$ for $x \in \mathbb{R},\ y \geqq 1$ and all $a \geqq a_0$.*

Proof. (1) The continuity of u_a can be derived easily, using the subadditivity of ω_a. Further, u_a is harmonic on $\mathbb{C} \setminus \mathbb{R}$, since we may interchange differentiation and integration.

(2) The function $g_a(z) := \omega_a(|z|) - u_a(z)$, defined for $\operatorname{Im} z \geqq 0$ is continuous there and subharmonic in the interior by (1). Further $g_a|_{\mathbb{R}} \equiv 0$ and $g_a(z) = o(|z|)$. Hence the Phragmén-Lindelöf principle for the upper half plane implies $g_a \leqq 0$. This implies (2) because of the symmetry properties of ω_a and u_a.

(3) Since ω_a ist subadditive, an obvious change of variables gives

$$u_a(x+iy) = \frac{|y|}{\pi} \int_{-\infty}^{+\infty} \frac{\omega_a(x+\tau)}{\tau^2+y^2} d\tau \leqq \frac{|y|}{\pi} \int_{-\infty}^{+\infty} \frac{\omega_a(x)+\omega_a(\tau)}{\tau^2+y^2} d\tau = \omega_a(x) + u_a(iy).$$

(4) The definition of ω_a implies for $a \geqq 2$ and $y \geqq 1$:

$$\begin{aligned} u_a(iy) &= \frac{y}{\pi \log a} \int_0^a \frac{2t}{t^2+y^2} dt + \frac{2ay}{\pi \log a} \left(\int_a^\infty \frac{1}{t^2+y^2} dt + \int_a^\infty \frac{\log \frac{t}{a}}{t^2+y^2} dt \right) \\ &\leqq \frac{y}{\pi \log a} [2 \log a + \log 2 + 2 + 2] = \frac{y}{\pi} \left[2 + \frac{4}{\log a} + \frac{\log 2}{\log a} \right], \end{aligned}$$

as standard estimates show.

(5) Because of $u_a(x+iy) = u_a(-x+i|y|)$ it suffices to prove (5) for $x \geqq 0$ and $y > 0$. Assume that x and $y \in \mathbb{R}$ have this property and fix $a > x$. Then the definition of w_a implies

$$(*) \qquad \pi u_a(x+iy) = \frac{1}{\log a} \int_{-a}^{a} \frac{y|t|}{(t-x)^2+y^2} dt + \frac{ay}{\log a} \int_{|t| \geqq a} \frac{1+\log\left(\frac{|t|}{a}\right)}{(t-x)^2+y^2} dt .$$

A direct computation shows that the first integral on the right hand side of $(*)$ converges to $2y$ as a tends to infinity and that the second one tends to zero, which proves (5).

(6) Define $\sigma : \mathbb{R} \to \mathbb{R}$ by $\sigma_a(x) := u_a(x+i)$. Then it is easily seen that σ_a is an even continuous function which has the same properties as ω_a and that the harmonic extension of σ_a equals $u_a(x+i(1+y))$ in the upper half plane. Hence it follows as in the proof of (2) that

$$u_a(x+i+iy) \geqq \sigma_a(|x+iy|) \geqq \sigma_a(|x|) = u_a(x+i).$$

Obviously this implies (6), since u_a is differentiable at $x+i$.

(7) From Meise and Taylor [5], 2.3, and (4) we get the existence of $a_0 \geqq 2$ such that

$$\left| \frac{\partial u_a}{\partial y}(x+iy) \right| \leqq \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\omega_a(t)}{t^2+y^2} dt = \frac{1}{y} u_a(iy) \leqq \frac{3}{\pi} < 1 \text{ for all } x \in \mathbb{R}, y \geqq 1, a \geqq a_0.$$

Definition 2.5. For $a \geqq 2$ define $v_a : \mathbb{C} \to \mathbb{R}$ by

$$v_a(z) = u_a(z+i) \quad \text{for} \quad \operatorname{Im} z \geqq 0 \quad \text{and} \quad v_a(z) = u_a(z-i) \quad \text{for} \quad \operatorname{Im} z \leqq 0.$$

Proposition 2.6. *For $a \geqq 2$ the functions v_a have the following properties:*

(1) *v_a is continuous and subharmonic on $\mathbb{C}$,*
(2) *$v_a(z) \geqq \omega_a(|z|)$ for all $z \in \mathbb{C}$,*
(3) *$v_a(x+iy) \leqq \omega_a(x) + v_a(iy)$ for all $x, y \in \mathbb{R}$,*
(4) *there exists $C > 0$ satisfying $v_a(iy) \leqq C(|y|+1)$ for all $a \geqq 3$, $y \in \mathbb{R}$,*
(5) *$\lim_{a \to \infty} v_a(x+iy) = \frac{2}{\pi}(|y|+1)$ for all $x \in \mathbb{R}$, $y \in \mathbb{R} \setminus \{0\}$,*
(6) *there exists $a_0 \geqq 2$ such that for all $a \geqq a_0$ the function $z \mapsto |\operatorname{Im} z| - v_a(z)$ is subharmonic on $\mathbb{C}$.*

Proof. (1) and (6): The continuity of v_a follows from the symmetry properties of u_a. To see that v_a is subharmonic, note first that v_a is harmonic on $\mathbb{C} \setminus \mathbb{R}$. Next observe that Green's formula implies that for each $\varphi \in \mathscr{D}(\mathbb{C})$ satisfying $\varphi \geqq 0$ we have

$$\int_{\mathbb{C}} v_a \Delta\varphi d\lambda(z) = 2 \int_{\mathbb{R}} \frac{\partial u_a}{\partial y}(x+i)\varphi(x)dx \geqq 0$$

because of Lemma 2.4 (6). By the same argument, 2.4 (7) implies (6).

(2)–(5): These statements are easy consequences of the definition of v_a and the corresponding assertions for u_a.

From [7], 2.9, we recall the following lemma.

Lemma 2.7. *For $n \in \mathbb{N}$ let $B_n := \{z \in \mathbb{C}^n : |z| \leqq 1\}$. There exists $H \in C(B_n) \cap \mathrm{PSH}(B_n)$ having the following properties:*

(1) $H(z) \leqq |\mathrm{Im}\, z|$ *for* $|z| \leqq 1$ (3) $H(x) \leqq 0$ *for* $x \in \mathbb{R}^n,\ |x| \leqq 1$

(2) $H(z) \leqq |\mathrm{Im}\, z| - \frac{1}{2}$ *for* $|z| = 1$ (4) $H(iy) \geqq 0$ *for* $y \in \mathbb{R}^n,\ |y| \leqq 1$.

3. Right inverses in half spaces. In [9], 3.5, we proved a sufficient condition for an operator $P(D)$ to admit a right inverse on $\mathscr{E}(H)$, H an open half space in $\mathbb{R}^n$. In this section we show that this condition is necessary, whenever the polynomial P is dominated by its principal part P_m. The proof is based on the characterization of the existence of right inverses in terms of Phragmén-Lindelöf conditions on the zero variety of $V(P)$ and on the results of Hörmander [3] on the characteristic Cauchy problem. To formulate the Phragmén-Lindelöf condition, we recall the following definitions.

Definition 3.1. (a) For $P \in \mathbb{C}[z_1, \ldots, z_n] \setminus \mathbb{C}$ and a convex compact set K in $\mathbb{R}^n$ we let

$$V(P) := \{z \in \mathbb{C}^n : P(z) = 0\} \quad \text{and} \quad h_K : x \mapsto \sup_{y \in K} \langle x, y \rangle.$$

(b) Let V be an algebraic variety in $\mathbb{C}^n$. A function $u : V \to [-\infty, \infty[$ is called plurisubharmonic on V if it is locally bounded above and plurisubharmonic at the regular points $V_{\mathrm{reg}} \subset V$. Further we assume that

$$u(z) = \limsup_{\xi \to z,\, \xi \in V_{\mathrm{reg}}} u(\xi) \quad \text{for} \quad z \in V_{\mathrm{sing}}.$$

By PSH(V) we denote all functions that are plurisubharmonic on V in this sense.

Theorem 3.2. *Let $P \in \mathbb{C}[z_1, \ldots, z_n] \setminus \mathbb{C}$ be homogeneous and let Ω be a convex open subset of $\mathbb{R}^n$. Then the following conditions are equivalent:*

(1) *$P(D) : \mathscr{E}(\Omega) \to \mathscr{E}(\Omega)$ admits a continuous linear right inverse,*

(2) *$P(D) : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ admits a continuous linear right inverse,*

(3) *$V(P)$ satisfies the following Phragmén-Lindelöf condition* $\mathrm{PL}(\Omega)$*: For each convex compact $K \subset \Omega$ there exists $K' \subset \Omega$ convex and compact such that each $u \in PSH(V(P))$ satisfying (α) and (β) also satisfies (γ):*

(α) $u(z) \leqq h_K(\mathrm{Im}\, z) + o(|z|),\ z \in V(P)$,

(β) $u(z) \leqq 0,\ z \in V(P) \cap \mathbb{R}^n$,

(γ) $u(z) \leqq h_{K'}(\mathrm{Im}\, z),\ z \in V(P)$.

Proof. (1) and (2) are equivalent by [6], 2.7. Since Ω is convex, it follows from Definition 2.8 in [6] and [6], 4.5, that (1) is equivalent to the Phragmén-Lindelöf condition stated in Definition 4.1 in [6] (which unfortunately contains misprints: read h_η in (b) and h_δ in (c) instead of h_ε). Since P is homogeneous, it follows from [8], 3.3, that this Phragmén-Lindelöf condition is equivalent to PL(Ω) in (3).

Next we want to derive from Theorem 3.2 that for each $N \in S^{n-1}$ and each homogeneous polynomial $P \in \mathbb{C}[z_1, \ldots, z_n] \setminus \mathbb{C}$ for which $P(D)$ admits a continuous linear right inverse on $\mathscr{E}(H_+(N))$, $P(D)$ is an evolution operator with respect to $\overline{H_\pm(N)}$ in the sense of Hörmander [4], Def. 12.8.16. To do so we use that by Hörmander [3], Thm. 4.2, for homogeneous polynomials this property is equivalent to the fact that for each $x \in \mathbb{R}^n$ the complex polynomial $t \mapsto P(x + tN)$ is either identically zero or has only real zeros. Further we apply the functions v_a defined in 2.5 to construct plurisubharmonic functions on $V(P)$ that are needed to use the condition PL(Ω) from Theorem 3.2.

Lemma 3.3. *Let $P \in \mathbb{C}[z_1, \ldots, z_n]$ be homogeneous and irreducible. If $V(P)$ satisfies* PL$(H_+(N))$ *for some $N \in \mathbb{R}^n \setminus \{0\}$ then for each $x \in \mathbb{R}^n$ the complex polynomial $t \mapsto P(x + tN)$ is either identically zero or has only real zeros.*

Proof. Without restriction we may assume $N = e_n$. Arguing by contradiction, we assume that there exist $x_0 \in \mathbb{R}^n$ and $t_0 \in \mathbb{C} \setminus \mathbb{R}$ such that $P(x_0 + t_0 e_n) = 0$ and $P(x_0 + te_n) \not\equiv 0$. Then there exists also a regular point $(\xi', \tau(\xi')) \in V(P) \cap (\mathbb{R}^{n-1} \times \mathbb{C})$ which satisfies $\xi' \neq 0$ and $\operatorname{Im} \tau(\xi') \neq 0$. Without restriction we may assume

$$\text{(1)} \qquad \operatorname{Im} \tau(\xi') < 0 \quad \text{and} \quad |\xi'| = 1.$$

Then we choose $0 < \delta < 1$ so small that the component of $V(P)$ which contains $(\xi', \tau(\xi'))$ is the graph of a holomorphic function $\tau : B_\delta(\xi') \to \mathbb{C}$, where $B_\delta(\xi') = \{w \in \mathbb{C}^{n-1} : |w - \xi'| \leqq \delta\}$, and such that

$$\text{(2)} \qquad \operatorname{Im} \tau(w) < \frac{1}{2} \operatorname{Im} \tau(\xi') < 0 \quad \text{for all} \quad w \in B_\delta(\xi').$$

Let

$$\text{(3)} \qquad C_1 := \max\left\{1 + \left|\frac{\tau(w)}{w}\right| : w \in B_\delta(\xi')\right\}.$$

Next let C denote the constant from Proposition 2.6 (4) and choose $\beta, \varepsilon > 0$ so small that

$$\text{(4)} \qquad \beta C C_1 (1 + \delta) < \frac{\delta}{2} \quad \text{and} \quad \varepsilon < \frac{2\beta}{\pi}.$$

Further, let a_0 denote the number from Proposition 2.6 (6) and define for $a \geqq a_0$ the functions $\varphi_a : \mathbb{C}^n \to \mathbb{R}$ by

$$\text{(5)} \qquad \varphi_a(z_1, \ldots, z_n) := \beta\left[v_a(z_n) + C_1\left(\sum_{j=1}^{n-1} |\operatorname{Im} z_j| - v_a(z_j)\right)\right] + \varepsilon \operatorname{Im} z_n - \beta C,$$

where v_a denotes the function defined in 2.5. By Proposition 2.6 (1) and (6), φ_a is plurisubharmonic on $\mathbb{C}^n$.

To estimate φ_a in $(w, \tau(w))$, where $w \in B_{\delta R}(R\xi')$ for $R \geqq 1$, we first note that the subadditivity of the function ω_a, defined in 1.1, 2.6 (2) and the definition of C_1 in (3) imply for $r \geqq 1$

$$\omega_a(|\tau(rw)|) = \omega_a(r|\tau(w)|) \leqq \omega_a(C_1 r|w|) \leqq C_1\omega_a(r|w|)$$
$$\leqq C_1 \sum_{j=1}^{n-1} \omega_a(r|w_j|) \leqq C_1 \sum_{j=1}^{n-1} v_a(rw_j).$$

Using 2.6 (3) and (4) we obtain from this for $w \in B_{\delta R}(R\xi') = RB_\delta(\xi')$

$$\begin{aligned}
&v_a(\tau(w)) + C_1 \sum_{j=1}^{n-1} \big(|\mathrm{Im}\ w_j| - v_a(w_j)\big)\\
&\quad\leqq \omega_a(|\tau(w)|) + v_a(i\ \mathrm{Im}\ \tau(w)) + C_1 \sum_{j=1}^{n-1} \big(|\mathrm{Im}\ w_j| - v_a(w_j)\big)\\
&\quad\leqq C_1 \sum_{j=1}^{n-1} v_a(w_j) + C(1 + |\mathrm{Im}\ \tau(w)|) + C_1 \sum_{j=1}^{n-1} \big(|\mathrm{Im}\ w_j| - v_a(w_j)\big)\\
&\quad= C|\mathrm{Im}\ \tau(w)| + C + C_1 \sum_{j=1}^{n-1} |\mathrm{Im}\ w_j|
\end{aligned}$$

and hence

$$\varphi_a(w, \tau(w)) \leqq \beta[C|\mathrm{Im}\ \tau(w)| + C_1 \sum_{j=1}^{n-1} |\mathrm{Im}\ w_j|] + \varepsilon\ \mathrm{Im}\ \tau(w). \tag{6}$$

Next consider the compact subset K of $H_+(N)$, defined as

$$K := \{x \in \mathbb{R}^n : x_n = \varepsilon,\ |x_j| \leqq \beta C_1 \text{ for } 1 \leqq j \leqq n-1\}.$$

Its support function h_K is given by

$$h_K(\mathrm{Im}\ z) = \varepsilon\ \mathrm{Im}\ z_n + (\beta C_1 + 1) \sum_{j=1}^{n-1} |\mathrm{Im}\ z_j|.$$

Then note that for $w \in B_{\delta R}(R\xi')$ we get from (3)

$$|\mathrm{Im}\ \tau(w)| = R\left|\mathrm{Im}\ \tau\Big(\frac{w}{R}\Big)\right| \leqq RC_1\left|\frac{w}{R}\right| < C_1(1+\delta)R. \tag{7}$$

Now let H denote the function from Lemma 2.7. Using (6), (7), 2.7 (2) and (4) we get for each $w \in \mathbb{C}^{n-1}$ satisfying $|w - R\xi'| = \delta R$:

$$\begin{aligned}
&\varphi_a(w, \tau(w)) + \delta R H\Big(\frac{w - R\xi'}{\delta R}\Big)\\
&\quad\leqq \beta C C_1(1+\delta)R + \beta C_1 \sum_{j=1}^{n-1} |\mathrm{Im}\ w_j| + \varepsilon \mathrm{Im}\ \tau(w) + |\mathrm{Im}\ w| - \delta R/2\\
&\quad= h_K(\mathrm{Im}\ (w, \tau(w)) + \Big(\beta C C_1(1+\delta) - \frac{\delta}{2}\Big)R < h_K(\mathrm{Im}\ (w, \tau(w)).
\end{aligned}$$

Consequently, we can define $U_a : V(P) \to \mathbb{R}$ by

$$\max\Big\{\varphi_a(w, \tau(w)) + \delta R H\Big(\frac{w - R\xi'}{\delta R}\Big),\ h_K(\mathrm{Im}\ (w, \tau(w)))\Big\}$$

for $\{(w, \tau(w)) : w \in B_{\delta R}(R\xi')\}$ and by $h_K \circ \mathrm{Im}$ otherwise. Then U_a satisfies (α) of $\mathrm{PL}(H_+(N))$ for the given set K, since it equals $h_K(\mathrm{Im}\, z)$ outside a compact subset of $V(P)$. From (6) and 2.7 (3) it follows that for $w \in B_{\delta R}(R\xi')$ satisfying $(w, \tau(w)) \in \mathbb{R}^n$, we have

$$\varphi_a(w, \tau(w)) + \delta R H\left(\frac{w - R\xi'}{\delta R}\right) \leqq 0.$$

Hence U_a satisfies the conditions (α) and (β) of $\mathrm{PL}(H_+(N))$. Since $V(P)$ satisfies $\mathrm{PL}(H_+(N))$ by hypothesis, there exists a convex compact set $K' \subset H_+(N)$ such that

(8) $$U_a(z) \leqq h_{K'}(\mathrm{Im}\, z) \quad \text{for all} \quad z \in V(P).$$

To apply (8) at $(R\xi', \tau(R\xi')) = R(\xi', \tau(\xi'))$ note that $\xi' \in \mathbb{R}^{n-1}$ and $\mathrm{Im}\,(\tau(R\xi')) = R\,\mathrm{Im}\,\tau(\xi') < 0$ by our choice of ξ'. This implies $h_{K'}\big(\mathrm{Im}\,(R\xi', \tau(R\xi'))\big) \leqq 0$ since $K \subset H_+(N)$. Using this, and 2.7 (4) we get

(9) $$\beta\left[v_a(\tau(R\xi')) - C_1 \sum_{j=1}^{n-1} v_a(R\xi_j')\right] + \varepsilon\, \mathrm{Im}\, \tau(R\xi') - \beta C$$
$$\leqq U_a(R\xi', \tau(R\xi')) \leqq h_{K'}(\mathrm{Im}\,(R\xi', \tau(R\xi'))) \leqq 0.$$

If we now let a tend to infinity in (9), then 2.6 (5) implies

(10) $$\beta\frac{2}{\pi}\left[(|\mathrm{Im}\,\tau(R\xi')| + 1) - C_1(n-1)\right] + \varepsilon R\, \mathrm{Im}\,\tau(\xi') - \beta C \leqq 0.$$

Dividing (10) by R and then passing to the limit $R \to \infty$ we get

$$0 \geqq \frac{2\beta}{\pi}|\mathrm{Im}\,\tau(\xi')| + \varepsilon \mathrm{Im}\,\tau(\xi') = \left(\frac{2\beta}{\pi} - \varepsilon\right)|\,\mathrm{Im}\,\tau(\xi')|.$$

Since $\dfrac{2\beta}{\pi} - \varepsilon$ is positive by (4) and since $\mathrm{Im}\,\tau(\xi') \neq 0$, this is a contradiction. Hence our assumption was false and the proof is complete.

To apply Lemma 3.3, we recall the following definitions from Hörmander [4], 10.4.1 and 10.4.4.

Definition 3.4. For $P \in \mathbb{C}[z_1, \ldots, z_n]$ let

$$\widetilde{P}(x,t) := \left(\sum_{\alpha \in \mathbb{N}_0^n} |P^{(\alpha)}(x)|^2\, t^{2|\alpha|}\right)^{1/2}, \quad x \in \mathbb{R}^n,\ t > 0.$$

P is said to dominate $Q \in \mathbb{C}[z_1, \ldots, z_n]$ if $\lim\limits_{t \to \infty} \sup\limits_{x \in \mathbb{R}^n} \dfrac{\widetilde{Q}(x,t)}{\widetilde{P}(x,t)} = 0$.

Theorem 3.5. *For $P \in \mathbb{C}[z_1, \ldots, z_n] \setminus \mathbb{C}$ let P_m denote the principal part of P and assume that P_m dominates $P - P_m$. Then the following assertions are equivalent for $N \in S^{n-1}$:*

(1) *$P(D)$ admits a right inverse on $\mathscr{E}(H_+(N))$,*
(2) *$P_m(D)$ admits a right inverse on $\mathscr{E}(H_+(N))$,*
(3) *$P(D)$ admits a right inverse on $\mathscr{E}(\mathbb{R}^n)$ and there exists a fundamental solution E for $P(D)$ satisfying* Supp $E \subset \overline{H_-(N)}$.

Proof. (1) $\Rightarrow$ (2): This follows from [6], 2.10 and 4.5, in connection with [8], 4.1.

(2) $\Rightarrow$ (3): By Theorem 3.2 the variety $V(P_m)$ satisfies $\mathrm{PL}(H_+(N))$. Hence Lemma 3.3 and Hörmander [3], Thm. 4.2 (resp. Hörmander [4], Def. 12.8.16), imply that $P_m(D)$ is an evolution operator with respect to $\overline{H_\pm(N)}$, in the sense of [4], Def. 12.8.16. By hypothesis, P_m dominates $P - P_m$. Hence it follows from [3], Thm. 4.1 (resp. [4], Thm. 12.8.17), that $P(D)$ is an evolution operator with respect to $\overline{H_\pm(N)}$. In particular $P(D)$ admits a fundamental solution $E \in \mathscr{D}'(\mathbb{R}^n)$ that satisfies $E \subset \overline{H_-(N)}$.

(3) $\Rightarrow$ (1): This holds by [9], Prop. 4.

The following example shows that Theorem 3.5 may hold or as well fail if the principal part of P does not dominate the lower order terms in P.

Example 3.6. For $\lambda \in \mathbb{C}$ satisfying $|\lambda| = 1$, define

$$P_\lambda(z_1, z_2, z_3) := z_1^2 - z_2^2 + \lambda z_3.$$

Then it follows from [9], 2.8, that $P_\lambda(D)$ admits a right inverse on $\mathscr{E}(H)$, $H := \{x \in \mathbb{R}^3 : x_3 > 0\}$, if and only if $\lambda = \pm 1$. Obviously, $z_1^2 - z_2^2$ does not dominate λz_3.

References

[1] U. Franken, Extension of zero-solutions of partial differential operators with dominating principal part defined on compact, convex sets with non-empty interior. J. Reine Angew. Math., to appear.

[2] U. Franken and R. Meise, Continuous linear right inverses for homogeneous linear partial differential operators on bounded convex open sets and extension of zero-solutions. "Functional Analysis", S. Dierolf, S. Dineen and P. Domanski, eds., Walter de Gruyter & Co., Berlin-New York 153–161 (1966).

[3] L. Hörmander, On the characteristic Cauchy problem. Ann. Math. **88,** 341–370 (1968).

[4] L. Hörmander, The Analysis of Linear Partial Differential Operators II. Berlin-Heidelberg-New York.

[5] R. Meise and B. A. Taylor, A decomposition lemma and its applications to spaces of ultradifferentiable functions. Math. Nachr. **142,** 45–72 (1989).

[6] R. Meise, B. A. Taylor and D. Vogt, Characterization of the linear partial differential operators with constant coefficients that admit a continuous linear right inverse. Ann. Inst. Fourier **40,** 619–655 (1990).

[7] R. Meise, B. A. Taylor and D. Vogt, Extremal plurisubharmonic functions of linear growth on algebraic varieties. Math. Z. **219,** 515–537 (1995).

[8] R. Meise, B. A. Taylor and D. Vogt, Phragmén-Lindelöf principles on algebraic varieties. J. Amer. Math. Soc., to appear.

[9] R. Meise, B. A. Taylor and D. Vogt, Continuous linear right inverses for partial differential operators of order 2 and fundamental solutions in half spaces. Manuscr. Math. **90,** 449–464 (1996).

[10] V. P. Palamodov, A criterion for splitness of differential complexes with constant coefficients, Geometrical and Algebraical Aspects in Several Complex Variables. C. A. Bernstein and D. C. Struppa, eds., EditEl 265–291 (1991).

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