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Intersection multiplicities over Gorenstein rings

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Abstract. Let R be a complete local ring of dimension d over a perfect field of prime characteristic p, and let M be an R-module of finite length and finite projective dimension. S. Dutta showed that the equality $\lim_{n\to\infty} \frac{\ell(F_R^n(M))}{p^{nd}} = \ell(M)$ holds when the ring R is a complete intersection or a Gorenstein ring of dimension at most 3. We construct a module over a Gorenstein ring R of dimension five for which this equality fails to hold. This then provides an example of a nonzero Todd class $\tau_3(R)$, and of a bounded free complex whose local Chern character does not vanish on this class.

1 Introduction

Let R be a complete local ring of dimension d over a perfect field of prime characteristic p, and let $G_\bullet$ be a bounded complex of free modules with finite length homology. In [Du] S. Dutta introduced the *limit multiplicity*

$$\chi_\infty(G_\bullet) = \lim_{n\to\infty} \frac{\chi(F^n(G_\bullet))}{p^{nd}},$$

where $F^n(-)$ denotes the nth iteration of the Frobenius functor. C. Szpiro and L. Peskine showed in [PS2] that the equality

$$\chi(F^n(G_\bullet)) = p^{nd}\chi(G_\bullet)$$

holds for a graded complex over a graded ring, and Szpiro conjectured that this would hold in general over a Cohen-Macaulay ring, see [Sz]. In the specific case that $G_\bullet$ is the resolution of a module M of finite length and finite projective dimension, the conjecture then asserts that

$$\ell(F^n(M)) = p^{nd}\ell(M).$$

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Dutta showed that this equality does hold if the ring R is a complete intersection or a Gorenstein ring of dimension at most 3, see [Du]. In [Ro3] P. Roberts constructed a counterexample to Szpiro's conjecture over a Cohen-Macaulay ring of dimension three using the famous example of negative Serre intersection multiplicity due to Dutta, Hochster and McLaughlin, [DHM]. The question however remained open for Gorenstein rings, and the main aim of this paper is to demonstrate that the conjecture is false in general over Gorenstein rings. Specifically, we construct a module M of finite length and finite projective dimension over a Gorenstein ring R of dimension five such that

$$\lim_{n\to\infty} \frac{\ell(F_R^n(M))}{p^{5n}} \neq \ell(M).$$

Using techniques similar to those used in [DHM], we first construct a module N of finite length and of finite projective dimension over the hypersurface

$$A = K[U, V, W, X, Y, Z]_m/(UX + VY + WZ),$$

where m is the ideal (U, V, W, X, Y, Z), such that $\chi(N, A/P) = -2$, where P is the prime ideal (u, v, w). (Lower case letters will denote the images of the corresponding variables.) We believe this is of independent interest since it is an example of two modules, one of whom has finite projective dimension, with a nonvanishing intersection multiplicity where the sum of the dimensions of the modules is $\dim A - 2$. It should be pointed out that if N_1 and N_2 are two modules, each of which has finite projective dimension over A, then the condition

$$\dim N_1 + \dim N_2 < \dim A$$

does imply $\chi(N_1, N_2) = 0$ by the main theorems of [Ro1] or [GS].

The limit multiplicity has an interpretation in terms of localized Chern characters and the local Riemann-Roch formula, see [Ro3] or [Ro4]. The results mentioned above then provide an example of a nonzero Todd class $\tau_3(R)$ over a Gorenstein ring R of dimension five, and of a bounded free complex whose local Chern character does not vanish on this class. In [Ku] K. Kurano does obtain a Gorenstein ring R of dimension five with a nonzero Todd class $\tau_3(R)$, but it is not known if there is a free complex whose local Chern character does not vanish on this Todd class.

2 Background

In this section we give a brief summary of the relevant terminology as well as record some well-known facts about χ and χ_∞ that we shall find useful later in our work.

For two modules M and N over a local ring (R, m) such that $M \otimes_R N$ has finite length and M has finite projective dimension, the Serre intersection multiplicity is defined as

$$\chi(M, N) = \sum_{i=0}^{\dim R} (-1)^i \ell(\mathrm{Tor}_i^R(M, N))$$

where $\ell(-)$ denotes the length. This definition does agree with the geometric notion of intersection multiplicity, see [Ser]. For a bounded complex $G_\bullet$

$$0 \to G_s \to \cdots \to G_1 \to G_0 \to 0$$

with finite length homology, we define

$$\chi(G_\bullet) = \sum_{i=0}^{s} (-1)^i \ell(H_i(G_\bullet)).$$

Let R be a ring of positive characteristic p and dimension d. Using the Frobenius endomorphism f of R (which takes r to r^p for $r \in R$), Peskine and Szpiro defined the Frobenius functor $F_R(-)$. This functor takes an R-module M to

$$F_R(M) = M \otimes_R {}^f R,$$

where ${}^f R$ is R viewed as a module over itself with a left action via the Frobenius endomorphism, and a right action via the usual multiplication. We use $F_R^n(-)$ (or simply $F^n(-)$) to denote the nth iteration of the Frobenius functor. For a bounded free complex $G_\bullet$ with finite length homology, Dutta defined

$$\chi_\infty(G_\bullet) = \lim_{n\to\infty} \frac{\chi(F^n(G_\bullet))}{p^{nd}}.$$

If $G_\bullet$ is a finite free resolution of an R-module M of finite length and finite projective dimension, then

$$\chi(G_\bullet) = \ell(M) \qquad \text{and} \qquad \chi_\infty(G_\bullet) = \lim_{n\to\infty} \frac{\ell(F_R^n(M))}{p^{nd}}.$$

For the second assertion note that

$$H_i(F_R^n(G_\bullet)) = \mathrm{Tor}_i^R(M, {}^{f^n} R) = 0$$

for all $i \geq 1$ and $n \geq 0$, by [PS1, Theorem 1.7].

Lemma 2.1. *Let R be an integral domain of characteristic $p > 0$ and let S be a module-finite extension ring which has rank r as an R-module. Then for any bounded complex $G_\bullet$ of free R-modules which has finite length homology,*

$$\chi_\infty(G_\bullet \otimes_R S) = r \cdot \chi_\infty(G_\bullet).$$

Proof. Since S is an R-module of rank r, there exists a short exact sequence

$$0 \longrightarrow R^r \longrightarrow S \longrightarrow T \longrightarrow 0$$

where T is a torsion R-module and thus has dimension less than d. Since $\chi(F_R^n(G_\bullet) \otimes_R -)$ is additive on short exact sequences, we get

$$\chi(F_R^n(G_\bullet) \otimes_R S) = r \cdot \chi(F_R^n(G_\bullet)) + \chi(F_R^n(G_\bullet) \otimes_R T). \qquad (*)$$

Since the module T has dimension less than the dimension of R,

$$\lim_{n \to \infty} \frac{\chi(F_R^n(G_\bullet) \otimes_R T)}{p^{nd}} = 0$$

by [Sei, Proposition 1]. We obtain the desired equality by dividing the equation $(*)$ by p^{nd} and forming the appropriate limits since

$$F_R^n(G_\bullet) \otimes_R S = F_S^n(G_\bullet \otimes_R S).$$

□

We will also use the following lemma, which is similar to Proposition 2.4 of [DHM].

Lemma 2.2. *Let (R, m, K) be a local ring, and let M be a finitely generated R-module. Suppose that P is an ideal of R such that R/P is a regular ring. Then M has finite projective dimension if and only if* $\operatorname{Tor}_i^R(M, R/P) = 0$ *for* $i \gg 0$.

Proof. Since R/P is regular, the residue field K has a finite resolution by free R/P-modules. Then $\operatorname{Tor}_i^R(M, R/P) = 0$ for $i \gg 0$ if and only if $\operatorname{Tor}_i^R(M, K) = 0$ for $i \gg 0$. However $\operatorname{Tor}_i^R(M, K) = 0$ for $i \gg 0$ if and only if M has finite projective dimension. □

3 Overview of the construction

We summarize the work that will be carried out in Sections 4 and 5 and explain how this provides the example we are aiming for.

In Section 4 we construct a module N of length 55 and finite projective dimension over the local hypersurface

$$A = K[U, V, W, X, Y, Z]_m/(UX + VY + WZ),$$

where m is the maximal ideal (U, V, W, X, Y, Z), such that N has a nonzero intersection multiplicity with A/P where P is the prime ideal (u, v, w). Specifically, we have

$$\chi(N, A/P) = \sum_{i=0}^{5} (-1)^i \ell(\operatorname{Tor}_i^A(N, A/P)) = -2.$$

In Section 5 we construct a Gorenstein normal domain R which is a module finite extension of A and for which there is an exact sequence of A-modules

$$0 \longrightarrow A^3 \longrightarrow R \longrightarrow P \longrightarrow 0. \qquad (**)$$

Note that the ring R has rank 4 as an A-module. Consider the R-module $M = N \otimes_A R$. We claim that for this module

$$\lim_{n\to\infty} \frac{\ell(F_R^n(M))}{p^{5n}} \neq \ell(M).$$

To see this, let $F_\bullet$ be a finite free resolution of N over A. Since $A \hookrightarrow R$ is a module-finite extension, the complex $G_\bullet = F_\bullet \otimes_A R$ has finite length homology. Furthermore since R is Cohen-Macaulay, the complex $G_\bullet$ is acyclic by the Acyclicity Lemma of Peskine and Szpiro, [PS1, Lemma 1.8]. Hence $G_\bullet$ provides a finite free resolution of M as an R-module. To compute the length of M we use the additivity of $\chi(F_\bullet \otimes_A -)$ on the exact sequence $(**)$. This gives

$$\ell(M) = \chi(G_\bullet) = \chi(F_\bullet \otimes_A R) = 3\chi(F_\bullet) + \chi(F_\bullet \otimes_A P).$$

The additivity also gives

$$\chi(F_\bullet \otimes_A P) = \chi(F_\bullet) - \chi(F_\bullet \otimes_A A/P),$$

and so

$$\begin{aligned} \ell(M) &= 4\chi(F_\bullet) - \chi(F_\bullet \otimes_A A/P) = 4\ell(N) - \chi(N, A/P) \\ &= 4 \cdot 55 - (-2) = 222. \end{aligned}$$

On the other hand, since R has rank 4 as an A-module, Lemma 2.1 gives

$$\lim_{n\to\infty} \frac{\ell(F_R^n(M))}{p^{5n}} = \chi_\infty(G_\bullet) = \chi_\infty(F_\bullet \otimes_A R) = 4\,\chi_\infty(F_\bullet).$$

Since A is a hypersurface and $F_\bullet$ is a finite resolution of N, we have $\chi_\infty(F_\bullet) = \ell(N)$ by [Du, Theorem 1.9]. Therefore,

$$\lim_{n\to\infty} \frac{\ell(F_R^n(M))}{p^{5n}} = 4 \cdot 55 = 220.$$

4 A module of finite projective dimension

Consider the local hypersurface

$$A = K[U, V, W, X, Y, Z]_m/(UX + VY + WZ)$$

where U, V, W, X, Y and Z are indeterminates over a field K of arbitrary characteristic, and m is the maximal ideal (U, V, W, X, Y, Z). We construct a module N of finite length and finite projective dimension over A, which has a nonzero intersection multiplicity with the module A/P, where P denotes the prime ideal $(u, v, w)A$. Specifically, we construct N such that

$$\chi(N, A/P) = \sum_{i=0}^{5}(-1)^i \ell(\mathrm{Tor}_i^A(N, A/P)) = -2.$$

The following complex is an minimal free resolution of A/P:

$$\cdots \xrightarrow{\phi_3} A^4 \xrightarrow{\phi_4} A^4 \xrightarrow{\phi_3} A^4 \xrightarrow{\phi_2} A^3 \xrightarrow{\phi_1} A \longrightarrow 0.$$

The maps in this complex are given by the matrices:

$$\phi_1 = \begin{pmatrix} u & v & w \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} x & 0 & -w & v \\ y & w & 0 & -u \\ z & -v & u & 0 \end{pmatrix},$$

$$\phi_3 = \begin{pmatrix} 0 & u & v & w \\ u & 0 & z & -y \\ v & -z & 0 & x \\ w & y & -x & 0 \end{pmatrix}, \text{ and } \phi_4 = \begin{pmatrix} 0 & x & y & z \\ x & 0 & -w & v \\ y & w & 0 & -u \\ z & -v & u & 0 \end{pmatrix}.$$

The modules $\mathrm{Tor}_i^A(N, A/P)$ may be computed by tensoring the above complex with the module N. If N has length 55, the resulting complex may be viewed as

$$\xrightarrow{\alpha} K^{220} \xrightarrow{\beta} K^{220} \xrightarrow{\alpha} K^{220} \xrightarrow{\theta_2} K^{165} \xrightarrow{\theta_1} K^{55} \longrightarrow 0$$

where $\theta_1, \theta_2, \alpha$ and β are matrices over K. The i th homology of this complex is $\mathrm{Tor}_i^A(N, A/P)$, and will vanish for $i \geq 3$ provided the sum of the ranks of the matrices α and β is 220. In this case the module N has finite projective dimension by Lemma 2.2, and an easy calculation shows that

$$\begin{aligned} &\chi(N, A/P) \\ &\quad = \ell(\mathrm{Tor}_0^A(N, A/P)) - \ell(\mathrm{Tor}_1^A(N, A/P)) + \ell(\mathrm{Tor}_2^A(N, A/P)) \\ &\quad = 55 - 165 + 220 - \mathrm{rank}(\alpha) = 110 - \mathrm{rank}(\alpha). \end{aligned}$$

In our construction, the matrix α will have rank 112 and β will have rank 108.

As in [DHM], we regard a module of finite length over A as a finite dimensional vector space over K. The action of the generators of the ring can then be treated as commuting nilpotent endomorphism of this vector space. We shall denote the endomorphisms given by the action of u, v, w, x, y, z by the matrices $\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6$, respectively. Note that the matrices must satisfy the relation

$$\psi_i\psi_j = \psi_j\psi_i, \text{ for all } i \text{ and } j,$$

corresponding to commutativity, and the relation

$$\psi_1\psi_4 + \psi_2\psi_5 + \psi_3\psi_6 = 0,$$

corresponding to the defining equation of the hypersurface.

The module of finite length and finite projection that we construct is annihilated by $m^3 + (x, y, z)m$. Consequently ψ_i may be written in block form as

$$\psi_i = \begin{pmatrix} 0 & 0 & a_i & c_i \\ 0 & 0 & 0 & d_i \\ 0 & 0 & 0 & b_i \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ for } i = 1, \ldots, 6.$$

Furthermore we set

$$a_4 = a_5 = a_6 = 0 \text{ and } \quad b_4 = b_5 = b_6 = 0.$$

Since

$$\psi_i\psi_j = \begin{pmatrix} 0 & 0 & 0 & a_ib_j \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

the relation $\psi_1\psi_4 + \psi_2\psi_5 + \psi_3\psi_6 = 0$ is easily seen to be satisfied.

The matrices a_1, a_2, a_3 are

$$a_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix},$$

where 1 denotes the 4×4 identity matrix, and 0 denotes the 4×4 zero matrix. For b_i we take

$$b_1 = \begin{pmatrix} 1\,0\,0\,0\,0\,0 \\ 0\,1\,0\,0\,0\,0 \\ 0\,0\,1\,0\,0\,0 \end{pmatrix}, \quad b_2 = \begin{pmatrix} 0\,1\,0\,0\,0\,0 \\ 0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,1\,0 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 0\,0\,1\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0 \\ 0\,0\,0\,0\,0\,1 \end{pmatrix}$$

where 1 denotes the 4×4 identity matrix, and 0 denotes the 4×4 zero matrix. Note that for all i and j we have $a_ib_j = a_jb_i$, and so the commutativity relations $\psi_i\psi_j = \psi_j\psi_i$ do hold.

After interchanging certain rows and columns, the matrix α is

$$\begin{pmatrix}
0 & a_1 & a_2 & a_3 & 0 & c_1 & c_2 & c_3 \\
a_1 & 0 & 0 & 0 & c_1 & 0 & c_6 & -c_5 \\
a_2 & 0 & 0 & 0 & c_2 & -c_6 & 0 & c_4 \\
a_3 & 0 & 0 & 0 & c_3 & c_5 & -c_4 & 0 \\
0 & 0 & 0 & 0 & 0 & d_1 & d_2 & d_3 \\
0 & 0 & 0 & 0 & d_1 & 0 & d_6 & -d_5 \\
0 & 0 & 0 & 0 & d_2 & -d_6 & 0 & d_4 \\
0 & 0 & 0 & 0 & d_3 & d_5 & -d_4 & 0 \\
0 & 0 & 0 & 0 & 0 & b_1 & b_2 & b_3 \\
0 & 0 & 0 & 0 & b_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b_3 & 0 & 0 & 0
\end{pmatrix}.$$

The rank of the matrix α is easily seen to be the sum of 40 and the rank of the submatrix

$$\alpha_1 = \begin{pmatrix}
d_1 & d_2 & d_3 \\
0 & d_6 & -d_5 \\
-d_6 & 0 & d_4 \\
d_5 & -d_4 & 0 \\
b_1 & b_2 & b_3
\end{pmatrix}.$$

Similarly, after deleting rows and columns of zeros and interchanging certain rows and columns, the matrix β reduces to

$$\begin{pmatrix}
0 & 0 & 0 & 0 & c_4 & c_5 & c_6 \\
0 & -a_3 & a_2 & c_4 & 0 & -c_3 & c_2 \\
a_3 & 0 & -a_1 & c_5 & c_3 & 0 & -c_1 \\
-a_2 & a_1 & 0 & c_6 & -c_2 & c_1 & 0 \\
0 & 0 & 0 & 0 & d_4 & d_5 & d_6 \\
0 & 0 & 0 & d_4 & 0 & -d_3 & d_2 \\
0 & 0 & 0 & d_5 & d_3 & 0 & -d_1 \\
0 & 0 & 0 & d_6 & -d_2 & d_1 & 0 \\
0 & 0 & 0 & 0 & 0 & -b_3 & b_2 \\
0 & 0 & 0 & 0 & b_3 & 0 & -b_1 \\
0 & 0 & 0 & 0 & -b_2 & b_1 & 0
\end{pmatrix}.$$

This rank of this matrix is the sum of 12 and the rank of the submatrix

$$\beta_1 = \begin{pmatrix} 0 & c_4 & c_5 & c_6 \\ 0 & d_4 & d_5 & d_6 \\ d_4 & 0 & -d_3 & d_2 \\ d_5 & d_3 & 0 & -d_1 \\ d_6 & -d_2 & d_1 & 0 \\ 0 & 0 & -b_3 & b_2 \\ 0 & b_3 & 0 & -b_1 \\ 0 & -b_2 & b_1 & 0 \end{pmatrix}.$$

We next let $c_1 = c_2 = c_3 = c_4 = c_5 = 0$ and

$$c_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where 1 denotes the 4×4 identity matrix, and 0 denotes the 4×4 zero matrix. It remains to exhibit matrices d_i for $1 \leq i \leq 6$ such that the matrices α and β have ranks 112 and 108 respectively. We let $d_1 = d_2 = 0$ and

$$d_3 = \begin{pmatrix}
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0
\end{pmatrix},$$

$$d_4 = \begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0
\end{pmatrix},$$

$$d_5 = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix},$$

and

$$d_6 = \begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0
\end{pmatrix}.$$

Of course, any choice of matrices d_3, d_4, d_5 and d_6 in general position will serve our purpose. That the matrices exhibited above do have the required rank properties is an elementary, though tedious, verification.

5 The construction of the Gorenstein ring

We now construct a Gorenstein normal domain R which is an extension of a free A-module by the prime ideal P. The construction is carried out in the case that A is a hypersurface over a field of characteristic 2.

Consider the ring $R = A[a, b, c, d, e]$ where

$$a = \sqrt{uyz},\ b = \sqrt{vxz},\ c = \sqrt{wxy},\ d = \sqrt{uvw},\ e = \sqrt{vwyz}.$$

The ring R is a normal domain; in fact, it is the integral closure of A in the field $L(\sqrt{uyz},\ \sqrt{vxz})$ where L is the fraction field of A. This furthermore shows that the ring R has rank 4 as an A-module.

To show that R is a Cohen-Macaulay ring, we work with the system of parameters $u, x, v, y, w - z$ and show that the multiplicity of the ideal $I = (u, x, v, y, w - z)$ is 8 and that this equals the length of the K vector space R/I.

Consider the extension $K[u, x, v, y, w-z] \subseteq R$. The degree of this extension may be computed by examining the corresponding extension of fraction fields, and so is easily seen to be 8.

The following relations show that R is generated as an A-module by the elements $1, a, b, c, d, e$.

$$\begin{array}{lll} a^2 = uyz, & ab = ze + vyz, & ac = ye + wyz, \\ ad = ue, & ae = dyz, & b^2 = vxz, \\ bc = xe, & bd = ve + vwz, & be = vzc, \\ c^2 = wxy, & cd = we + vwy, & ce = wyb, \\ d^2 = uvw, & de = vwa, & e^2 = vwyz. \end{array}$$

The relations amongst the elements of R also include

$$\begin{array}{lll} ux + vy + wz, & xa + yb + zc, & wb + vc + xd, \\ wa + uc + yd, & va + ub + zd. & \end{array}$$

Consequently the images of the following elements form a generating set for R/I as a K vector space:

$$1,\ z,\ a,\ b,\ c,\ d,\ e,\ ze.$$

Hence the length of R/I is less than or equal to 8, but since the multiplicity of the ideal I was earlier computed to be 8, the length must be precisely 8. This shows that R is a Cohen-Macaulay ring, and furthermore that it is Gorenstein, since the socle in R/I is of dimension one, being generated by the image of ze. It is not difficult to verify that the relations listed above are precisely the relations amongst the generators of R.

Lastly, it may be verified that there is an exact sequence of A-modules

$$0 \longrightarrow A^3 \longrightarrow R \longrightarrow P \longrightarrow 0$$

where the map $R \to P$ is determined by

$$1 \to 0,\ a \to u,\ b \to v,\ c \to w,\ d \to 0,\ e \to 0.$$

6 Local Chern characters

Let X be a closed subset of $\operatorname{Spec} R$ where the ring R is a d-dimensional homomorphic image of a regular local ring. For each integer i, let $Z_i(X)$ denote the free $\mathbb{Q}$-module generated by cycles of the form $[R/P]$ for P in X such that $\dim R/P = i$. For a prime Q with $\dim R/Q = i+1$ and an element x of R such that $x \notin Q$, define

$$\operatorname{div}(x, Q) = \sum_{\dim R/P = i} \ell((R/(Q + xR))_P)[R/P].$$

Let $B_i(X)$ denote the subgroup of $Z_i(X)$ generated by elements of the form $\operatorname{div}(x, Q)$ for Q in X. The i th graded piece of the Chow group of X is $A_i(X) = Z_i(X)/B_i(X)$, and the Chow group of X is

$$A_*(X) = \bigoplus_{i=1}^{d} A_i(X).$$

Let $G_\bullet$ be a bounded free complex of R-modules, and let $Z \subset \operatorname{Spec} R$ be the support of this complex. The local Chern character $\operatorname{ch}(G_\bullet)$ is the sum

$$\operatorname{ch}(G_\bullet) = \operatorname{ch}_d(G_\bullet) + \operatorname{ch}_{d-1}(G_\bullet) + \cdots + \operatorname{ch}_0(G_\bullet)$$

where, for each k,

$$\operatorname{ch}_k(G_\bullet) : A_i(X) \to A_{i-k}(X \cap Z)$$

is a $\mathbb{Q}$-module homomorphism. (For precise definitions and properties we refer the reader to [Fu].) We consider the special case of the local Riemann-Roch formula where the homology modules of the complex $G_\bullet$ are of finite length: for any finitely generated R-module N, there is an element

$$\tau(N) = \tau_d(N) + \cdots + \tau_0(N) \in A_*(\operatorname{Supp}(N)),$$

called the *Todd class* of N, such that

$$\chi(G_\bullet \otimes N) = \operatorname{ch}(G_\bullet)(\tau(N)).$$

Note that since the support of $G_\bullet$ is the closed point of $\operatorname{Spec} R$, $\operatorname{ch}(G_\bullet)(\tau(N))$ is an element of $A_0(\operatorname{Spec}(R/m)) \cong \mathbb{Q}$. If $G_\bullet$ is the resolution of a module M of finite length and finite projective dimension, the local Riemann-Roch formula then gives

$$\ell(M) = \operatorname{ch}(G_\bullet)(\tau(R)) = \sum_{i=0}^{d} \operatorname{ch}_i(G_\bullet)(\tau_i(R)).$$

Now suppose in addition that R is a complete local ring over a perfect field of prime characteristic p. Since the local Chern characters are compatible with finite maps, one can show that

$$\lim_{n\to\infty} \frac{\ell(F_R^n(M))}{p^{nd}} = \operatorname{ch}_d(G_\bullet)(\tau_d(R)),$$

see, [Ro4, Proposition 12.7.1]. If the ring R is a complete intersection then $\tau_i(R) = 0$ for all $i < d$, and so

$$\ell(M) = \operatorname{ch}(G_\bullet)(\tau(R)) = \operatorname{ch}_d(G_\bullet)(\tau_d(R)) = \lim_{n\to\infty} \frac{\ell(F_R^n(M))}{p^{nd}}.$$

For a Gorenstein ring R the duality property shows that $\tau_{d-i}(R) = 0$ for all odd numbers i. In the specific case that R is a Gorenstein ring of dimension 3 and $G_\bullet$ is the resolution of a module M of finite length and finite projective dimension, this then gives

$$\ell(M) = \operatorname{ch}(G_\bullet)(\tau(R)) = \operatorname{ch}_3(G_\bullet)(\tau_3(R)) + \operatorname{ch}_1(G_\bullet)(\tau_1(R)).$$

The operator $\operatorname{ch}_1(G_\bullet)$ can be identified with the MacRae invariant, and is known to vanish whenever the module M has codimension greater than one, see [Ro2, Theorem 3], and also [Fo,Ma]. In [Ku] Kurano gave examples of Gorenstein rings R for which $\tau_i(R) \neq 0$ for some $i < d$. However it is not known in the case of Kurano's examples if there exists a bounded free complex $G_\bullet$ with finite length homology for which $\mathrm{ch}_i(G_\bullet)(\tau_i(R))$ does not vanish for some $i < d$. We would also like to point out that, using the example in [DHM] of a module with negative intersection multiplicity, Roberts did construct an example of a Cohen-Macaulay ring R of dimension 3 such that $\mathrm{ch}_2(G_\bullet)(\tau_2(R)) \neq 0$ for a bounded free complex $G_\bullet$ with finite length homology, which is, in fact, the resolution of a module of finite length and finite projective dimension.

In our example, where R is a Gorenstein ring of dimension 5 and $G_\bullet$ is the finite free resolution of the module M, we have

$$\ell(M) = \mathrm{ch}_5(G_\bullet)(\tau_5(R)) + \mathrm{ch}_3(G_\bullet)(\tau_3(R)) = 222.$$

(Recall that $\mathrm{ch}_1(G_\bullet) = 0$ by [Ro2, Theorem 3].) On the other hand,

$$\mathrm{ch}_5(G_\bullet)(\tau_5(R)) = \lim_{n\to\infty} \frac{\ell(F_R^n(M))}{p^{5n}} = 220,$$

and so, we must have

$$\mathrm{ch}_3(G_\bullet)(\tau_3(R)) = 2.$$

Thus our example provides an example of a Todd class $\tau_3(R)$ over a five dimensional Gorenstein ring R, and of a bounded free complex with a local Chern character that does not vanish on this class. Furthermore, we may also conclude that in our example, where $p = 2$, we have the formula

$$\ell(F_R^n(M)) = 220 \cdot 2^{5n} + 2 \cdot 2^{3n} \quad \text{for all } n \geq 0$$

by the following proposition, which follows from the compatibility of local Chern characters with finite maps: see the proof of [Ro4, Proposition 12.7.1].

Proposition 6.1 (Roberts). *Let R be a Noetherian local ring of positive characteristic p and dimension d. Suppose that the residue field of R is perfect and that the Frobenius endomorphism is a finite map. Then for any bounded free complex $G_\bullet$ with finite length homology, we have*

$$\mathrm{ch}(F_R^n(G_\bullet))(\tau(R)) = \sum_{i=0}^{d} p^{ni}\,\mathrm{ch}_i(G_\bullet)(\tau_i(R))$$

for all $n \geq 0$.

7 Further consequences of the example

Let (R, m) be a complete local ring of dimension d, and let M be an R-module of finite length. The length $\ell(F^n(M))$ is equal to $p^{nd}\ell(M)$ if the ring R is regular, and this may be viewed as a special case of the following result, [Du, Theorem 1.9]:

Theorem 7.1 (Dutta). *Let (R, m) be a complete intersection ring of dimension d, and let M be an R-module of finite length. Then*

$$\ell(F_R^n(M)) \geq \ell(M)p^{nd},$$

and equality holds if the module M has finite projective dimension.

C. Miller has obtained a converse to this theorem in [Mi], which then gives a characterisation of modules of finite length and finite projective dimension:

Theorem 7.2 (Miller). *Let (R, m) be a complete intersection ring of dimension d, and let M be an R-module of finite length. Then the following statements are equivalent:*
1) $\ell(F_R^n(M)) = \ell(M)p^{nd}$ *for all* $n \geq 0$.
2) *The module M has finite projective dimension.*
3) $\lim_{n\to\infty} \frac{\ell(F_R^n(M))}{p^{nd}} = \ell(M)$.

A natural question raised by these theorems is whether there is a relationship between $\ell(F_R^n(M))$ and $\ell(M)p^{nd}$ if the ring R is Gorenstein, but is not a complete intersection. The examples we have constructed already show that over a Gorenstein ring R of dimension 5, the equality $\ell(F_R^n(M)) = \ell(M)p^{5n}$ need not hold for a module M of finite length and finite projective dimension. We next show that the inequality $\ell(F_R^n(M)) \geq \ell(M)p^{nd}$ which holds for *all* modules M of finite length over a complete intersection ring, fails to hold over Gorenstein rings even when the finite length module M has finite projective dimension.

In Section 4 we constructed a module N over the hypersurface

$$A = K[U, V, W, X, Y, Z]_m/(UX + VY + WZ)$$

which has finite length and finite projective dimension and has the property that

$$\chi(N, A/P) = -2,$$

where P is the prime ideal (u, v, w). Let Q denote the prime ideal (x, v, w), and consider the short exact sequence

$$0 \longrightarrow A/Q \longrightarrow A/(v, w) \longrightarrow A/P \longrightarrow 0,$$

where the first map is multiplication by the element u. Since the elements v and w form a regular sequence in the ring A, the Koszul resolution gives that

$$\chi(N, A/(v, w)) = \sum_{i=0}^{2}(-1)^i H_i(v, w; N),$$

where $H_i(v, w; N)$ denotes the i th Koszul homology module. Since $\dim(N) < 2$, we then have $\sum_{i=0}^{2}(-1)^i H_i(v, w; N) = 0$ by [Ser, Theorem 1, Chapter IV]. Consequently,

$$\chi(N, A/Q) = -\chi(N, A/P) = 2.$$

Consider the automorphism σ of A which switches x and u, and fixes v, w, y and z. Let N' denote the module N now viewed as an A-module by restriction of scalars via σ. We then have

$$\chi(N', A/P) = \chi(N, A/Q) = 2.$$

The same argument as in Section 3 then shows that the R-module $M' = N' \otimes_A R$ has length

$$\ell(M') = 4\ell(N') - \chi(N', A/P) = 218,$$

whereas

$$\lim_{n\to\infty} \frac{\ell(F_R^n(M'))}{p^{5n}} = 220.$$

Remark 7.3. For a Cohen-Macaulay local ring A, let Æ(A) denote the Grothendieck group of A-modules which have finite length and finite projective dimension. If $x_1, \ldots, x_d$ is a system of parameters for A, then $[A/(x_1, \ldots, x_d)]$ is an element of Æ(A). The intersection theory of the modules of the form $A/(x_1, \ldots, x_d)$ is well understood: if M is a finitely generated A-module with $\dim M < d$, then

$$\chi(A/(x_1, \ldots, x_d), M) = 0$$

by [Ser, Theorem 1, Chapter IV]. Consequently it is of interest to understand the group $G(A)$ which is the quotient of Æ(A) by the subgroup generated by all modules of the form $A/(x_1, \ldots, x_d)$ where $x_1, \ldots, x_d$ is a system of parameters for A. Using K-theoretic methods, M. Levine has computed this group for certain hypersurfaces, see [Le, §4]. We would like to point out that in the case where A is the hypersurface

$$A = K[U, V, W, X, Y, Z]_m/(UX + VY + WZ),$$

and N is the module of finite length and finite projective dimension constructed in Section 4, the fact that $\chi(N, A/(u, v, w)) \neq 0$ shows that $[N]$ is a nonzero element of the group $G(A)$.

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