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Mirror duality and string-theoretic Hodge numbers

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Abstract. We prove in full generality the mirror duality conjecture for string-theoretic Hodge numbers of Calabi–Yau complete intersections in Gorenstein toric Fano varieties. The proof is based on properties of intersection cohomology.

1 Introduction

The first author has conjectured that the polar duality of reflexive polyhedra induces the mirror involution for Calabi–Yau hypersurfaces in Gorenstein toric Fano varieties [2]. The second author has proposed a more general duality which conjecturally induces the mirror involution for Calabi–Yau *complete intersections* in Gorenstein toric Fano varieties [7]. The most general form of the combinatorial duality which includes mirror constructions of physicists for rigid Calabi–Yau manifolds was formulated by both authors in [4].

The main purpose of our paper is to show that all proposed combinatorial dualities agree with the following Hodge-theoretic property of mirror symmetry predicted by physicists:

If two smooth n-dimensional Calabi–Yau manifolds V and W form a mirror pair, then their Hodge numbers satisfy the relation

$$h^{p,q}(V) = h^{n-p,q}(W), \quad 0 \leqq p,\ q \leqq n\,. \tag{1}$$

A verification of this property becomes rather non-trivial if we do not make restrictions on the dimension n. The main difficulty is the necessity to work with singular Calabi–Yau varieties whose singularities in general do not admit any crepant desingularization. This difficulty was the motivation for

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introduction of so called *string-theoretic Hodge numbers* $h_{\rm st}^{p,q}(V)$ for singular V [6]. The string-theoretic Hodge numbers $h_{\rm st}^{p,q}(V)$ coincide with the usual Hodge numbers $h^{p,q}(V)$ if V is smooth, and with the usual Hodge numbers of a crepant desingularization $\hat{V}$ of V if such a desingularization exists. Therefore the property (1) must be reformulated as follows:

Let (V, W) be a mirror pair of singular n-dimensional Calabi–Yau varieties. Then the string-theoretic Hodge numbers satisfy the duality:

$$h_{\rm st}^{p,q}(V) = h_{\rm st}^{n-p,q}(W), \quad 0 \leqq p, \ q \leqq n \, . \tag{2}$$

The string-theoretic Hodge numbers for Gorenstein algebraic varities with toroidal or quotient singularities were introduced and studied in [6]. It was also conjectured in [6] that the conbinatorial construction of mirror pairs of Calabi–Yau complete intersections in Gorenstein toric Fano varieties satisfies the duality (2). This conjecture has been proved in [6] for mirror pairs of Calabi–Yau hypersurfaces of arbitrary dimension that can be obtained by the Greene–Plesser construction [19]. Some other results supporting this conjecture have been obtained in [2, 5, 27]. Additional evidence in favor of the conjecture has been received by explicit computations of instanton sums using generalized hypergeometric functions [3, 20, 22, 24].

The paper is organized as follows:

In Sect. 2, we introduce a polynomial invariant $B(P; u, v)$ of an Eulerian partially ordered set P using results of Stanley [31]. For our purposes, their most important property is the relation between $B(P; u, v)$ and $B(P^*; u, v)$, where P^* is the dual to P Eulerian poset (Theorem 2.13). It seems that the polynomials $B(P; u, v)$ have independent interest in combinatorics[1].

In Sect. 3, we give an explicit formula for the polynomial $E(Z; u, v)$ which describes the mixed Hodge structure of an affine hypersurface Z in an algebraic torus $\mathbf{T}$ (Theorem 3.24). We remark the following: the explicit formula for $E(Z; 1, 1)$ is due to Bernstein, Khovanskiî and Kushnirensko [21, 23]; the computation of the polynomial $E(Z; t, 1)$ which describes the Hodge filtration on $H_c^*(Z)$ is due to Danilov and Khovanskiî [10] (see also [1]); the polynomial $E(Z; t, t)$ which describes the weight filtration on $H_c^*(Z)$ has been computed by Denef and Loeser [13].

In Sect. 4, we derive an explicit formula for the polynomial $E_{\rm st}(V; u, v)$ where V is a Calabi–Yau complete intersection in a Gorenstein toric Fano variety (Theorem 4.14). The coefficients of $E_{\rm st}(V; u, v)$ are equal up to a sign to string-theoretic Hodge numbers of V. Since our formula is written in terms of B-polynomials as a sum over pairs of lattice points contained in the corresponding pair of dual to each other reflexive Gorenstein cones C and $\check{C}$, the mirror duality for string-theoretic Hodge numbers becomes immediate consequence of the duality for B-polynomials after the transposition $C \leftrightarrow \check{C}$ (Theorem 4.15). Following some recent development of ideas of Witten [33] by Morisson and Plesser [25], we conjecture that the formula obtained in this

[1] We are grateful to R. Stanley who point out us that another proof of Theorem 2.13 could be obtained from the results which were used in his proof of a conjecture of G. Kalai [32] Sect. 8.

paper gives the spectrum of the abelian gauge theory in two dimensions which could be constructed from any pair $(C, \check{C})$ of two dual to each other reflexive Gorenstein cones.

2 Combinatorial polynomials of Eulerian posets

Let P be a finite poset (i.e., finite partially ordered set). Recall that the Möbius function $\mu_P(x, y)$ of a poset P is a unique integer valued function on $P \times P$ such that for every function $f : P \to A$ with values in an abelian group A the following *Möbius inversion formula* holds:

$$f(y) = \sum_{x \leqq y} \mu_P(x, y) g(x), \quad \text{where} \quad g(y) = \sum_{x \leqq y} f(x) .$$

From now on we always assume that the poset P has a unique minimal element $\hat{0}$, a unique maximal element $\hat{1}$, and that every maximal chain of P has the same length d which will be called the *rank of* P. For any $x \leqq y$ in P, define the interval

$$[x, y] = \{z \in P : x \leqq z \leqq y\} .$$

In particular, we have $P = [\hat{0}, \hat{1}]$. Define the rank function $\rho : P \to \{0, 1, \ldots, d\}$ of P by setting $\rho(x)$ equal to the length of any saturated chain in the interval $[\hat{0}, x]$.

Definition 2.1 [31] *A poset P as above is said to be* **Eulerian** *if for any $x \leqq y$ $(x, y \in P)$ we have*

$$\mu_P(x, y) = (-1)^{\rho(y)-\rho(x)} .$$

Remark 2.2 It is easy to see that any interval $[x, y] \subset P$ in an Eulerian poset P is again an Eulerian poset with the rank function $\rho(z) - \rho(x)$ for any $z \in [x, y]$. If an Eulerian poset P has rank d, then the dual poset P^* is again an Eulerian poset with the rank function $\rho^*(x) = d - \rho(x)$.

Example 2.3 Let C be an d-dimensional finite convex polyhedral cone in $\mathbf{R}^d$ such that $-C \cap C = \{0\} \in \mathbf{R}^d$. Then the poset P of faces of C satisfies all the assumptions above with the maximal element C, the minimal element $\{0\}$, and the rank function ρ which is equal to the dimension of the corresponding face. It is easy to show that P is an Eulerian poset of rank d.

Definition 2.4 [31] *Let $P = [\hat{0}, \hat{1}]$ be an Eulerian poset of rank d. Define two polynomials $G(P, t)$, $H(P, t) \in \mathbf{Z}[t]$ by the following recursive rules*:

$$G(P, t) = H(P, t) = 1 \quad \text{if } d = 0 ;$$

$$H(P, t) = \sum_{\hat{0} < x \leqq \hat{1}} (t-1)^{\rho(x)-1} G([x, \hat{1}], t) \quad (d > 0) ,$$

$$G(P, t) = \tau_{<d/2}((1-t) H(P, t)) \quad (d > 0) ,$$

where $\tau_{<r}$ denotes the truncation operator $\mathbf{Z}[t] \to \mathbf{Z}[t]$ which is defined by

$$\tau_{<r}\left(\sum_i a_i t^i\right) = \sum_{i<r} a_i t^i .$$

Theorem 2.5 [31] *Let P be an Eulerian poset of rank $d \geqq 1$. Then*

$$H(P,t) = t^{d-1} H(P,t^{-1}) .$$

Proposition 2.6 *Let P be an Eulerian poset of rank $d \geqq 0$. Then*

$$t^d G(P,t^{-1}) = \sum_{\hat{0} \leqq x \leqq \hat{1}} (t-1)^{\rho(x)} G([x,\hat{1}],t) .$$

Proof. The case $d = 0$ is obvious. Using 2.5, we obtain

$$(t-1)H(P,t) = t^d G(P,t^{-1}) - G(P,t) \quad (d > 0) .$$

Now the statement follows from the formula for $H(P,t)$ in 2.4. □

Definition 2.7 *Let P be an Eulerian poset of rank d. Define the polynomial $B(P;u,v) \in \mathbf{Z}[u,v]$ by the following recursive rules*:

$$B(P;u,v) = 1 \quad \text{if } d = 0 ,$$

$$\sum_{\hat{0} \leqq x \leqq \hat{1}} B([\hat{0},x];u,v) u^{d-\rho(x)} G([x,\hat{1}],u^{-1}v) = G(P,uv) .$$

Example 2.8 Let P be the boolean algebra of rank $d \geqq 1$. Then $G(P,t) = 1$, $H(P,t) = 1 + t + \cdots + t^{d-1}$, and $B(P;u,v) = (1-u)^d$.

Example 2.9 Let $C \subset \mathbf{R}^3$ be a 3-dimensional finite convex polyhedral cone with k 1-dimensional faces ($-C \cap C = \{0\} \in \mathbf{R}^3$), P the Eulerian poset of faces of C. Then $G(P,t) = 1 + (k-3)t$, $H(P,t) = 1 + (k-2)t + t^2$, and

$$B(P;u,v) = 1 - (k-(k-3)v)u + (k-(k-3)v)u^2 - u^3 .$$

We notice that $B(P;u,v)$ satisfies the relation

$$B(P;u,v) = (-u)^3 B(P;u^{-1},v)$$

which is a consequence of the selfduality $P \cong P^*$ and a more general property 2.13.

Proposition 2.10 *Let P be an Eulerian poset of rank $d > 0$. Then $B(P;u,v)$ has the following properties*:

(i) $B(P;u,1) = (1-u)^d$ *and* $B(P;1,v) = 0$;

(ii) *the degree of $B(P;u,v)$ with respect to v is less than $d/2$.*

Proof. The statement (i) follows immediately from 2.6 and the recursive definition of $B(P;u,v)$. In order to prove (ii) we use induction on d. By assumption, the degree of $B([\hat{0},x];u,v)$ with respect to v is less than $\rho(x)/2$. On the other hand, the v-degree of $G([x,\hat{1}];u^{-1}v)$ is less than $(d-\rho(x))/2$ (see 2.4). It remains to apply the recursive formula of 2.7. □

Proposition 2.11 *Let P be an Eulerian poset of rank d. Then B-polynomials of intervals $[\hat{0},x]$ and $[x,\hat{1}]$ satisfy the following relation*:

$$\sum_{\hat{0}\leqq x\leqq \hat{1}} B([\hat{0},x];u^{-1},v^{-1})(uv)^{\rho(x)}(v-u)^{d-\rho(x)} = \sum_{\hat{0}\leqq x\leqq \hat{1}} B([x,\hat{1}];u,v)(uv-1)^{\rho(x)} .$$

Proof. Let us substitute u^{-1},v^{-1} instead of u,v in the recursive relation 2.7. We obtain

$$\sum_{\hat{0}\leqq x\leqq \hat{1}} B([\hat{0},x];u^{-1},v^{-1})u^{-d+\rho(x)}G([x,\hat{1}],uv^{-1}) = G(P,u^{-1}v^{-1}) . \tag{3}$$

By 2.6, we have

$$G(P,u^{-1}v^{-1}) = (uv)^{-d}\sum_{\hat{0}\leqq x\leqq \hat{1}} (uv-1)^{\rho(x)}G([x,\hat{1}],uv) \tag{4}$$

and

$$\begin{aligned} G([x,\hat{1}],uv^{-1}) &= \sum_{x\leqq y\leqq \hat{1}} (u^{-1}v-1)^{\rho(y)-\rho(x)}u^{d-\rho(x)}v^{\rho(x)-d}G([y,\hat{1}],u^{-1}v) \\ &= \sum_{x\leqq y\leqq \hat{1}} u^{d-\rho(y)}v^{\rho(x)-d}(v-u)^{\rho(y)-\rho(x)}G([y,\hat{1}],u^{-1}v) . \end{aligned} \tag{5}$$

By 2.7, we also have

$$G([x,\hat{1}],uv) = \sum_{x\leqq y\leqq \hat{1}} u^{d-\rho(y)}B([x,y];u,v)G([y,\hat{1}],u^{-1}v) . \tag{6}$$

By substitution (6) in (4), and two equations (4), (5) in (3) we obtain:

$$\begin{aligned} &\sum_{\hat{0}\leqq x\leqq y\leqq \hat{1}} B([\hat{0},x];u^{-1},v^{-1})u^{\rho(x)-\rho(y)}v^{\rho(x)-d}(v-u)^{\rho(y)-\rho(x)}G([y,\hat{1}],u^{-1}v) \\ &= \sum_{\hat{0}\leqq x\leqq y\leqq \hat{1}} B([x,y];u,v)u^{-\rho(y)}v^{-d}(uv-1)^{\rho(x)}G([y,\hat{1}],u^{-1}v) . \end{aligned} \tag{7}$$

Now we use induction on d. It is easy to see that the equation (7) and the induction hypothesis for $y<\hat{1}$ immediately imply the statement of the proposition. □

Proposition 2.12 *The B-polynomials are uniquely determined by the relation* 2.11, *by the property of v-degree from* 2.10(ii), *and by the initial condition* $B(P;u,v)=1$ *if* $d=0$.

Proof. Indeed, if we know $B([x,y];u,v)$ for all $\rho(y)-\rho(x)<d$, then we know all terms in 2.11 except for $B(P;u,v)$ on the right hand side and $B(P;u^{-1},v^{-1})(uv)^d$ on the left hand side. Because the v-degree of $B(P;u,v)$ is less than $d/2$, the possible degrees of monomials with respect to variable v in $B(P;u,v)$ and $B(P;u^{-1},v^{-1})(uv)^d$ do not coincide. This allows us to determine $B(P;u,v)$ uniquely. □

Theorem 2.13 *Let P be an Eulerian poset of rank d, P^* be the dual Eulerian poset. Then*

$$B(P;u,v)=(-u)^d B(P^*;u^{-1},v)\,.$$

Proof. We set

$$Q(P;u,v)=(-u)^d B(P^*;u^{-1},v)\,.$$

It is clear that $Q(P;u,v)=1$ and v-degree of $Q(P;u,v)$ is the same as v-degree of $B(P;u,v)$. By 2.12, it remains to establish the same recursive relations for $Q(P;u,v)$ as for $B(P;u,v)$ in 2.11. The last property follows from straightforward computations. Indeed, the equality

$$\sum_{\hat{0}\leqq x\leqq \hat{1}} Q([\hat{0},x];u^{-1},v^{-1})(uv)^{\rho(x)}(v-u)^{d-\rho(x)}=\sum_{\hat{0}\leqq x\leqq \hat{1}} Q([x,\hat{1}];u,v)(uv-1)^{\rho(x)} \tag{8}$$

is equivalent to the relation 2.11 for $B(P^*;u,v^{-1})$:

$$\sum_{\hat{0}\leqq x\leqq \hat{1}} B([x,\hat{1}]^*;u^{-1},v)(uv^{-1})^{d-\rho(x)}(v^{-1}-u)^{\rho(x)}$$

$$=\sum_{\hat{0}\leqq x\leqq \hat{1}} B([\hat{0},x]^*;(u,v^{-1})(uv^{-1}-1)^{d-\rho(x)}\,,$$

because

$$Q([x,\hat{1}];u,v)=(-u)^{d-\rho(x)}B([x,\hat{1}]^*;u^{-1},v)$$

and

$$Q([\hat{0},x];u^{-1},v^{-1})=(-u)^{-\rho(x)}B([\hat{0},x]^*;u,v^{-1})\,.$$ □

3 *E*-polynomials of toric hypersurfaces

Let M and N be two free abelian groups of rank d which are dual to each other; i.e., $N=\mathrm{Hom}(M,\mathbf{Z})$. We denote by

$$\langle *,*\rangle : M\times N\to \mathbf{Z}$$

the canonical bilinear pairing, and by $M_{\mathbf{R}}$ (resp. by $N_{\mathbf{R}}$) the real scalar extensions of M (resp. of N).

Definition 3.1 *A subset $C \subset M$ is called a d-dimensional rational convex polyhedral cone with vertex $\{0\} \in M$ if there exists a finite set $\{e_1, \ldots, e_k\} \subset M$ such that*

$$C = \{\lambda_1 e_1 + \cdots + \lambda_k e_k \in M_{\mathbf{R}} : \text{where } \lambda_i \in \mathbf{R}_{\geqq 0}\ (i = 1, \ldots, k)\}$$

and $-C + C = M_{\mathbf{R}}$, $-C \cap C = \{0\} \in M$.

Remark 3.2 If $C \subset M$ is a d-dimensional rational convex polyhedral cone with vertex $\{0\} \in M$, then the dual cone

$$\check{C} = \{z \in N_{\mathbf{R}} : \langle e_i, z\rangle \geqq 0 \text{ for all } i \in \{1, \ldots, k\}\}$$

is also a d-dimensional rational convex polyhedral cone with vertex $\{0\}$ in the dual space $N_{\mathbf{R}}$. Moreover, there exists a canonical bijective correspondence $F \leftrightarrow F^*$ between faces $F \subset C$ and faces $F^* \subset \check{C}$ ($\dim F + \dim F^* = d$):

$$F \mapsto F^* := \{z \in \check{C} : \langle z', z\rangle = 0 \text{ for all } z' \in F\}$$

which reverses inclusion relation between faces.

Let P be the Eulerian poset of faces of a d-dimensional rational convex polyhedral cone $C \subset M_{\mathbf{R}}$ with vertex in $\{0\}$. For convenience of notations, we use elements $x \in P$ as indices and denote by C_x the face of C corresponding to $x \in P$, in particular, we have $C_{\hat{0}} = \{0\}$, $C_{\hat{1}} = C$, and $\rho(x) = \dim C_x$. The dual Eulerian poset P^* can be identified with the poset of faces C_x^* of the dual cone $\check{C} \subset N_{\mathbf{R}}$.

Definition 3.3 *A d-dimensional cone C ($d \geqq 1$) as in* 3.1 *is called* **Gorenstein** *if there exists an element $n_C \in N$ such that $\langle z, n_C\rangle > 0$ for any nonzero $z \in C$, and all vertices of the $(d-1)$-dimensional convex polyhedron*

$$\Delta(C) = \{z \in C : \langle z, n_C\rangle = 1\}$$

belong to M. This polyhedron will be called the **supporting polyhedron of** *C. For convenience, we consider $\{0\}$ as a 0-dimensional Gorenstein cone with the supporting polyhedron $\Delta(\{0\}) := \emptyset$. For any $m \in C \cap M$, we define the* **degree of** *m as*

$$\deg m = \langle m, n_C\rangle\,.$$

Remark 3.4 It is clear that any face C_x of a Gorenstein cone is again a Gorenstein cone with the supporting polyhedron

$$\Delta(C_x) = \{z \in C_x : \langle z, n_C\rangle = 1\}\,.$$

Now we recall standard facts from the theory of toric varieties [9, 11, 26] and fix our notations:

Let $\mathbf{P}(C)$ be the $(d-1)$-dimensional projective toric variety associated with a Gorenstein cone C. By definition,

$$\mathbf{P}(C) = \operatorname{Proj} \mathbf{C}[C \cap M]$$

where $\mathbf{C}[C \cap M]$ is a graded semigroup algebra over $\mathbf{C}$ of lattice points $m \in C \cap M$. Each face $C_x \subset C$ of positive dimension defines an irreducible projective toric subvariety

$$\mathbf{P}(C_x) = \operatorname{Proj} \mathbf{C}[C_x \cap M] \subset \mathbf{P}(C)$$

which is a compactification of a $(\rho(x) - 1)$-dimensional algebraic torus

$$\mathbf{T}_x := \operatorname{Spec} \mathbf{C}[M_x] ,$$

where $M_x \subset M$ is the subgroup of all lattice points $m \in (-C_x + C_x) \cap M$ such that $\langle m, n_C \rangle = 0$. Moreover, the multiplicative group law on $\mathbf{T}_x$ extends to a regular action of $\mathbf{T}_x$ on $\mathbf{P}(C_x)$ so that one has the natural stratification

$$\mathbf{P}(C_x) = \bigcup_{\hat{0} < y \leqq x} \mathbf{T}_y$$

by $\mathbf{T}_x$-orbits $\mathbf{T}_y$. We denote by $\mathcal{O}_{\mathbf{P}(C)}(1)$ the ample tautological sheaf on $\mathbf{P}(C)$. In particular, lattice points in $\Delta(C)$ can be identified with a torus invariant basis of the space of global sections of $\mathcal{O}_{\mathbf{P}(C)}(1)$. We denote by $\overline{Z}$ the set of zeros of a generic global section of $\mathcal{O}_{\mathbf{P}(C)}(1)$ and set

$$Z_x := \overline{Z} \cap \mathbf{T}_x \quad (\hat{0} < x \leqq \hat{1}) .$$

Thus we have the natural stratification:

$$\overline{Z} = \bigcup_{\hat{0} < x \leqq \hat{1}} Z_x ,$$

where each Z_x is a smooth affine hypersurface in $\mathbf{T}_x$ defined by a generic Laurent polynomial with the Newton polyhedron $\Delta(C_x)$.

Definition 3.5 *Define two functions*

$$S(C_x, t) := (1 - t)^{\rho(x)} \sum_{m \in C_x \cap M} t^{\deg m}$$

and

$$T(C_x, t) := (1 - t)^{\rho(x)} \sum_{m \in Int(C_x) \cap M} t^{\deg m} ,$$

where $Int(C_x)$ denotes the relative interior of $C_x \subset C$.

The following statement is a consequence of the Serre duality (see [10, 1]):

Proposition 3.6 *$S(C_x, t)$ and $T(C_x, t)$ are polynomials satisfying the relation*

$$S(C_x, t) = t^d T(C_x, t^{-1}) .$$

Definition 3.7 [10] *Let X be a quasi-projective algebraic variety over* $\mathbf{C}$. *For each pair of integers (p,q), one defines the following generalization of Euler characteristic*:

$$e^{p,q}(X) = \sum_k (-1)^k h^{p,q}(H_c^k(X)),$$

where $h^{p,q}(H_c^k(X))$ is the dimension of the (p,q)-component of the mixed Hodge structure of $H_c^k(X)$ [12]. *The sum*

$$E(X;u,v) := \sum_{p,q} e^{p,q}(X)u^p v^q$$

*is called **E-polynomial of** X.*

Next statement is also due to Danilov and Khovanskiî (see [10] Sect. 4, or another approach in [1]):

Proposition 3.8 *We set $E(Z_{\hat{0}};t,1) := (t-1)^{-1}$. Then*

$$E(Z_x;t,1) = \frac{(t-1)^{\rho(x)-1} + (-1)^{\rho(x)}S(C_x,t)}{t}$$

for $\rho(x) \geqq 0$.

The purpose of this section is to give an explicit formula for E-polynomials of affine hypersurfaces $Z_x \subset \mathbf{T}_x$. Following the method of Denef and Loeser [13] combined with ideas of Danilov and Khovanskiî [10], we compute $E(Z_x;u,v)$ using intersection cohomology (with the middle perversity) introduced by Goresky and MacPherson [17]. Recall that intersection cohomology $IH^*(X)$ of a quasi-projective algebraic variety X of pure dimension n over an algebraically closed field K can be defined as hypercohomology of the so called *intersection complex* $IC_X^\bullet$ which is uniquely determined as an object of the derived category $D^b(X)$. In the case $\operatorname{char} K > 0$ the intersection complex $IC_X^\bullet$ with l-adic coefficients carries a natural weight filtration which has been studied by Beilinson, Bernstein, Deligne and Gabber using the theory of perverse sheaves [8]. There exists the following explicit construction of $IC_X^\bullet$ proposed by Deligne:

Let $X = Z^0 \supset Z^1 \supset Z^2 \supset \cdots \supset Z^n \supset Z^{n+1} = \emptyset$ be an irreducible stratified complex algebraic variety of dimension n; i.e., Z^k are closed subvarieties, the strata $S^k = Z^k \backslash Z^{k+1}$ are smooth complex algebraic locally closed subvarieties of codimension k in X, and the open subset S^0 is dense in X. Denote by $\mathscr{F}$ a constant sheaf on S^0 with coefficients in some field $\mathbf{F}$ (the field $\mathbf{F}$ is usually one of the following: $\mathbf{Q}, \mathbf{Q}_l, \mathbf{R}$, or $\mathbf{C}$). Then the intersection complex $IC_X^\bullet(\mathbf{F})$ with coefficients in $\mathbf{F}$ can be defined as

$$IC_X^\bullet(\mathbf{F}) := \tau_{<n} R_{i_n} \cdots \tau_{<1} R_{i_1} \mathscr{F},$$

where $i_k : X \backslash Z^k \to X \backslash Z^{k+1}$ is the open inclusion and $\tau_{<k}$ truncates sheaf cohomology in degrees $\geqq k$. The cohomology $\mathscr{H}^i(IC_X^\bullet(\mathbf{F}))$ are constructible sheaves which do not depend on the choice of a stratification. Without lost of generality we can often assume that the sheaves $\mathscr{H}^j(IC_X^\bullet(\mathbf{F}))$ are locally constant along all connected components X_i^k of strata S^k.

Definition 3.9 *Let X be a quasi-projective algebraic variety over $K, X = \bigcup_i X_i$ a stratification of X by pairwise disjoint smooth irreducible locally closed strata X_i such that the cohomology sheaf $\mathscr{H}^j(IC_X^\bullet(\mathbf{F}))$ is locally constant along X_i for every $j \geqq 0$. Assume that for every stratum X_i we have*:

(i) $\mathscr{H}^j(IC_X^\bullet(\mathbf{F}))|_{X_i} = 0$ *for all odd values of* j;

(ii) *the Tate twisted sheaves* $\mathscr{H}^{2k}j(IC_X^\bullet(\mathbf{F}))(-k)|_{X_i}$ *are direct sums of copies of the constant sheaf* $\mathbf{F}$ *on* X_i.

In this situation, we define for every stratum $X_i \subset X$ the polynomial

$$G_{\text{int}}(X_i, t) := \sum_{k \geqq 0} \dim_{\mathbf{F}} \mathscr{H}^{2k}(IC_X^\bullet(\mathbf{F}))_s t^k ,$$

where $\mathscr{H}^{2k}(IC_X^\bullet(\mathbf{F}))_s$ is the stalk of $\mathscr{H}^{2k}(IC_X^\bullet(\mathbf{F}))$ over some closed point $s \in X_i$.

Remark 3.10 It follows immediately from the construction of Deligne that

$$\deg G_{\text{int}}(X_i, t) < \operatorname{codim} X_i/2 .$$

The mixed Hodge structure on intersection cohomology of algebraic varieties over $\mathbf{C}$ has been introduced by M. Saito using the theory of mixed Hodge modules [28, 29, 30]. In particular, one has the following property:

Theorem 3.11 *Let $X = \bigcup_i X_i$ be a stratified quasi-projective algebraic variety over $\mathbf{C}$. Then the hypercohomology groups with compact supports of $IC_X^\bullet$ and its restrictions to strata $X_i \subset X$ have natural mixed Hodge structures.*

Definition 3.12 *Let $X = \bigcup_i X_i$ a stratified quasi-projective variety. We call the polynomial*

$$E_{\text{int}}(X; u, v) := \sum_k (-1)^k h^{p,q}(IH_c^k(X)) u^p v^q$$

the **intersection cohomology *E*-polynomial of** X.

Let $IH_c^\bullet(X/X_i)$ the hypercohomology of the restriction of $IC_X^\bullet$ to X_i. We call the polynomial

$$E_{\text{int}}(X/X_i; u, v) := \sum_k (-1)^k h^{p,q}(IH_c^k(X/X_i)) u^p v^q$$

the **intersection cohomology *E*-polynomial of the stratum** $X_i \subset X$.

From M. Saito's theory, one immediatelly obtains:

Theorem 3.13 *Let $X = \bigcup_i X_i$ be a stratified quasi-projective algebraic variety over $\mathbf{C}$. Then*

$$E_{\text{int}}(X; u, v) = \sum_i E_{\text{int}}(X/X_i; u, v) .$$

Moreover, if the stratification of X satisfies the conditions (i), (ii) *in* 3.9, *then*

$$E_{\text{int}}(X; u, v) = \sum_i E(X_i; u, v) \cdot G_{\text{int}}(X_i, uv) .$$

Following ideas in [8, 16] for the l-adic version of the intersection cohomology of algebraic varieties over K in the case $\operatorname{char} K > 0$, M. Saito has proved the following purity theorem for varieties over $\mathbf{C}$ (see a generalized version of the purity theorem for links in [14]):

Theorem 3.14 *Let X be a projective algebraic variety over $\mathbf{C}$. Then the mixed Hodge structure in $IH^j(X)$ is pure of weight j.*

Corollary 3.15 *Let X be a projective algebraic variety. Then*

$$h^{p,q}(IH^{p+q}(X)) = (-1)^{p+q} e^{p,q}_{\text{int}}(X)\,,$$

where the numbers $e^{p,q}_{\text{int}}(X)$ are the coefficients of the intersection cohomology E-polynomial

$$E_{\text{int}}(X; u, v) = \sum_{p,q} e^{p,q}_{\text{int}}(X) u^p v^q\,.$$

The following statement has been discovered by Bernstein, Khovanskiî and MacPherson (see two independent proofs in [13] and [15]):

Theorem 3.16 *Let*

$$\mathbf{P}(C) = \bigcup_{\hat{0} < x \leqq \hat{1}} \mathbf{T}_x$$

be a projective toric variety with the natural stratification by the torus orbits $\mathbf{T}_x$. Then this stratification satisfies the condition (i), (ii) *in* 3.9 *and*

$$G_{\text{int}}(\mathbf{T}_x, t) = G([x, \hat{1}], t)\,.$$

In particular, one has

$$E_{\text{int}}(\mathbf{P}(C); u, v) = \sum_{\hat{0} < x \leqq \hat{1}} (uv - 1)^{\rho(x)-1} G([x, \hat{1}], uv) = H(P, uv)\,.$$

Corollary 3.17 *Let $\overline{W} \subset \mathbf{P}(C)$ be a hypersurface that meets transversally all toric strata $\mathbf{T}_x \subset \mathbf{P}(C)$ that it intersects* ($\overline{W}$ *is not assumed to be ample*). *Then*

$$E_{\text{int}}(\overline{W}; u, v) = \sum_{\hat{0} < x \leqq \hat{1}} E(W_x; u, v) G([x, \hat{1}], uv)\,,$$

where $W_x = \overline{W} \cap \mathbf{T}_x$ $(\hat{0} < x \leqq \hat{1})$.

Proof. Let $IC^\bullet_{\mathbf{P(C)}}$ (resp. $IC^\bullet_{\overline{W}}$) be the intersection complex which is obtained by the construction of Deligne applied to the natural stratification of $\mathbf{P}(C)$ by $\mathbf{T}_x$ (resp. of $\overline{W}$ by W_x). Since the stratification of $\mathbf{P}(C)$ by $\mathbf{T}_x$ is locally isomorphic in analytic topology to the stratification of $\overline{W} \times \mathbf{A}^1$ by $W_x \times \mathbf{A}^1$, the restriction of $IC^\bullet_{\mathbf{P(C)}}$ to $\overline{W}$ coincides with $IC^\bullet_{\overline{W}}$, and the restrictions of the cohomology sheaves $\mathscr{H}^i(IC^\bullet_{\mathbf{P(C)}})$ to $\overline{W}$ coincide with $\mathscr{H}^i(IC^\bullet_{\overline{W}})$. By 3.16, $IC^\bullet_{\overline{W}}$ satisfies the conditions (i), (ii) in 3.9 with respect to the stratification by W_x and

$$G_{\text{int}}(W_x, t) = G_{\text{int}}(\mathbf{T}_x, t)\,.$$

Now the statement follows from 3.13. □

Applying 3.8, we obtain:

Corollary 3.18

$$E_{\mathrm{int}}(\overline{Z}; t, 1) = \sum_{\hat{0} < x \leqq \hat{1}} \left(\frac{(t-1)^{\rho(x)-1} + (-1)^{\rho(x)} S(C_x, t)}{t} \right) G([x, \hat{1}], t)\,.$$

Definition 3.19 *Define* $H_{\mathrm{Lef}}(P,t)$ *to be the polynomial of degree* $(d-2)$ *with the following properties*:

(i) $H_{\mathrm{Lef}}(P,t) = t^{d-2} H_{\mathrm{Lef}}(P, t^{-1})$;

(ii) $\tau_{\leqq (d-2)/2} H_{\mathrm{Lef}}(P,t) = \tau_{\leqq (d-2)/2} H(P,t)$.

Proposition 3.20

$$H_{\mathrm{Lef}}(P,t) = (1-t)^{-1}(G(P,t) - t^{d-1} G(P, t^{-1}))\,.$$

Proof. Let us set

$$Q(P,t) := (1-t)^{-1}(G(P,t) - t^{d-1} G(P, t^{-1}))\,.$$

We check that the properties 3.19(i)–(ii) are satisfied for $Q(P,t)$. Indeed 3.19(i) follows immediately from the definition of $Q(P,t)$. If

$$H(P,t) = \sum_{0 \leqq i \leqq d-1} h_i t^i$$

and

$$G(P,t) = h_0 + \sum_{1 \leqq i < d/2} (h_i - h_{i-1}) t^i\,,$$

then

$$Q(P,t) = h_0 \frac{1 - t^{d-1}}{1-t} + \sum_{1 \leqq i < d/2} (h_i - h_{i-1}) \frac{t^i - t^{d-1-i}}{1-t}\,.$$

This shows (ii) and the fact that $Q(P,t)$ is a polynomial. □

Proposition 3.21 *Define* $E^{\mathrm{prim}}_{\mathrm{int}}(\overline{Z}; u, v)$ *to be the polynomial*

$$E^{\mathrm{prim}}_{\mathrm{int}}(\overline{Z}; u, v) := E_{\mathrm{int}}(\overline{Z}; u, v) - H_{\mathrm{Lef}}(P, uv)\,.$$

Then $E^{\mathrm{prim}}_{\mathrm{int}}(\overline{Z}; u, v)$ *is a homogeneous polynomial of degree* $(d-2)$.

Proof. By the Lefschetz theorem for intersection cohomology [18], we have isomorphisms

$$IH^i(\mathbf{P}(C)) \cong IH^i(\overline{Z}), \quad (0 \leqq i < d-2)$$

and the short exact sequence

$$0 \to IH^{d-2}(\mathbf{P}(C)) \to IH^{d-2}(\overline{Z}) \to IH^{d-2}_{\mathrm{prim}}(\overline{Z}) \to 0\,,$$

where $IH^{d-2}_{\mathrm{prim}}(\overline{Z})$ denotes the primitive part of intersection cohomology of $\overline{Z}$ in degree $(d-2)$. By purity Theorem 3.14, the Hodge structure of $IH^{d-2}_{\mathrm{prim}}(\overline{Z})$ is pure. On the other hand, it follows from the Poincaré duality for intersection cohomology that $E^{\mathrm{prim}}_{\mathrm{int}}(\overline{Z}; u, v)$ is the E-polynomial of this Hodge structure. □

Theorem 3.22 *We set* $E(Z_{\hat{0}}; u, v) := (uv-1)^{-1}$. *Then E-polynomials* $E(Z_x; u, v)$ *of affine toric hypersurfaces satisfy the following recursive relation*

$$\sum_{\hat{0} \leqq x \leqq \hat{1}} (E(Z_x; u, v) - (uv)^{-1}(uv-1)^{\rho(x)-1}) G([x, \hat{1}], uv)$$
$$= v^{d-2} \sum_{\hat{0} \leqq x \leqq \hat{1}} (u^{-1}v)(-1)^{\rho(x)} S(C_x, uv^{-1}) G([x, \hat{1}], uv^{-1}) .$$

Proof. By 3.18 and 3.20, we have

$$E_{\rm int}^{\rm prim}(\overline{Z}; t, 1) = E_{\rm int}(\overline{Z}; t, 1) - H_{\rm Lef}(P, t)$$
$$= \sum_{\hat{0} < x \leqq \hat{1}} t^{-1}((t-1)^{\rho(x)-1} + (-1)^{\rho(x)} S(C_x, t)) G([x, \hat{1}], t)$$
$$-(1-t)^{-1}(G(P, t) - t^{d-1} G(P, t^{-1})) .$$

Using 2.6, we obtain

$$\sum_{\hat{0} < x \leqq \hat{1}} t^{-1}(t-1)^{\rho(x)-1} G([x, \hat{1}], t) = t^{-1}(t-1)^{-1}(t^d G(P, t^{-1}) - G(P, t)) .$$

This yields

$$E_{\rm int}^{\rm prim}(\overline{Z}; t, 1) = \sum_{\hat{0} \leqq x \leqq \hat{1}} t^{-1}(-1)^{\rho(x)} S(C_x, t) G([x, \hat{1}], t) . \tag{9}$$

On the other hand, by 3.17 and 3.20, we have

$$E_{\rm int}^{\rm prim}(\overline{Z}; u, v) = E_{\rm int}(\overline{Z}; u, v) - H_{\rm Lef}(P, uv)$$
$$= \sum_{\hat{0} < x \leqq \hat{1}} E(Z_x; u, v) G([x, \hat{1}], uv)$$
$$-(1-uv)^{-1}(G(P, uv) - (uv)^{d-1} G(P, (uv)^{-1})) .$$

Using 2.6, we obtain

$$\sum_{\hat{0} \leqq x \leqq \hat{1}} (uv)^{-1}(uv-1)^{\rho(x)-1} G([x, \hat{1}], uv) = (uv)^{d-1}(uv-1)^{-1} G(P, (uv)^{-1}) .$$

This yields

$$E_{\rm int}^{\rm prim}(\overline{Z}; u, v) = \sum_{\hat{0} \leqq x \leqq \hat{1}} (E(Z_x; u, v) - (uv)^{-1}(uv-1)^{\rho(x)-1}) G([x, \hat{1}], uv) . \tag{10}$$

By 3.21, we have

$$E_{\rm int}^{\rm prim}(\overline{Z}; u, v) = v^{d-2} E_{\rm int}^{\rm prim}(\overline{Z}; uv^{-1}, 1) .$$

It remains to combine (9) and (10). □

Definition 3.23 *Let m be a lattice point in $C \cap M$. We denote by $x(m)$ the minimal element among $x \in P$ such that the face $C_x \subset C$ contains m. The interval $[x(m), \hat{1}] \subset P$ parametrizes the set of all faces of C containing m. We identify the dual interval $[x(m), \hat{1}]^*$ with the Eulerian poset of all faces $C_x^* \subset \check{C}$ such that $\langle m, z \rangle = 0$ for all $z \in C_x^*$.*

Theorem 3.24 *Let us set $Z := Z_{\hat{1}}$. Then there exists the following explicit formula for $E(Z; u, v)$ in terms of B-polynomials*:

$$E(Z;u,v) = \frac{(uv-1)^{d-1}}{uv} + \frac{(-1)^d}{uv} \sum_{m \in C \cap M} (v-u)^{\rho(x(m))} B([x(m),\hat{1}]^*; u, v) \left(\frac{u}{v}\right)^{\deg m} .$$

Proof. By induction, E-polynomials are uniquely determined from the recursive formula 3.22. Therefore, it suffices to show that the functions

$$\frac{(uv-1)^{\rho(x)-1}}{uv} + \frac{(-1)^{\rho(x)}}{uv} \sum_{m \in C_x \cap M} (v-u)^{\rho(x(m))} B([x(m),x]^*; u, v) \left(\frac{u}{v}\right)^{\deg m}$$

satisfy the same recursive formula as polynomials $E(Z_x; u, v)$. Indeed, let us substitute these functions instead of E-polynomials in the left hand side of 3.22 and expand

$$(-1)^{\rho(x)} S(C_x, uv^{-1}) = \left(\frac{u}{v} - 1\right)^{\rho(x)} \sum_{m \in C_x \cap M} \left(\frac{u}{v}\right)^{\deg m}$$

on the right hand side of 3.22. Now we choose a lattice point $m \in C \cap M$, collect terms containing $(u/v)^{\deg m}$ in right and left hand sides, and use the equality (2.6)

$$\sum_{x(m) \leqq x \leqq \hat{1}} \left(\frac{u}{v} - 1\right)^{\rho(x)} G([x,\hat{1}], uv^{-1}) = \left(\frac{u}{v} - 1\right)^{\rho(x(m))} \left(\frac{u}{v}\right)^{d-\rho(x(m))} G([x(m),\hat{1}], u^{-1}v)$$

on the right hand side. By the duality (2.13)

$$B([x(m),x]^*; u, v) = (-u)^{\rho(x)-\rho(m(x))} B([x(m),x]; u^{-1}, v) ,$$

it remains to establish the recursive relation:

$$\frac{(v-u)^{\rho(x(m))}}{uv} \sum_{x(m) \leqq x \leqq \hat{1}} (-1)^{\rho(x)} (-u)^{\rho(x)-\rho(m(x))} B([x(m),x]; u^{-1}, v) G([x,\hat{1}], uv)$$
$$= \left(\frac{u}{v} - 1\right)^{\rho(x(m))} \frac{v^{d-1}}{u} \left(\frac{u}{v}\right)^{d-\rho(x(m))} G([x(m),\hat{1}], u^{-1}v)$$

which is equivalent to the recursive relation in 2.7 after the substitution u^{-1} instead of u. □

4 Mirror duality

Let $\overline{M}$ and $\overline{N} = \mathrm{Hom}(\overline{N}, \mathbf{Z})$ be dual to each other free abelian groups of rank $\overline{d}$, $\overline{M}_{\mathbf{R}}$ and $\overline{N}_{\mathbf{R}}$ the real scalar extensions of $\overline{M}$ and $\overline{N}$, $\langle *,* \rangle : \overline{M} \times \overline{N} \to \mathbf{Z}$ the natural pairing.

Definition 4.1 [4] *Let $C \subset \overline{M}_{\mathbf{R}}$ be a $\overline{d}$-dimensional Gorenstein cone. The cone C is called* **reflexive** *if the dual cone $\check{C} \subset N_{\mathbf{R}}$ is also Gorenstein*; *i.e.*, *there exists a lattice element $m_{\check{C}} \in M$ such that all vertices of the supporting polyhedron $\Delta(\check{C}) = \{z \in \check{C} : \langle m_{\check{C}}, z \rangle = 1\}$ are contained in M. In this case*, *we call $r = \langle m_{\check{C}}, n_C \rangle$ the* **index** *of C.*

Definition 4.2 [2] *Let M be a free abelian group of rank d. A d-dimensional polyhedron in $M_{\mathbf{R}}$ with vertices in M is called* **reflexive** *if it can be identified with a supporting polyhedron of some* $(d+1)$*-dimensional reflexive Gorenstein cone of index* 1.

Recall the definition of string-theoretic Hodge numbers of an algebraic variety X with at most Gorenstein toroidal singularities [6]:

Definition 4.3 [6] *Let $X = \bigcup_{i \in I} X_i$ be a k-dimensional stratified algebraic variety over* $\mathbf{C}$ *with at most Gorenstein toroidal singularities such that for any $i \in I$ the singularities of X along the stratum X_i of codimension k_i are defined by a k_i-dimensional finite rational polyhedral cone σ_i*; *i.e.*, *X is locally isomorphic to*

$$\mathbf{C}^{k-k_i} \times U_{\sigma_i}$$

at each point $x \in X_i$ where U_{σ_i} is a k_i-dimensional affine toric variety which is associated with the cone σ_i (see [9]). *Then the polynomial*

$$E_{\mathrm{st}}(X; u, v) := \sum_{i \in I} E(X_i; u, v) \cdot S(\sigma_i, uv)$$

is called the **string-theoretic *E*-polynomial of *X***. *If we write $E_{\mathrm{st}}(X; u, v)$ in form*

$$E_{\mathrm{st}}(X; u, v) = \sum_{p,q} a_{p,q} u^p v^q ,$$

then the numbers $h_{\mathrm{st}}^{p,q}(X) := (-1)^{p+q} a_{p,q}$ are called the **string-theoretic Hodge numbers of *X***.

Remark 4.4 Comparing with 3.13, 3.16 and 3.17, the definition of the string-theoretic Hodge numbers looks as if there were a complex $ST_X^{\bullet}$ whose hypercohomology groups have natural Hodge structure which assumed to be pure if X is compact. We remark that the construction of such a complex $ST_X^{\bullet}$ (an analog of the intersection complex) is still an open problem.

Let $V = D_1 \cap \cdots \cap D_r$ be a generic Calabi–Yau complete intersection of r semi-ample divisors $D_1, \ldots, D_r$ in a d-dimensional Gorenstein toric Fano variety $\mathbf{X}$ ($k \geqq r$). According to [4], there exists a d-dimensional reflexive polyhedron Δ and its decomposition into a Minkowski sum

$$\Delta = \Delta_1 + \cdots + \Delta_r ,$$

where each lattice polyhedron Δ_i is the supporting polyhedron for global sections of a semi-ample invertible sheaf $\mathscr{L}_i \cong \mathscr{O}_{\mathbf{X}}(D_i)$ $(i = 1, \ldots, r)$.

Definition 4.5 [7] *Denote by $E_1, \ldots, E_k$ the closures of $(d-1)$-dimensional torus orbits in $\mathbf{X}$ and set $I := \{1, \ldots, k\}$. A decompostion into a Minkowski sum $\Delta = \Delta_1 + \cdots + \Delta_r$ as above is called a* **nef-partition** *if there exists a decomposition of I into a disjoint union of r subsets $I_j \subset I$ $(j = 1, \ldots, r)$ such that*

$$\mathscr{O}(D_j) \cong \mathscr{O}\left(\sum_{l \in I_j} E_l\right), \quad (j = 1, \ldots, r)$$

Now we put $\overline{M} = \mathbf{Z}^r \oplus M$, $\overline{d} = d + r$, and define the $\overline{d}$-dimensional cone $C \subset \overline{M}_{\mathbf{R}}$ as

$$C := \{(\lambda_1, \ldots, \lambda_r, \lambda_1 z_1 + \cdots + \lambda_r z_r) \in \overline{M}_{\mathbf{R}} : \lambda_i \in \mathbf{R}_{\geqq 0},\ z_i \in \Delta_i,\ i = 1, \ldots, r\} .$$

We extend the pairing $\langle \cdot\, , \cdot \rangle : M \times N \to \mathbf{Z}$ to the pairing between $\overline{M}$ and $\overline{N} := \mathbf{Z}^r \oplus N$ by the formula

$$\langle (a_1, \ldots, a_r, m), (b_1, \ldots, b_r, n) \rangle = \sum_{i=1}^{r} a_i b_i + \langle m, n \rangle .$$

Theorem 4.6 [7, 4] *Let $\Delta = \Delta_1 + \cdots + \Delta_r$ be a nef-partition. Then it defines canonically a d-dimensional reflexive polyhedron $\nabla \subset N_{\mathbf{R}}$ and a nef-partition $\nabla = \nabla_1 + \cdots + \nabla_r$ which are uniquely determined by the property that*

$$\check{C} := \{(\lambda_1, \ldots, \lambda_r, \lambda_1 z_1 + \cdots + \lambda_r z_r) \in \overline{N}_{\mathbf{R}} : \lambda_i \in \mathbf{R}_{\geqq 0},\ z_i \in \nabla_i,\ i = 1, \ldots, r\}$$

is the dual reflexive Gorenstein cone $\check{C} \subset \overline{N}_{\mathbf{R}}$.

Definition 4.7 [7] *The nef-partition $\nabla = \nabla_1 + \cdots + \nabla_r$ as in* 4.6 *is called* **the dual nef-partition**.

We set

$$\mathbf{Y} := \mathbf{P}(\mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_r) .$$

Recall the standard construction of the reduction of complete intersection $V \subset \mathbf{X}$ to a hypersurface $\tilde{V} \subset \mathbf{Y}$ [4]. Let π be the canonical projection $\mathbf{Y} \to \mathbf{X}$ and $\mathscr{O}_{\mathbf{Y}}(-1)$ the tautological Grothendieck sheaf on $\mathbf{Y}$. Since

$$\pi_* \mathscr{O}_{\mathbf{Y}}(1) = \mathscr{L}_1 \oplus \cdots \oplus \mathscr{L}_r ,$$

we obtain the isomorphism

$$H^0(\mathbf{Y}, \mathscr{O}_{\mathbf{Y}}(1)) \cong H^0(\mathbf{X}, \mathscr{L}_1) \oplus \cdots \oplus H^0(\mathbf{X}, \mathscr{L}_r) .$$

Assume that D_i is the set of zeros of a global section $s_i \in H^0(\mathbf{X}, \mathscr{L}_i)$ $(1 \leqq i \leqq r)$. We define $\tilde{V}$ as the zero set of the global section $s \in H^0(\mathbf{Y}, \mathscr{O}_{\mathbf{Y}})$ which corresponds to the r-tuple $(s_1, \ldots, s_r)$ under above isomorphism. Our main interest is the following standard property ([4]):

Proposition 4.8 *The restriction of* π *on* $\mathbf{Y}\backslash\tilde{V}$ *is a locally trivial* $\mathbf{C}^{r-1}$*-bundle in Zariski topology over* $\mathbf{X}\backslash V$.

Let us set

$$\mathbf{P} = \operatorname{Proj} \bigoplus_{i \geqq 0} H^0(\mathbf{Y}, \mathcal{O}_{\mathbf{Y}}(i)) .$$

The following statement is contained in [4]:

Proposition 4.9 *The tautological sheaf* $\mathcal{O}_{\mathbf{Y}}(1)$ *is semi-ample and the natural toric morphism*

$$\alpha : \mathbf{Y} \to \mathbf{P}$$

is crepant. Moreover, $\mathcal{O}_{\mathbf{Y}}(r)$ *is the anticanonical sheaf of* $\mathbf{Y}$, $\mathbf{P}$ *is a Gorenstein toric Fano variety, and* $\overline{Z} := \alpha(\tilde{V})$ *is an ample hypersurface in* $\mathbf{P}$.

There is the following explicit formula for $E_{\mathrm{st}}(V; u, v)$ in terms of $E_{\mathrm{st}}(\mathbf{P}; u, v)$ and $E_{\mathrm{st}}(\overline{Z}; u, v)$:

Theorem 4.10

$$E_{\mathrm{st}}(V; u, v) = ((uv - 1)((uv)^r - 1)^{-1})E_{\mathrm{st}}(\mathbf{P}; u, v) - (uv)^{1-r}E_{\mathrm{st}}(\mathbf{P}\backslash\overline{Z}; u, v) .$$

Proof. Since V is transversal to all toric strata in $\mathbf{X}$ we have:

$$E_{\mathrm{st}}(V; u, v) = E_{\mathrm{st}}(\mathbf{X}; u, v) - E_{\mathrm{st}}(\mathbf{X}\backslash V; u, v) .$$

Using the $\mathbf{CP}^{r-1}$-bundle structure of $\mathbf{Y}$ over $\mathbf{X}$, we obtain:

$$E_{\mathrm{st}}(\mathbf{X}; u, v) = ((uv)^r - 1)^{-1}(uv - 1)E_{\mathrm{st}}(\mathbf{Y}; u, v) .$$

By 4.8, we also have

$$E_{st}(\mathbf{X}\backslash V; u, v) = (uv)^{1-r}E_{\mathrm{st}}(\mathbf{Y}\backslash\tilde{V}; u, v) .$$

Since birational crepant toric morphisms do not change string-theoretic Hodge numbers (see [6]), by 4.9, we conclude

$$E_{\mathrm{st}}(\mathbf{Y}; u, v) = E_{\mathrm{st}}(\mathbf{P}; u, v), \quad E_{\mathrm{st}}(\mathbf{Y}\backslash\tilde{V}; u, v) = E_{\mathrm{st}}(\mathbf{P}\backslash\overline{Z}; u, v) . \qquad \square$$

Definition 4.11 *Let* $C \subset \overline{M}_{\mathbf{R}}$ *be a reflexive Gorenstein cone,* $\check{C} \subset \overline{N}_{\mathbf{R}}$ *the dual reflexive Gorenstein cone. We define*

$$\Lambda(C, \check{C}) := \{(m, n) \in \overline{M} \oplus \overline{N} : m \in C,\ n \in \check{C},\ \text{and } \langle m, n \rangle = 0\} .$$

Definition 4.12 *Let* (m, n) *be an element of* $\Lambda(C, \check{C})$. *We define the Eulerian poset* $P_{(m,n)}$ *as the subset of all faces* $C_x \subset C$ *such that* C_x *contains* m *and* $\langle z, n \rangle = 0$ *for all* $z \in C_x$. *We denote by* $\rho(x^*(n))$ *the dimension of the intersection of* C *with the hyperplane* $\langle z, n \rangle = 0$.

Remark 4.13 The dual Eulerian poset $P^*_{(m,n)}$ can be identified with the subset of all faces $C^*_x \subset \check{C}$ such that C^*_x contains n and $\langle m, z \rangle = 0$ for all $z \in C^*_x$.

Theorem 4.14 *Let us set* $\overline{d} = d + r$ *and*

$$A_{(m,n)}(u,v) = \frac{(-1)^{\rho(x^*(n))}}{(uv)^r}(v-u)^{\rho(x(m))}B(P^*_{(m,n)};u,v)(uv-1)^{\overline{d}-\rho(x^*(n))} .$$

Then

$$E_{\rm st}(V;u,v) = \sum_{(m,n)\in\Lambda(C,\check{C})} \left(\frac{u}{v}\right)^{\deg m} A_{(m,n)}(u,v) \left(\frac{1}{uv}\right)^{\deg n}$$

Proof. By Definition 4.3,

$$\begin{aligned} E_{\rm st}(\mathbf{P};u,v) &= \sum_{\hat{0}<x\leqq\hat{1}} (uv-1)^{\rho(x)-1} S(C_x^*, uv) \\ &= \sum_{\hat{0}<x\leqq\hat{1}} (uv-1)^{\rho(x)-1}(uv-1)^{\overline{d}-\rho(x)} T(C_x^*, (uv)^{-1}) \\ &= (uv-1)^{\overline{d}-1} \sum_{\hat{0}<x\leqq\hat{1}} \left(\sum_{n\in Int(C_x^*)\,\cap\,\overline{N}} (uv)^{-\deg n} \right) \\ &= (uv-1)^{\overline{d}-1} \sum_{n\in\partial\check{C}\,\cap\,\overline{N}} (uv)^{-\deg n} , \end{aligned}$$

where $\partial\check{C} = \check{C}\backslash Int(\check{C})$ is the boundary of $\check{C}$. Since $\overline{N}\cap Int(\check{C}) = p + \overline{N}\cap\check{C}$ and $\deg p = r$, we conclude:

$$\begin{aligned} E_{\rm st}(\mathbf{P};u,v) &= (1-(uv)^{-r})(uv-1)^{\overline{d}-1} \sum_{n\in\check{C}\cap\overline{N}} (uv)^{-\deg n} \\ &= ((uv)^r - 1)(uv-1)^{\overline{d}-1} \sum_{n\in Int(\check{C})\cap\overline{N}} (uv)^{-\deg n} . \end{aligned}$$

On the other hand,

$$E_{\rm st}(\mathbf{P}\backslash\overline{Z};u,v) = E_{\rm st}(\mathbf{P};u,v) - E_{\rm st}(\overline{Z};u,v) .$$

By Definition 4.3 and Theorem 3.24,

$$\begin{aligned} &E_{\rm st}(\overline{Z};u,v) \\ &= \sum_{\hat{0}<x\leqq\hat{1}} \left(\frac{(uv-1)^{\rho(x)-1}}{uv} \right) S(C_x^*, uv) \\ &+ \sum_{\hat{0}<x\leqq\hat{1}} \left(\frac{(-1)^{\rho(x)}}{uv} \sum_{m\in C_x\,\cap\,\overline{M}} (v-u)^{\rho(x(m))} B([x(m),x]^*;u,v) \left(\frac{u}{v}\right)^{\deg m} \right) S(C_x^*, uv) \\ &= (uv)^{-1} E_{\rm st}(\mathbf{P};u,v) \\ &+ \sum_{\hat{0}<x\leqq\hat{1}} \left(\frac{(-1)^{\rho(x)}}{uv} \sum_{m\in C_x\cap\overline{M}} (v-u)^{\rho(x(m))} B([x(m),x]^*;u,v) \left(\frac{u}{v}\right)^{\deg m} \right) S(C_x^*, uv) . \end{aligned}$$

By 4.10,

$$\begin{aligned}
&E_{\mathrm{st}}(V;u,v)\\
&= ((uv-1)((uv)^r-1)^{-1}-(uv)^{1-r}+(uv)^{-r})E_{\mathrm{st}}(\mathbf{P},u,v)\\
&+\sum_{\hat{0}<x\leqq\hat{1}}\left(\frac{(-1)^{\rho(x)}}{(uv)^r}\sum_{m\in C_x\cap\overline{M}}(v-u)^{\rho(x(m))}B([x(m),x]^*;u,v)\left(\frac{u}{v}\right)^{\deg m}\right)S(C_x^*,uv)\\
&=(uv)^{-r}(uv-1)^{\overline{d}}\sum_{n\in Int(\check{C})\cap\overline{N}}(uv)^{-\deg n}\\
&+\sum_{\hat{0}<x\leqq\hat{1}}\left(\frac{(-1)^{\rho(x)}}{(uv)^r}\sum_{m\in C_x\cap\overline{M}}(v-u)^{\rho(x(m))}B([x(m),x]^*;u,v)\left(\frac{u}{v}\right)^{\deg m}\right)S(C_x^*,uv)\\
&=\sum_{\hat{0}\leqq x\leqq\hat{1}}\left(\frac{(-1)^{\rho(x)}}{(uv)^r}\sum_{m\in C_x\cap\overline{M}}(v-u)^{\rho(x(m))}B([x(m),x]^*;u,v)\left(\frac{u}{v}\right)^{\deg m}\right)S(C_x^*,uv)\,.
\end{aligned}$$

It remains to use the formula

$$S(C_x^*,uv)=(uv-1)^{\overline{d}-\rho(x)}\sum_{n\,\in\,Int(C_x^*)\cap\overline{N}}(uv)^{-\deg n}\quad(\hat{0}\leqq x\leqq\hat{1})$$

and notice that $\rho(x)=\rho(x^*(n))$ if n is an interior lattice point of C_x^* (see 4.12). □

Theorem 4.15 *Let V be a $(d-r)$-dimensional Calabi–Yau complete intersection defined by a nef-partition $\Delta=\Delta_1+\cdots+\Delta_r$, W a $(d-r)$-dimensional Calabi–Yau complete intersection defined by the dual nef-partition $\nabla=\nabla_1+\cdots+\nabla_r$. Then*

$$E_{\mathrm{st}}(V;u,v)=(-u)^{d-r}E_{\mathrm{st}}(W;u^{-1},v)\,,$$

i.e.,

$$h_{\mathrm{st}}^{p,q}(V)=h_{\mathrm{st}}^{d-r-p,q}(W)\quad 0\leqq p,q\leqq d-r\,.$$

Proof. If we use the duality between two $\overline{d}$-dimensional reflexive Gorenstein cones $C\subset\overline{M}_{\mathbf{R}}$ and $\check{C}\subset\overline{N}_{\mathbf{R}}$ 4.6, then the statement of Theorem follows immediatelly from the explicit formula in 4.14 and from the duality for B-polynomials 2.13. □

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