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# A Construction of $H_4$ without Miracles*

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**Abstract.** We present a new construction of the root system $\mathcal{H}_4$.

## 1. Introduction

The reflection group $H_4$ is the symmetry group of two four-dimensional regular polytopes, the 600-cell and its dual, the 120-cell. It has a noncrystallographic root system $\mathcal{H}_4$ consisting of 120 elements, the vertices of the 600-cell. See Section 8.5 of [C].

In this note we present an explicit construction of $\mathcal{H}_4$ that requires very little in the way of tedious checking, nor much in the way of miracles. It is analogous to constructing a crystallographic root system as the set of short vectors in a suitable lattice. The difference here is that no lattice is available, so as a substitute we use a finitely generated group that is not discrete. Our construction also has the benefit of demonstrating in an obvious way the fact (perhaps not widely known) that $\mathcal{H}_4$ includes a copy of the root system $\mathcal{D}_4$, and hence that there is a corresponding inclusion of the Weyl group $D_4$ as a subgroup of $H_4$.

Before proceeding, we briefly describe what is probably the "standard" construction, as found in Exercise VI.4.12 of [B] and Section 2.13 of [H]. In fact this construction is due to [W], although explicit coordinates for the 120-cell and 600-cell go back at least to Schläfli in the 1850s and Schoute in the 1900s [C, Section 8.9]. One shows that every finite subgroup of $\mathbf{H}^*$ (the multiplicative group of the quaternions) that includes $-1$ is a root system, and then one miraculously produces a suitable 120-element subgroup that meets the requirements, the so-called *icosian group* [CS]. To verify that the icosians do form a group (or simply a root system) is rather tedious.

The icosian group can be rendered less mysterious by noting that the alternating group of degree five has a three-dimensional representation as rotational symmetries of the icosahedron. Lifting this from $SO(3)$ to Spin(3) yields a 120-element group, a

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double cover of the alternating group. However, Spin(3) is isomorphic to the unit-norm subgroup of $\mathbf{H}^*$, so the 120-element group embeds in $\mathbf{H}^*$. Admittedly, this may make the standard construction seem *more* miraculous, not less, and in any case leaves aside the unpleasant task of identifying explicit coordinates.

## 2. A New Construction

Let $A$ be a subring of $\mathbf{R}$ (the real field), and $L$ an $A$-submodule of the Euclidean space $\mathbf{R}^n$ with inner product $\langle\ ,\ \rangle$.

**Proposition.** *If $\Phi$ is any finite subset of $L_2 = \{\alpha \in L : \langle \alpha, \alpha \rangle = 2\}$ that is maximal with respect to the property that $\langle \alpha, \beta \rangle \in A$ for all $\alpha, \beta \in \Phi$, then $\Phi$ is a root system.*

*Proof.* Let $\alpha, \beta \in \Phi$. The reflection of $\alpha$ through the hyperplane orthogonal to $\beta$ is $\gamma = \alpha - \langle \alpha, \beta \rangle \beta$. Hence $\gamma \in L_2$, since $L$ is an $A$-module and reflections preserve length. Furthermore, the inner product of $\gamma$ with any other member of $\Phi$ is clearly in $A$, since $\alpha$ and $\beta$ have this property. Therefore $\gamma \in \Phi$ by maximality. Hence every reflection through a hyperplane orthogonal to a member of $\Phi$ permutes $\Phi$, so $\Phi$ is a root system. □

Now, to construct $\mathcal{H}_4$, let $a = 2\cos(\pi/5) = (1+\sqrt{5})/2$ denote the golden ratio, $A = \mathbf{Z}[a]$, and $\varepsilon_1, \dots, \varepsilon_4$ an orthonormal basis of $\mathbf{R}^4$. Let

$$\begin{aligned} L &= \{a_1\varepsilon_1 + \dots + a_4\varepsilon_4 : a_i \pm a_j \in A \text{ for all } i, j\} \\ &= \{\alpha \in \mathbf{R}^4 : \langle \alpha, \beta \rangle \in A \text{ for all } \beta \in \mathcal{D}_4\}, \end{aligned}$$

where $\mathcal{D}_4 = \{\pm\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le 4\}$ denotes the usual realization of the root system for the Weyl group $D_4$.

There are only finitely many members of $L_2$. Indeed, every member of $L$ has the form

$$\alpha = \tfrac{1}{2}(m_1 + an_1)\varepsilon_1 + \dots + \tfrac{1}{2}(m_4 + an_4)\varepsilon_4,$$

with $m_i, n_i \in \mathbf{Z}$, $m_1 = \dots = m_4$ mod 2 and $n_1 = \dots = n_4$ mod 2, and the Diophantine equation $\langle \alpha, \alpha \rangle = 2$ involves a positive definite quadratic form in the variables $m_i, n_i$. In fact, it is easy to check that the members of $L_2$ consist of all signed permutations of

$$\begin{aligned} \alpha_1 &= \varepsilon_1 + \varepsilon_2, \\ \alpha_2 &= \tfrac{1}{2}(1-a)(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + \tfrac{1}{2}(1+a)\varepsilon_4, \\ \alpha_3 &= \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + (a - \tfrac{1}{2})\varepsilon_4, \\ \alpha_4 &= \tfrac{1}{2}a(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + (\tfrac{1}{2}a - 1)\varepsilon_4, \end{aligned}$$

for a total of $24 + 64 + 64 + 64 = 216$ vectors.

We cannot fit all of $L_2$ into a single root system of the sort described by the proposition, since it is not true that $\langle \alpha, \beta \rangle \in A$ for all $\alpha, \beta \in L_2$. For if $\alpha = a_1\varepsilon_1 + \dots + a_4\varepsilon_4$ is any member of $L_2$, then the result of negating the coordinate $\varepsilon_1$ yields $\alpha' = \alpha - 2a_1\varepsilon_1 \in L_2$.

Hence $\langle\alpha,\alpha'\rangle = 2-2a_1^2$, and therefore $\langle\alpha,\alpha'\rangle \in A$ if and only if $a_1^2 \in A/2$. However, one can easily check that the coordinates of $\alpha_2, \alpha_3, \alpha_4$ do not have this property, so any subset of $L_2$ whose pairwise inner products belong to $A$ can contain at most half of the signed permutations of $\alpha_2$, $\alpha_3$, and $\alpha_4$, leaving a maximum size of $24+32+32+32 = 120$.

On the other hand, the inner product of every $\alpha \in \mathcal{D}_4$ with every $\beta \in L_2$ belongs to $A$, by construction of $L$. Hence every subset $\Phi$ of $L_2$ that is maximal with respect to $A$-valued inner products must necessarily include all of $\mathcal{D}_4$, and must also form a root system, by the proposition. It follows that the root system $\Phi$ must be a union of $D_4$-orbits, with $D_4$ acting as an index-two subgroup of the group of all signed permutations of the coordinates. Therefore, by checking that the pairwise inner products of $\alpha_2, \alpha_3, \alpha_4$ belong to $A$, we may immediately conclude that

$$\mathcal{H}_4 := \mathcal{D}_4 \ \cup\ D_4\alpha_2 \ \cup\ D_4\alpha_3 \ \cup\ D_4\alpha_4$$

is a root system of order 120. A set of simple roots is given by $\varepsilon_2-\varepsilon_1$, $\varepsilon_3-\varepsilon_2$, $\varepsilon_4-\varepsilon_3$, and $\frac{1}{2}(a+1)\varepsilon_1 - \frac{1}{2}(a-1)(\varepsilon_2+\varepsilon_3+\varepsilon_4)$, as can be verified by checking that the matrix of inner products is consistent with the geometry implied by the Coxeter diagram of $H_4$.

## References


[B] N. Bourbaki, *Groupes et Algèbres de Lie*, Chapters IV–VI, Masson, Paris, 1981.

[CS] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 2nd edn., Springer-Verlag, New York, 1993.

[C] H. S. M. Coxeter, *Regular Polytopes*, 3rd edn., Dover, New York, 1974.

[H] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press, Cambridge, 1990.

[W] E. Witt, Spiegelungsgruppen und Aufzählung halbeinfacher Liescher Ringe, *Abh. Math. Sem. Univ. Hamburg* **14** (1941), 289–322.


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