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Theory of Residence-Time Control by Output Feedback

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Abstract. The problem of residence-time control by the observer-based output feedback is formulated and solved for the case of linear systems with small additive input noise. Both noiseless and noisy measurements are considered. In the noiseless measurements case, it is shown that the fundamental bounds on the achievable residence time depend on the nonminimum phase zeros of the system. In the noisy measurements case, the achievable residence time is shown to be always bounded, and an estimate of this bound is given. Controller design techniques are presented. The development is based on the asymptotic large deviations theory.

1. Introduction

Consider the following Ito stochastic system

$$\begin{aligned} dx &= (Ax + Bu)dt + \epsilon C dw \\ y &= Dx, \end{aligned} \tag{1}$$

where $x \in I\!R^n$, $u \in I\!R^m$, $y \in I\!R^p$, $w(t)$ is a standard r-dimensional Brownian motion, A, B, C, D are matrices of appropriate dimensionality, and $0 < \epsilon \ll 1$. For a given u, the behavior of system (1) in a bounded domain $\Psi \subset I\!R^p$ can be characterized by the first passage time (Freidlin and Wentzell, 1984)

$$\bar{\tau}^\epsilon(u) = \inf\{t \geq 0 : y(t,u) \in \partial\Psi | y(0,u) \in \Psi\}$$

($\partial\Psi$ is the boundary of Ψ), or by its average value

$$\bar{\tau}^\epsilon(u) = E[\tau^\epsilon(u)].$$

The $\bar{\tau}^\epsilon(u)$ is referred to as the (average) residence time of system (1) in Ψ.

Assume that control specifications of system (1) are given in the form of an aiming (pointing) problem: maintain $y(t)$ in a given domain $\Psi \subset I\!R^p$ during a specified time interval $[0,T]$, $T < \infty$. In terms of the average residence time, this problem has the form

$$\bar{\tau}^{\epsilon}(u) \geq T. \tag{2}$$

Technical examples of this problem can be found elsewhere (Meerkov and Runolfsson, 1988).

To accomplish problem (2) the feedback control approach can be utilized. Meerkov and Runolfsson (1988, 1989) address this problem under the assumption that all states x are available for control and u is chosen as

$$u = Kx. \tag{3}$$

In Meerkov and Runolfsson (1988), it was assumed that $D = I$, i.e., the pointing of states has been considered, and the general case of output aiming has been analyzed in Meerkov and Runolfsson (1989). It has been shown that, from the point of view of satisfying problem (2), all systems (1) can be partitioned into two groups: weakly and strongly residence-time controllable. Roughly speaking, system (1) is weakly residence-time controllable (*wrt*-controllable) if there exists $T^* < \infty$ such that the closed-loop system (1),(3) satisfies problem (2) for $T < T^*$ and some K and does not satisfy problem (2) for $T > T^*$ and any K. System (1) is strongly residence-time controllable (*srt*-controllable) if $T^* = \infty$. It has been shown (Meerkov and Runolfsson, 1988) that system (1) with $D = I$ is *wrt*-controllable if and only if (A, B) is stabilizable and *srt*-controllable if and only if $\operatorname{Im} C \subseteq \operatorname{Im} B$. It has been shown (Meerkov and Runolfsson, 1989) that system (1) *wrt*-controllable in states can, if fact, be *srt*-controllable in outputs $y \neq x$. In particular, it was shown that a single input-single output (SISO) system (1) is *srt*-controllable if and only if all nonminimum phase zeros of $G_s(s) \triangleq D(sI - A)^{-1}B$ coincide with nonminimum phase zeros of $G_n(s) \triangleq D(sI - A)^{-1}C$. This means, of course, that minimum phase plants are pointable with any precision whereas nonminimum phase ones may or may not be, depending on the location of the right half plane zeros of $G_n(s)$.

In the present article, we address problem (2) under the assumption that only (measured) outputs are available for control and, therefore, the output feedback has to be utilized. To simplify the problem, we consider here the observer-based output feedback, i.e., controllers of the form

$$\begin{aligned} u &= K\hat{x} \\ \dot{\hat{x}} &= A\hat{x} + Bu + L(z - E\hat{x}) \end{aligned} \tag{4}$$

if the measured output,

$$z = Ex, \; z \in \mathbb{R}^q, \; E \in \mathbb{R}^{q \times n},$$

is noise free, or of the form

$$\begin{aligned} u &= K\hat{x} \\ d\hat{x} &= (A\hat{x} + Bu)dt + L(dz - E\hat{x}dt) \end{aligned} \tag{5}$$

if the measured output,

$$dz = Exdt + \epsilon F dw_1,$$

is noisy. Here $w_1(t)$ is a q-dimensional standard Brownian motion and $0 < \epsilon \ll 1$. In each case, (4) and (5), the problem is to choose the pair (K,L) so that problem (2) is satisified.

To this end, in this article we derive the following results:

1. System (1) with feedback (4) is *srt*-controllable if and only if the system is invertible and minimum phase in an appropriate sense.
2. If this is not the case, the maximal achievable residence time T^* for system (1), (4) coincides with that for system (1), (3) if and only if $G_{nl}(s) \triangleq E(sI - A)^{-1}C$ is left invertible and minimum phase; otherwise the output controllers lead to a smaller residence time.
3. System (1) with feedback (5) is never *srt*-controllable. Thus, the measurement noise has a much more severe effect on the residence time than the input noise.
4. The observer gain L that ensures the largest possible residence time in system (1), (5) coincides with that of the corresponding Kalman filter. Thus, the Kalman filter is optimal not only with respect to the standard performance measure, i.e., the mean-square estimation error, but also from the point of view of the residence time.
5. The feedback gain K that ensures the largest possible residence time in system (1), (5) is dependent on the optimal value of L mentioned above. Thus, although the separation principle does not take place, the situation here can be characterized as semiseparation: the optimal observations do not depend on optimal control, but the optimal control does depend on optimal observations. As a result, the maximal achievable residence time for controllers derived in this article is larger than that for LQG-designed systems.

The remainder of this article is organized as follows. In section 2, some mathematical preliminaries are discussed. In sections 3 to 5 system (1) with controllers (4) and (5), respectively, is considered, and in section 6 an illustrative example is given. In section 7, the conclusions are formulated. The proofs are given in the appendix.

2. Preliminaries

In this section, the notion of logarithmic residence time, i.e., the main tool of asymptotic analysis of system (1) with controllers (3)–(5), is introduced and utilized for a precise formulation of problem (2).

Consider the linear Ito system

$$\begin{aligned} dx &= Axdt + \epsilon C dw \\ y &= Dx \end{aligned} \tag{6}$$

where, as before, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $w(t)$ is a standard r-dimensional Brownian motion and $0 < \epsilon \ll 1$. Let $\Psi \subset \mathbb{R}^p$ be again a bounded domain with the origin in its interior and a smooth boundary $\partial\Psi$. Define

$$\Omega_0 \triangleq \{x \in \mathbb{R}^n \colon y = Dx \in \Psi\}, \tag{7}$$

$$\Omega \triangleq \{x \in \mathbb{R}^n \colon De^{At}x \in \Psi,\ t \geq 0\}. \tag{8}$$

Assume that $x(0) = x_0 \in \Omega_0$ and introduce the first passage time as

$$\tau^\epsilon(x_0) \triangleq \inf\{t \geq 0\colon y(t, x_0) \in \partial\Psi\}, \tag{9}$$

where $y(t, x_0)$ is the solution of system (6). The following theorem was proved in Meerkov and Runolfsson (1989).

Theorem 1. Suppose A is Hurwitz and (A, C) is disturbable, i.e., rank $[C\ AC \ldots A^{n-1}C] = n$. Then uniformly for all x_0 belonging to compact subsets of Ω, we have

$$\lim_{\epsilon \to 0} \epsilon^2 \ln \bar{\tau}^\epsilon (x_0) = \hat{\mu}, \tag{10}$$

where, as before, $\bar{\tau}^\epsilon(x_0) = E_{x_0} \tau^\epsilon (x_0)$ and

$$\begin{aligned} \hat{\mu} &= \min_{y \in \partial\Psi} \frac{1}{2} y^T N y, \\ N &= (DXD^T)^{-1},\ AX + XA^T + CC^T = 0. \end{aligned} \tag{11}$$

Constant $\hat{\mu}$ is referred to as the logarithmic residence time of system (6) in Ψ.
Let $\bar{y}(t, x_0, \hat{x}_0, K, L)$ be the solution of the deterministic system

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} &= \begin{bmatrix} A & BK \\ LE & A + BK - LE \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}, \quad \begin{bmatrix} x(0) \\ \hat{x}(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ \hat{x}_0 \end{bmatrix}, \\ y &= Dx \end{aligned} \tag{12}$$

and define

$$\Omega(K,L) = \left\{ \begin{bmatrix} x_0 \\ \hat{x}_0 \end{bmatrix} \in \mathbb{R}^{2n} \middle| \bar{y}(t, x_0, \hat{x}_0, K, L) \in \Psi,\ t \geq 0 \right\}. \tag{13}$$

Then, with regard to control system (1) and controllers (4) or (5), theorem 1 allows us to conclude that for sufficiently small ϵ and $[x_0 \hat{x}_0]^T \in \Omega(K,L)$, problem (2) can be replaced by an alternative problem of selecting the pair (K,L) such that

$$\hat{\mu}(\Psi;K,L) > \mu \tag{14}$$

where $\hat{\mu}(\Psi;K,L)$ is the logarithmic residence time of the closed-loop system (1), (4) or (1), (5) and $\mu = \epsilon^2 \ln T$. This is the problem solved in this article.

As it was pointed out in the introduction, the solution of this problem is given in terms of the weak and strong residence-time controllability defined precisely below. In order to simplify the notations, we drop argument Ψ in problem (14).

Definition 1. 1. System (1) is called weakly residence-time controllable if for any bounded domain $\Psi \subset \mathbb{R}^p$ $(0 \in \Psi)$ there exists controller (4) (or (5)) such that $\hat{\mu}(K,L) > 0$; 2. System (1) is said to be strongly residence-time controllable if for any bounded $\Psi \subset \mathbb{R}^p (0 \in \Psi)$ and $\mu > 0$ there exists controller (4) (or (5)) such that $\hat{\mu}(K,L) > \mu$.

In what follows, we make the following assumptions:

Assumption 1. (A,C) is disturbable.

Assumption 2. (D,A) is detectable.

Assumption 3. $FF^T > 0$, and $w(t)$ and $w_1(t)$ are independent Brownian motions.

Assumption 4. Transfer matrices $G_s(s) = D(sI-A)^{-1}B$, $G_n(s) = D(sI-A)^{-1}C$ and $G_{n1}(s) = E(sI-A)^{-1}C$ have full normal rank.

3. Noiseless measurements case

Let $\mathcal{K} \triangleq \{K \in \mathbb{R}^{m\times n}: A + BK \text{ is Hurwitz}\}$, $\mathcal{L} \triangleq \{L \in \mathbb{R}^{n\times p}: A - LE \text{ is Hurwitz}\}$ and define the maximal logarithmic residence time of system (1), (4) or system (1), (5) in Ψ as

$$\hat{\mu}^* = \sup_{\substack{K\in\mathcal{K}\\ L\in\mathcal{L}}} \hat{\mu}(K,L). \tag{15}$$

Introduce the following hypotheses:

Hypothesis 1. $G_s(s)$ is right invertible and minimum phase.

Hypothesis 2. $G_{n1}(s)$ is left invertible and minimum phase.

Hypothesis 3. There exists an $m \times r$ rational matrix $U(s)$ with no poles in Re $s > 0$ such that $G_n(s) + G_s(s)U(s) = 0$.

Hypothesis 4. There exists a $p \times q$ rational matrix $V(s)$ with no poles in Re $s > 0$ such that $G_n(s) + V(s)G_{n1}(s) = 0$.

Theorem 2. System (1) is
1. weakly residence-time controllable by controller (4) if and only if (A,B) is stabilizable and (E,A) is detectable,
2. strongly residence-time controllable by controller (4) if and only if (A,B) is stabilizable, (E,A) is detectable, and either hypotheses 1 and 4 or hypotheses 2 and 3 are satisfied.

Proof. See the appendix.

Remark 1. As was shown in Meerkov and Rudolfsson (1989), hypothesis 3 is the condition for strong residence-time controllability with respect to the state-space feedback $u = Kx$. Furthermore, hypothesis 1 is a stronger condition than hypothesis 3. Thus, either hypothesis 4 or hypothesis 2 is the additional condition that has to be satisfied when the state-space feedback is replaced by the output feedback.

Remark 2. In the SISO case with $D = E$, theorem 2 implies that for strong residence-time controllability, $G_s(s)$ should be minimum phase.

A comparison of the fundamental bounds on the residence time achievable by state space (3) and output (4) feedback can be given as follows:

Consider the closed-loop system (1), (3), i.e.,

$$dx = (A + BK)x\,dt + \epsilon C dw, \tag{16}$$

and define as

$$\mu^* = \sup \mu(\Psi;K) \tag{17}$$

its maximal logarithmic residence time in Ψ.

Theorem 3. Equality $\hat{\mu}^* = \mu^*$ takes place if and only if $G_{nl}(s)$ has a left inverse with no poles in Re $s > 0$.

Proof. See the appendix.

4. Noisy measurements case

Theorem 4. Let P be the unique positive definite solution of the (Kalman filter) Riccati equation:

$$AP + PA^T + CC^T - PE^T(FF^T)^{-1}EP = 0. \tag{18}$$

Then the maximal logarithmic residence time of the closed-loop system (1), (5) in Ψ satisfies the bound

$$\hat{\mu}^* \leq \min_{y \in \partial \Psi} \frac{1}{2} y^T (DPD^T)^{-1} y. \tag{19}$$

Proof. See the appendix.

Remark. It follows, in particular, from theorem 4 that since the upper bound in expression (19) is always finite, system (1) with control (5) is never strongly residence-time controllable. Therefore, the measurement noise in control (5) has a greater limiting effect on the achievable residence time than the input noise in system (1).

Theorem 5. The upper bound in expression (19) is attained if and only if there exists a rational matrix $W(s)$ with no poles in Re $s > 0$ such that

$$G_l(s) + G_s(s) W(s) = 0, \tag{20}$$

where $G_s(s)$ is defined as previously and

$$G_l(s) = D(sI - A)^{-1} \hat{L}, \tag{21}$$

$$\hat{L} = PE^T (FF^T)^{-1}. \tag{22}$$

Proof. See the appendix.

Remark. Theorem 5 illustrates that the upper bound in expression (19) is attainable. Therefore, it is the best possible upper bound.

5. Design techniques

In the two previous sections, we have characterized the fundamental bounds on the achievable logarithmic residence time. In this section we develop the controller design techniques that achieve these bounds. First system (1) with control (5) is considered and then system (1) with control (4) is addressed. An example is given in section 6.

To select the pair $\{K,L\}$ that maximizes $\hat{\mu}(K,L)$, assume for simplicity that domain Ψ is an ellipsoid

$$\Psi = \{y \in I\!R^p: y^T S y \leq r^2, S = S^T > 0\}. \tag{23}$$

Let $W \in I\!R^{p \times p}$ be a nonsingular matrix such that $S = W^T W$. Then by direct calculations we obtain

$$\hat{\mu}(K,L) = \frac{r^2}{2\lambda_{\max}[W\, DX(K,L) D^T W^T]}, \tag{24}$$

where $X(K,L)$ is given by

$$\begin{bmatrix} A & BK \\ LE & A + BK - LE \end{bmatrix} \begin{bmatrix} X(K,L) & T(K,L) \\ T^T(K,L) & \hat{X}(K,L) \end{bmatrix} \tag{25}$$
$$+ \begin{bmatrix} X(K,L) & T(K,L) \\ T^T(K,L) & \hat{X}(K,L) \end{bmatrix} \begin{bmatrix} A & BK \\ LE & A + BK - LE \end{bmatrix}^T + \begin{bmatrix} CC^T & 0 \\ 0 & LFF^TL^T \end{bmatrix} = 0.$$

Therefore, the pair $\{K,L\}$ is optimal if and only if it minimizes the largest eigenvalue of $\Gamma(K,L) \triangleq W\,DX(K,L)D^TW^T$. The $\lambda_{\max}(\Gamma)$ can be characterized as follows:

Lemma 1. Let $\theta \geq 0$ be a scalar, $l \geq 1$ be an integer, and select $K_l \in \mathcal{K}$ and $L_l \in \mathcal{L}$ such that

$$\mathrm{Tr}\ \Gamma(K_l, L_l)^l \leq (1 + \theta) \inf \{\mathrm{Tr}\ \Gamma(K,L)^l | K \in \mathcal{K}, L \in \mathcal{L}\}. \tag{26}$$

Then

$$\lim_{l\to\infty} \lambda_{\max}(\Gamma(K_l,L_l)) = \inf\{\lambda_{\max}(\Gamma(\mathrm{K},,L))|K \in \mathcal{K}, L \in \mathcal{L}\}. \tag{27}$$

Proof. The proof of this lemma is similar to the proof of theorem 2.1 in Allwright and Mao (1982). We omit the details here.

Thus, in order to minimize $\lambda_{\max}(\Gamma)$, we need only to minimize $\mathrm{Tr}\ \Gamma(K,L)^l$, $l = 1,2,3....$ To accomplish this, introduce

$$J^l_\gamma(K,L) = \mathrm{Tr}\ \Gamma(K,L)^l + \gamma\ \mathrm{Tr}\ K\hat{X}(K,L)K^T, \tag{28}$$

where $\hat{X}(K,L)$ is given by equation (25).

Lemma 2. Assume that $K^\gamma_l \in \mathcal{K}$ and $L^\gamma_l \in \mathcal{L}$ minimize $J^l_\gamma(K,L)$. Then

$$\lim_{\gamma\to 0} J^l_\gamma(K^\gamma_l, L^\gamma_l) = \inf_{\substack{K\in\mathcal{K} \\ L \in \mathcal{L}}} \mathrm{Tr}\ \Gamma(K,L)^l. \tag{29}$$

Proof. The proof of this lemma is similar to the first part of the proof of the theorem in Kwakernaak and Sivan (1972). We omit the details here.

From lemmas 1 and 2 follow corollary 1.

Corollary 1. Assume that the pair (K^γ_l, L^γ_l) with $K^\gamma_l \in \mathcal{K}$ and $L^\gamma_l \in \mathcal{L}$ minimizes $J^l_\gamma(K,L)$. Then

$$\lim_{l\to\infty} \lim_{\gamma\to 0} \hat{\mu}(K_l^\gamma, L_l^\gamma) = \hat{\mu}^*. \tag{30}$$

Thus, K_l^γ and L_l^γ provide the solution to equation (15). A necessary condition for the optimality of (K_l^γ, L_l^γ) in the sense of functional (28) can be formulated as follows.

Theorem 6. Assume that $K_l^\gamma \in \mathcal{K}$ and $L_l^\gamma \in \mathcal{L}$. Then in order for (K_l^γ, L_l^γ) to minimize $J_\gamma^l(K,L)$ it is necessary that

$$L_l^\gamma = \hat{L} = PE^T(FF^T)^{-1}, \tag{31}$$

$$K_l^\gamma = -\frac{1}{\gamma} B^T Q_l^\gamma, \tag{32}$$

where P is given by equation (18) and

$$A^T Q_l^\gamma + Q_l^\gamma A + D^T W^T M_l^\gamma WD - \frac{1}{\gamma} Q_l^\gamma BB^T Q_l^\gamma = 0, \tag{33}$$

$$M_l^\gamma = l(W\, D(\hat{X}_l^\gamma + P)D^T W^T)^{l-1}, \tag{34}$$

$$(A + BK_l^\gamma)\, \hat{X}_l^\gamma + \hat{X}_l^\gamma (A + BK_l^\gamma)^T + \hat{L}\hat{L}^T = 0. \tag{35}$$

Proof. See the appendix.

Thus, in particular, the optimal observation gain is independent of optimal control, while the optimal control gain is a function of optimal observations.

Since equation (18) has a positive solution, $L_l^\gamma = \hat{L} \in \mathcal{L}$, $\forall\, \gamma, l$. The following lemma gives a condition for $K_l^\gamma \in \mathcal{K}$.

Lemma 3. Assume that $M_l^\gamma > 0$. Then $K_l^\gamma \in \mathcal{K}$.

Proof. See the appendix.

Remark. As follows from theorem 6, the optimal estimator gain $\hat{L}$ given in equation (31) is the Kalman filter gain. Thus, the Kalman filter is optimal in optimization problem (15). Moreover, consider the equation for the estimation error $e \overset{\Delta}{=} x - \hat{x}$:

$$de = (A - LE)dt + \epsilon C dw - \epsilon LF dw_1 \tag{36}$$

and define its logarithmic residence time in any domain $\Lambda \subset \mathbb{R}^n (0 \in \Lambda)$ as $\hat{\mu}(\Lambda; L)$. Then

$$\hat{\mu}(\Lambda; L) = \min_{e \in \partial\Lambda} \frac{1}{2} e^T P^{-1}(L) e, \tag{37}$$

where $P(L)$ is the positive definition solution of

$$(A - LE)P(L) + P(L)(A-LE)^T + CC^T + LFF^TL^T = 0. \quad (38)$$

Since P given by equation (18) satisfies the inequality

$$P \leq P(L), \ \forall L \in \mathcal{L}, \quad (39)$$

we conclude that

$$\hat{\mu}(\Lambda;\hat{L}) = \min_{e \in \partial\Lambda} \frac{1}{2} e^T P^{-1} e \geq \hat{\mu}(\Lambda;L), \ \forall L \in \mathcal{L}. \quad (40)$$

Thus, the Kalman filter is optimal in the sense of optimization of the estimation error residence time in every bounded domain of $I\!R^n$.

The optimal control law for system (1) with control (4) can be obtained from equations (31) and (32) by selecting $F = \alpha I$ and letting $\alpha \to 0$. Indeed, since the optimal estimator law for system (1), (5) is the Kalman filter, we know from optimal filtering theory that the optimal (singular) filter for system (1), (4) is obtained in the limit $\alpha \to 0$ (see, e.g., Kwakernaak and Sivan, 1972). Therefore, the maximal logarithmic residence time for system (1), (4) is given by

$$\hat{\mu}^* = \lim_{l \to \infty} \lim_{\gamma \to 0} \lim_{\alpha \to 0} \hat{\mu}(K_l^{\gamma,\alpha}, L_l^{\gamma,\alpha}), \quad (41)$$

where $L_l^{\gamma,\alpha}$ and $K_l^{\gamma,\alpha}$ are given by equations (31)–(35) with $FF^T = \alpha^2 I$.

6. Example

Consider the second-order system

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \epsilon \begin{bmatrix} 1 \\ 0 \end{bmatrix} \dot{w}, \\ y &= [0 \quad 1]x, \\ z &= [1 \quad 0]x + \epsilon F \dot{w}_1. \end{aligned} \quad (42)$$

For this system,

$$G_s(s) = \frac{s}{s^2 + 1}, \quad G_n(s) = \frac{-1}{s^2 + 1}, \quad G_{n1}(s) = \frac{s}{s^2 + 1}. \quad (43)$$

Thus, since $G_s(s) = G_{nl}(s)$ is minimum phase, this system is *srt*-controllable by controller (4) when $F \equiv 0$.

Assume that $F \neq 0$. Then, by theorem 4, the logarithmic residence time in the inverval $\Psi = (-a,b)$, $a,b > 0$, is bounded by

$$\min_{y \in \partial\Psi} \frac{1}{2} y^T (DPD^T)^{-1} y = \frac{(\min(a,b))^2}{2|F|}. \tag{44}$$

Furthermore, when $a = b$, the (sub)optimal controller can be calculated using equations (31)–(35) to be

$$\hat{L} = \begin{bmatrix} \frac{1}{|F|} \\ 0 \end{bmatrix}, \; K_l^\gamma = -[0 \quad K_2] \tag{45}$$

where $K_2 > 0$ satisfies the equation

$$\frac{K_2^2 \gamma}{l|F|^{l-1}} = \left(1 + \frac{|F|}{2K_2} \right)^{l-1}. \tag{46}$$

The logarithmic residence time with this control is

$$\hat{\mu}(K_l^\gamma, \hat{L}) = \frac{a^2}{2|F|} \cdot \frac{2K_2}{2K_2 + |F|}. \tag{47}$$

Note that $\hat{\mu}(K_l^\gamma, \hat{L})$ is the upper bound in equation (44) multiplied by the factor

$$\rho = \frac{2K_2}{2K_2 + |F|}. \tag{48}$$

Thus, in order to obtain logarithmic residence time as close as desired to the maximal value, equations (44) and (48) can be used to calculate the necessary K_2 (for a given ρ) and l and γ can be determined from equation (46).

As $\gamma \to 0$, equation (46) simplifies considerably. Indeed, in this case $K_2 \to \infty$ and, thus, for small γ, equation (46) becomes

$$\frac{K_2^2 \gamma}{l|F|^{l-1}} \simeq 1. \tag{49}$$

Therefore,

$$K_2 \simeq \sqrt{\frac{l}{\gamma}} \, |F|^{\frac{l-1}{2}}. \tag{50}$$

7. Conclusions

It is shown in this article that the observer-based output feedback can be efficiently used for pointing of linear systems subject to both input and measurement noise. The fundamental bounds on the achievable precision of pointing depend on the locations of the right half plane zeros of the various transfer functions involved. Roughly speaking, the best precision of pointing is obtained for minimum-phase systems. Any desired precision of aiming is attainable only if no measurement noise is present. Therefore, the effect of the measurement noise on the achievable precision of aiming is more detrimental than that of the input noise.

Appendix

Proof of theorem 2. The proof of point 1 in theorem 2 parallels the proof of theorem 3.1 in Meerkov and Runolfsson (1989). We omit the details here. In order to prove point 2, we first derive the inequality

$$\frac{r^2}{2 \operatorname{Tr} DX(K,L)D^T} \leq \hat{\mu}(K,L) \leq \frac{pR^2}{2 \operatorname{Tr} DX(K,L)D^T}, \tag{51}$$

where $K \in \mathcal{K}$, $L \in \mathcal{L}$. To get the left inequality, note that

$$\begin{aligned}\hat{\mu}(K,L) &\geq \frac{1}{2} \lambda_{\min}[(DX(K,L)D^T)^{-1}] \min_{y \in \partial\Psi} y^T y \\ &= \frac{r^2}{2\lambda_{\max}[DX(K,L)D^T]} \\ &\geq \frac{r^2}{2 \operatorname{Tr} DX(K,L)D^T}.\end{aligned}$$

For the right inequality, we have ($R^2 = \max_{y \in \partial\Psi} y^T y$, $B(0,R) = \{y \mid y^T y \leq R^2\}$)

$$\begin{aligned}\hat{\mu}(K,L) &= \min_{y \in \partial\Psi} \frac{1}{2} y^T (DX(K,L)D^T)^{-1} y \\ &\leq \min_{y \in \partial B(0,R)} \frac{1}{2} y^T (DX(K,L)D^T)^{-1} y \\ &= \frac{1}{2} \lambda_{\min}[(DX(K,L)D^T)^{-1}] R^2\end{aligned}$$

$$= \frac{R^2}{2\lambda_{\max}[DX(K,L)D^T]}$$

$$\leq \frac{pR^2}{2 \text{ Tr } DX(K,L)D^T} .$$

It follows from inequality (51) that $\hat{\mu}^* = \infty$ is equivalent to

$$\inf_{\substack{K \in \mathcal{K} \\ L \in \mathcal{L}}} \text{Tr } DX(K,L)D^T = 0. \tag{52}$$

Next note that it follows from linear quadratic theory (Kwakernaak and Sivan, 1972; Russell, 1979) that

$$\inf_{\substack{K \in \mathcal{K} \\ L \in \mathcal{L}}} \text{Tr } DX(K,L)D^T = \lim_{\substack{\gamma \to 0 \\ \alpha \to 0}} \text{Tr } DX(K^\gamma,L^\gamma)D^T \tag{53}$$

where

$$K^\gamma = -\frac{1}{\gamma} B^TQ^\gamma, \; A^TQ^\gamma + Q^\gamma A + D^TD - \frac{1}{\gamma} Q^\gamma BB^TQ^\gamma = 0, \tag{54}$$

$$L^\alpha = -\frac{1}{\alpha} P^\alpha E^T, \; AP^\alpha + P^\alpha A + CC^T - \frac{1}{\alpha} P^\alpha E^TEP^\alpha = 0, \tag{55}$$

Furthermore,

$$\lim_{\substack{\gamma \to 0 \\ \alpha \to 0}} \text{Tr } DX(K^\gamma,L^\gamma)D^T = \lim_{\substack{\gamma \to 0 \\ \alpha \to 0}} \text{Tr } (DP^\alpha D^T + \alpha L^{\alpha^T} Q^\gamma L^\alpha)$$

$$= \lim_{\substack{\gamma \to 0 \\ \alpha \to 0}} \text{Tr } (C^TQ^\gamma C + \gamma\, K^\gamma P^\alpha K^{\gamma^T}). \tag{56}$$

Therefore, with $\tilde{C} = \lim_{\alpha \to 0} \sqrt{\alpha}\; L^\alpha$ and $\hat{D} = \lim_{\alpha \to 0} \sqrt{\gamma}\; K^\gamma$, we have

$$\inf_{\substack{K \in \mathcal{K} \\ L \in \mathcal{L}}} \text{Tr } DX(K,L)D^T = \text{Tr } (DP^0D^T + \tilde{C}^TQ^0\tilde{C})$$

$$= \text{Tr } (\hat{D}P^0\hat{D}^T + C^TQ^0C). \tag{57}$$

Each of the terms Tr DP^0D^T, Tr $\tilde{C}^TQ^0\tilde{C}$, Tr $\hat{D}P^0\hat{D}^T$, and Tr C^TQ^0C is nonnegative. Thus system (1) with control (4) is strongly residence-time controllable if and only if all four terms are zero.

It was shown in Meerkov and Runolfsson (1989) that Tr $C^TQ^0C = 0$ if and only if there exists a rational matrix $U(s)$, with no poles in Re $s > 0$, such that

$$G_n(s) + G_s(s)U(s) = 0. \tag{58}$$

Similarly, Tr $\tilde{C}^T Q^0 \tilde{C} = 0$, Tr $DP^0D^T = 0$, and Tr $\hat{D}P^0\hat{D}^T = 0$ if and only if there exist rational matrices $\tilde{U}(s)$, $V(s)$, and $\hat{V}(s)$, with no poles in Re $s > 0$, such that

$$\tilde{G}_n(s) + G_s(s)\tilde{U}(s) = 0, \tag{59}$$

$$G_n(s) + V(s)G_{n1}(s) = 0, \tag{60}$$

$$\hat{G}_n(s) + \hat{V}(s)G_{n1}(s) = 0, \tag{61}$$

where

$$\tilde{G}_n(s) = D(sI - A)^{-1}\tilde{C}, \tag{62}$$

$$\hat{G}_n(s) = \hat{D}(sI - A)^{-1}C. \tag{63}$$

Now, if hypothesis 1 is satisfied, then

$$U(s) = -G_s^{-1}(s)G_n(s) \text{ and } \tilde{U}(s) = -G_s^{-1}(s)\ \tilde{G}_n(s)\ (G_s^{-1}(s)$$

is the right inverse of $G_s(s)$) are both without poles in Re $s > 0$ and satisfy equations (58) and (59). Therefore, TrC^TQ^0C = Tr$\tilde{C}^TQ^0\tilde{C} = 0$. Furthermore, in this case $D^TD = \hat{D}^T\hat{D}$ (see, e.g., Kwakernaak and Sivan, 1972), and thus hypothesis 4 implies that $0 =$ Tr DP^0D^T $=$ Tr P^0D^TD $=$ Tr $P^0\hat{D}^T\hat{D}$ $=$ Tr $\hat{D}P^0\hat{D}^T$. Therefore, by equation (57) the system is strongly residence-time controllable. Similarly, if hypothesis 3 is satisfied, then $V(s) = -G_n(s)G_{n1}^{-1}(s)$ and $\hat{V}(s) = -G_n(s)G_{n1}^{-1}(s)$ are both without poles in Re $s > 0$ and, thus, Tr DP^0D^T = Tr $\hat{D}P^0\hat{D}^T = 0$. Furthermore, $CC^T = \tilde{C}\tilde{C}^T$ and, therefore, hypothesis 3 implies that $0 =$ Tr C^TQ^0C $=$ Tr $\tilde{C}^TQ^0\tilde{C}$. This proves the sufficiency part of the theorem.

Assume now that system (1), (4) is strongly residence-time controllable. Then equations (58)–(61) are satisfied, and, thus hypotheses 3 and 4 are true. Note that the existence of $U(s)$ such that equation (58) is satisfied and assumption 4 imply that $m \geq \min(p,r)$. Similarly, the existence of $V(s)$ and assumption 4 imply that $q \geq \min(p,r)$. Assume $p \leq r$. Then $m \geq p$ and, thus, $G_s(s)$ is right invertible. Similarly, if $p \geq r$, then $q \geq r$ and $G_{n1}(s)$ is left invertible. Next, it can be shown that equation (60) implies that $G_n(s)G_n^T(-s) = \tilde{G}_n(s)\tilde{G}_n^T(-s)$. Furthermore, $\tilde{G}_n(s)$ has no zeros in Re $s > 0$ (see, e.g., Shaked and Soroka, 1987). Similarly, equation (58) implies that $G_n^T(-s)G_n(s) = \hat{G}_n^T(-s)\hat{G}_n(s)$ and $\hat{G}_n(s)$ has no zeros in Re $s > 0$. Thus, if $p \leq r$, i.e., $G_s(s)$ is right invertible, then it follows from equations (60) and (59) that $G_s(s)$ has no zeros in Re $s > 0$. Thus hypothesis 1 is satisfied. Similarly, if $p \geq r$, then equations (58) and (61) imply that $G_{n1}(s)$ is left invertible and minimum phase, i.e., hypothesis 2 is true. Q.E.D.

Proof of theorem 3. Let $\mu(K)$ be the logarithmic residence time of equation (16). Then, obviously, for any $K \in \mathcal{K}$ and $L \in \mathcal{L}$, we have

$$\hat{\mu}(K,L) \leq \mu(K) \tag{64}$$

and, thus,

$$\sup_{L \in \mathcal{L}} \hat{\mu}(K,L) \leq \mu(K). \tag{65}$$

Furthermore, using a similar argument to the one in the proof of theorem 4 (see below), we have

$$\sup_{L \in \mathcal{L}} \hat{\mu}(K,L) = \lim_{\alpha \to 0} \hat{\mu}(K,L^{\alpha}), \tag{66}$$

$$L^{\alpha} = P^{\alpha}E^{T}, \; AP^{\alpha} + P^{\alpha}A^{T} + CC^{T} - \frac{1}{\alpha} P^{\alpha}E^{T}EP^{\alpha} = 0. \tag{67}$$

Thus, we want to show that left invertibility and minimum phase of $G_{nl}(s)$ is necessary and sufficient for

$$\begin{aligned} \lim_{\alpha \to 0} \hat{\mu}(K,L^{\alpha}) &= \min_{y \in \partial\Psi} \frac{1}{2} y^{T}(D(\hat{X}(K) + P^{0})D^{T})^{-1}y \\ &= \mu(K), \end{aligned} \tag{68}$$

where

$$(A + BK)\hat{X}(K) + \hat{X}(K)(A + BK) + \tilde{C}\tilde{C}^{T} = 0 \tag{69}$$

for all $K \in \mathcal{K}$. However, since

$$\mu(K) = \min \frac{1}{2} y^{T}(DX(K)D^{T})^{-1}y, \tag{70}$$

$$(A + BK)X(K) + X(K)(A + BK)^{T} + CC^{T} = 0, \tag{71}$$

it follows that equation (68) is true if and only if $DP^{0}D^{T} = 0$ and $CC^{T} = \tilde{C}\tilde{C}^{T}$. These are exactly the necessary and sufficient conditions for $G_{nl}(s)$ to be left invertible and minimum phase. Q.E.D.

Proof of theorem 4. It is straightforward to show that $X(K,L) \geq P$ (see, e.g., equation (75) below). Therefore, since

$$\hat{\mu}(K,L) = \min_{y \in \partial\Psi} \frac{1}{2} y^{T}(DX(K,L)D^{T})^{-1}y, \tag{72}$$

inequality (19) follows. Q.E.D.

Proof of theorem 5. The logarithmic residence time in a system with the optimal estimator gain $\hat{L} = PE^{T}(FF^{T})^{-1}$ is

$$\hat{\mu}(K,\hat{L}) = \min_{y \in \partial\Psi} \frac{1}{2} y^T (D(\hat{X}(K) + P)D^T)^{-1} y \tag{73}$$

where

$$(A + BK)\hat{X}(K) + \hat{X}(K)(A + BK)^T + \hat{L}\hat{L}^T = 0. \tag{74}$$

Thus, the upper bound (19) is attained if and only if $\inf_{K \in \mathcal{K}} \text{Tr } D\hat{X}(K)D^T = 0$. However, by the same argument as was used in the proof of theorem 2, this happens if and only if equation (20) is satisfied. Q.E.D.

Proof of theorem 6. Let $\hat{L}$ be the Kalman filter gain (31) and define

$$d\tilde{x} = (A\tilde{x} + BK\hat{x})\, dt + \hat{L}(dz - E\tilde{x}dt) \tag{75}$$

where $\tilde{x}$ is the estimate (5) for an arbitrary L. Then (Russell, 1979)

$$X(K,L) = \tilde{X}(K,L) + P \tag{76}$$

where P satisfies equation (18) and $\tilde{X}$ is given by

$$\begin{bmatrix} A & BK \\ LE & A + BE - LE \end{bmatrix} \begin{bmatrix} \tilde{X} & Z \\ Z^T & \hat{X} \end{bmatrix} + \begin{bmatrix} \tilde{X} & Z \\ Z^T & \hat{X} \end{bmatrix} \begin{bmatrix} A & BK \\ LE & A + BK - LE \end{bmatrix}^T + \begin{bmatrix} \hat{L}FF^T\hat{L}^T & \hat{L}FF^TL^T \\ LFF^T\hat{L}^T & LFF^TL^T \end{bmatrix} = 0. \tag{77}$$

Define

$$\begin{bmatrix} \tilde{X} & X_1 \\ X_1^T & X_2 \end{bmatrix} = \begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} \tilde{X} & Z \\ Z^T & \hat{X} \end{bmatrix} \begin{bmatrix} I & I \\ 0 & -I \end{bmatrix}. \tag{78}$$

Then

$$\begin{bmatrix} A + BK & -BK \\ 0 & A - LE \end{bmatrix} \begin{bmatrix} \tilde{X} & X_1 \\ X_1^T & X_2 \end{bmatrix} + \begin{bmatrix} \tilde{X} & X_1 \\ X_1^T & X_2 \end{bmatrix} \begin{bmatrix} A + BK & -BK \\ 0 & A - LE \end{bmatrix}^T + \begin{bmatrix} \hat{L}FF^T\hat{L}^T & \hat{L}FF^T(\hat{L} - L)^T \\ (\hat{L} - L)FF^T\hat{L} & (\hat{L} - L)FF^T(\hat{L} - L)^T \end{bmatrix} = 0. \tag{79}$$

In order to show that K_l^γ, $\hat{L}$ satisfy the necessary conditions for minimizing $J_\gamma^l(K,L)$, we have to show that ($F = [KL^T]^T$)

$$\frac{\partial J_\gamma^l(F)}{\partial F} = 0 \tag{80}$$

gives $F = F_l^\gamma = (K_l^\gamma \hat{L}^T)^T$. Equation (80) is equivalent to showing that

$$\frac{d}{d\epsilon} J_\gamma^l(F + \epsilon\Delta F)\Big|_{\epsilon=0} = \left\langle \frac{\partial J_\gamma^l}{\partial F}(F), \Delta F \right\rangle = \text{Tr}\ \Delta F^T \frac{\partial J_\gamma^l}{\partial F}(F) = 0 \tag{81}$$

for all $\Delta F = (\Delta K \Delta L^T)^T$. In order to simplify notation, we assume $WD = I$. Evaluating $\frac{d}{d\epsilon} J_\gamma^l(F + \epsilon\Delta F)\Big|_{\epsilon=0}$ gives

$$\frac{d}{d\epsilon} J_\gamma^l(F + \epsilon\Delta F)\Big|_{\epsilon=0} = l\ \text{Tr}\ X^{l-1} X' + \gamma\ \text{Tr}\ K\hat{X}'K^T + \gamma\ \text{Tr}\ \Delta K\hat{X}K^T + \gamma\ \text{Tr}\ K\hat{X}\Delta K^T \tag{82}$$

where

$$X' = \frac{d}{d\epsilon} X(K + \epsilon\Delta K, L + \epsilon\Delta L)\Big|_{\epsilon=0}, \tag{83}$$

$$\hat{X}' = \frac{d}{d\epsilon} \hat{X}(K + \epsilon\Delta K, L + \epsilon\Delta L)\Big|_{\epsilon=0}. \tag{84}$$

From equations (76) and (78) we get $\hat{X} = \tilde{X} - X_1^T - X_1 + X_2$ and

$$\hat{X}' = \tilde{X}' - X_1'^T - X_1' + X_2', \tag{85}$$

$$X' = \tilde{X}' \text{ (since } P = \text{const.)} \tag{86}$$

where

$$\tilde{X}' = \frac{d}{d\epsilon} \tilde{X}\Big|_{\epsilon=0}, \tag{87}$$

$$X_1' = \frac{d}{d\epsilon} X_1\Big|_{\epsilon=0}, \tag{88}$$

$$X_2' = \frac{d}{d\epsilon} X_2\Big|_{\epsilon=0}, \tag{89}$$

Using this in equation (82) gives

$$\frac{d}{d\epsilon} J_\gamma^l(F + \epsilon\Delta F)\Big|_{\epsilon=0} = l \operatorname{Tr} X^{l-1} \tilde{X}'$$

$$+ \gamma \operatorname{Tr} K^T K(\tilde{X}' - X_1'^T - X_1' + X_2') \qquad (90)$$
$$+ \gamma \operatorname{Tr} \hat{X}K^T\Delta K + \gamma \operatorname{Tr} \Delta K^T K\hat{X}.$$

From equation (79) we get the following equations for $\tilde{X}$, X_1, and X_2:

$$(A + BK)\tilde{X} + \tilde{X}(A + BK)^T - BKX_1 - X_1^T K^T B^T + \hat{L}FF^T\hat{L}^T = 0, \qquad (91)$$

$$(A + BK)X_1 + X_1(A - LE)^T - BKX_2 + \hat{L}FF^T(\hat{L} - L)^T = 0, \qquad (92)$$

$$(A - LE)X_2 + X_2(A - LE)^T + (\hat{L} - L)FF^T(\hat{L} - L)^T = 0. \qquad (93)$$

Thus, $\tilde{X}'$, X_1', and X_2' satisfy

$$(A + BK)\tilde{X}' + \tilde{X}'(A + BK)^T + B\Delta K\tilde{X} + \tilde{X}\Delta K^T B^T$$
$$- BKX_1' - X_1'^T K^T B^T - B\Delta KX_1 - X_1^T\Delta K^T B^T = 0, \qquad (94)$$

$$(A + BK)X_1' + X_1'(A - LE)^T + B\Delta KX_1 - X_1E^T\Delta L^T$$
$$- BKX_2' - B\Delta K_2X_2 - \hat{L}FF^T\Delta L^T = 0, \qquad (95)$$

$$(A - LE)X_2' + X_2'(A - LE)^T - \Delta LEX_2 - X_2E^T\Delta L^T$$
$$- (\hat{L} - L)^T FF^T\Delta L^T - \Delta LFF^T(\hat{L} - L)^T = 0. \qquad (96)$$

Next we rewrite equation (90) using equation (85) and the adjoint equation for equation (94). This gives

$$\frac{d}{d\epsilon} J_\gamma^l(F + \epsilon\Delta F)\Big|_{\epsilon=0} = \operatorname{Tr} (\tilde{X}QB + \gamma\hat{X}K^T - X_1QB)\Delta K$$
$$+ \operatorname{Tr} \Delta K^T(B^TQ\tilde{X} + \gamma K\hat{X} - B^TQX_1^T) - \operatorname{Tr}(QB + \gamma K^T)KX_1'$$
$$- \operatorname{Tr} X_1'^T K^T(B^TQ + \gamma K) + \gamma \operatorname{Tr} K^TKX_2' \qquad (97)$$

where

$$(A + BK)^TQ + Q(A + BK) + lX^{l-1} + \gamma K^TK = 0. \qquad (98)$$

Now, it follows from equation (96) and the last term in equation (97) that in order for equation (97) to be zero for any ΔL, it is necessary that $X_2' = 0$. Thus, $EX_2 + FF^T(\hat{L} - L)^T = 0$. Substituting $\hat{L} - L = -X_2E^T(FF^T)^{-1}$ into equation (93) gives $X_2 = 0$. Therefore

$L = \hat{L}$. Furthermore, with $L = \hat{L}$ and $X_2 = 0$, it follows from equation (92) that for any $K \in \mathcal{K}$ we have $X_1 = 0$. Therefore $\tilde{X} = \hat{X}$, and the first two terms on the right-hand side of equation (97) give $\gamma K + B^T Q = 0$. However, this makes the third and fourth terms in the right-hand side of equation (97) also equal to zero. Therefore, in order for equation (97) to be identically zero for any ΔF, we must have $L = \hat{L}$ and $K = (-1/\gamma)B^T Q$.

Finally, substituting $K = K_l^\gamma = (-1/\gamma)B^T Q$ into equation (98) gives equation (33), and equations (76) and (91) with $\tilde{X} = \hat{X}$ given in equation (35).

Proof of lemma 3. Note that if $M_l^\gamma > 0$, then $M_l^\gamma = N_l^{\gamma T} N_l^\gamma$ for some non-singular N_l^γ. Furthermore, since (D, A) is detectable, it follows that $(N_l^\gamma WD, A)$ is detectable. Thus, $Q_l^\gamma \geq 0$ and $K_l^\gamma \in \mathcal{K}$. Q.E.D.

Notes

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