*Journal of Elasticity* **75:** 187–192, 2004.



# Stresses in a Half Space Due to Newtonian Gravitation

J.R. BARBER
*Department of Mechanical Engineering, University of Michigan, Ann Arbor, MI 48109-2125, U.S.A. E-mail: jbarber@umich.edu*

Received 5 August 2004

**Abstract.** An efficient general solution is obtained for the problem of the elastic half space $z > 0$ with a traction-free surface experiencing gravitational attraction to an arbitrarily shaped body located in $z < 0$. Many components of the stress field can be written down immediately if the potential of the attracting body is known. Results are given for the case of attraction to a uniform sphere.

**Mathematics Subject Classifications (2000):** 74B05, 74L05.

**Key words:** Newtonian gravitation, asteroids, body forces.

## 1. Introduction

Consider the situation in which a massive asteroid passes close to the Earth. Particles of the Earth's crust will experience gravitational attractive forces directed towards the asteroid, resulting in a predominantly tensile elastic stress field that might have serious consequences, for example, in initiating earthquakes by reducing the compressive tractions between tectonic plates.

The stress field in the Earth can be determined by first constructing a potential to define the gravitational body forces [1, 2], finding a particular solution for the stress field corresponding to this potential [3, Section 18.5.1] and finally satisfying the condition that the Earth's surface be traction-free by superposing a series of stress fields derived from spherical harmonics. However, the problem is considerably simplified if the asteroid is very much smaller than the Earth and its closest approach to the surface is small compared with the Earth's radius, since in this case the affected region of the Earth can reasonably be approximated by a half space. In this Note, we shall develop an extremely efficient general solution for problems of this class.

In particular, we consider the problem of the traction-free elastic half space $z > 0$ of uniform density $\rho_0$ subjected to body forces $\mathbf{p}$ due to gravitational attraction to an arbitrary distribution of mass density $\rho(x, y, z)$ occupying part of the region $z < 0$.

## 2. The Gravitational Potential

The basic law of Newtonian gravitation states that two particles of mass $m_1, m_2$ respectively experience a mutual attractive force

$$\frac{Gm_1m_2}{R^2},$$

where $R$ is the distance between the particles and $G$ is the universal gravitational constant. If we locate a particle of mass $M$ at the origin, it will induce forces of attraction on any other material defined by the body force

$$p_R = -\frac{GM\rho_0}{R^2},$$

where $\rho_0$ is the density of the remaining material. The force $\mathbf{p}$ can in turn be written in terms of a body force potential $V$, where

$$\mathbf{p} = -\nabla V.$$

The field due to the massive particle is spherically symmetrical and hence

$$-\frac{\partial V}{\partial R} = -\frac{GM\rho_0}{R^2},$$

with solution

$$V = -\frac{GM\rho_0}{R}. \tag{1}$$

It is clear from equation (1) that $V$ is a harmonic function everywhere except at the origin as long as $\rho_0$ is constant, since $1/R$ is the Green's function for Laplace's equation. If the massive particle is replaced by a distribution of mass $\rho(x, y, z)$ occupying the region $z < 0$, it is easily shown [2, Section 69] that $V$ will be a solution of the Poisson equation

$$\nabla^2 V = -4\pi G\rho_0\rho(x, y, z). \tag{2}$$

For the problem under consideration, this implies that $V$ is a harmonic function in the region $z > 0$, since the mass distribution $\rho(x, y, z)$ is confined to the region $z < 0$. Notice that we do not include any self-gravitational body forces for the material of the half space, since the present solution describes only the *change* in the stress field due to the gravitational influence of the attracting body. In fact, the stress field in the Earth due to self-gravitational forces depends upon the scenario assumed for its formation [4, 5].

## 3. Particular Solution for the Stress Field

We next devise a strategy for determining a particular solution for the stresses in the half space due to the mass distribution $\rho(x, y, z)$. Following Barber [3, Section 18.5.1], we can write a particular solution for the displacement $\mathbf{u}$ in the form

$$2\mu\mathbf{u} = \nabla\phi,$$

where $\mu$ is the modulus of rigidity and the stress function $\phi$ is required to satisfy the equation

$$\nabla^2\phi = \frac{(1-2\nu)V}{(1-\nu)}. \tag{3}$$

Since $V$ is harmonic in $z > 0$, a convenient way to satisfy (3) is to define a new potential function $\chi$ such that

$$\nabla^2\chi = 0; \qquad \phi = z\frac{\partial\chi}{\partial z}. \tag{4}$$

Substituting (4) into (3) we then have

$$2\frac{\partial\chi}{\partial z} = \frac{(1-2\nu)V}{(1-\nu)}$$

or

$$V = \frac{2(1-\nu)}{(1-2\nu)}\frac{\partial\chi}{\partial z}. \tag{5}$$

The stress components can now be obtained from equations (18.45), (18.46) of [3] and are

$$\sigma_{xx} = \frac{\nu V}{(1-\nu)} + \frac{\partial^2\phi}{\partial x^2} = \frac{2\nu}{(1-2\nu)}\frac{\partial\chi}{\partial z} + z\frac{\partial^2\chi}{\partial x^2}, \tag{6}$$

$$\sigma_{yy} = \frac{\nu V}{(1-\nu)} + \frac{\partial^2\phi}{\partial y^2} = \frac{2\nu}{(1-2\nu)}\frac{\partial\chi}{\partial z} + z\frac{\partial^2\chi}{\partial y^2}, \tag{7}$$

$$\sigma_{zz} = \frac{\nu V}{(1-\nu)} + \frac{\partial^2\phi}{\partial z^2} = \frac{2(1-\nu)}{(1-2\nu)}\frac{\partial\chi}{\partial z} + z\frac{\partial^2\chi}{\partial z^2}, \tag{8}$$

$$\sigma_{xy} = \frac{\partial^2\phi}{\partial x\partial y} = z\frac{\partial^2\chi}{\partial x\partial y}, \tag{9}$$

$$\sigma_{yz} = \frac{\partial^2\phi}{\partial y\partial z} = \frac{\partial\chi}{\partial y} + z\frac{\partial^2\chi}{\partial y\partial z}, \tag{10}$$

$$\sigma_{zx} = \frac{\partial^2\chi}{\partial z\partial x} = \frac{\partial\chi}{\partial x} + z\frac{\partial^2\chi}{\partial z\partial x}. \tag{11}$$

## 4. The Traction-Free Half Space

The particular solution of equations (6)–(11) implies the existence of tractions

$$\sigma_{zx} = \frac{\partial\chi}{\partial x}; \qquad \sigma_{zy} = \frac{\partial\chi}{\partial y}; \qquad \sigma_{zz} = \frac{2(1-\nu)}{(1-2\nu)}\frac{\partial\chi}{\partial z}$$

on the surface $z = 0$. To remove these tractions, it is sufficient to superpose solution B of Green and Zerna [6] (see also [3, Table 19.1]), obtaining the stress field

$$\begin{aligned}
\sigma_{xx} &= \frac{2\nu}{(1-2\nu)}\frac{\partial\chi}{\partial z} + z\frac{\partial^2\chi}{\partial x^2} + z\frac{\partial^2\omega}{\partial x^2} - 2\nu\frac{\partial\omega}{\partial z}, \\
\sigma_{yy} &= \frac{2\nu}{(1-2\nu)}\frac{\partial\chi}{\partial z} + z\frac{\partial^2\chi}{\partial y^2} + z\frac{\partial^2\omega}{\partial y^2} - 2\nu\frac{\partial\omega}{\partial z}, \\
\sigma_{zz} &= \frac{2(1-\nu)}{(1-2\nu)}\frac{\partial\chi}{\partial z} + z\frac{\partial^2\chi}{\partial z^2} + z\frac{\partial^2\omega}{\partial z^2} - 2(1-\nu)\frac{\partial\omega}{\partial z}, \\
\sigma_{xy} &= z\frac{\partial^2\chi}{\partial x\partial y} + z\frac{\partial^2\omega}{\partial x\partial y}, \\
\sigma_{yz} &= \frac{\partial\chi}{\partial y} + z\frac{\partial^2\chi}{\partial y\partial z} + z\frac{\partial^2\omega}{\partial y\partial z} - (1-2\nu)\frac{\partial\omega}{\partial y}, \\
\sigma_{zx} &= \frac{\partial\chi}{\partial x} + z\frac{\partial^2\chi}{\partial z\partial x} + z\frac{\partial^2\omega}{\partial z\partial x} - (1-2\nu)\frac{\partial\omega}{\partial x}.
\end{aligned}$$

The corresponding tractions on $z = 0$ are

$$\begin{aligned}
\sigma_{zx} &= \frac{\partial\chi}{\partial x} - (1-2\nu)\frac{\partial\omega}{\partial x}, \\
\sigma_{zy} &= \frac{\partial\chi}{\partial y} - (1-2\nu)\frac{\partial\omega}{\partial y}, \\
\sigma_{zz} &= \frac{2(1-\nu)}{(1-2\nu)}\frac{\partial\chi}{\partial z} - 2(1-\nu)\frac{\partial\omega}{\partial z}
\end{aligned}$$

and it is clear by inspection that these expressions will all be zero if we choose $\omega$ such that $(1-2\nu)\omega = \chi$. Simpler final expressions are obtained if we achieve this result by defining a new potential function $\psi$ through

$$\psi = 2(1-\nu)\omega = \frac{2(1-\nu)\chi}{(1-2\nu)}, \tag{12}$$

after which the stress components are obtained as

$$\sigma_{xx} = z\frac{\partial^2\psi}{\partial x^2}, \qquad \sigma_{yy} = z\frac{\partial^2\psi}{\partial y^2}, \qquad \sigma_{zz} = z\frac{\partial^2\psi}{\partial z^2}, \tag{13}$$

$$\sigma_{xy} = z\frac{\partial^2\psi}{\partial x\partial y}, \qquad \sigma_{yz} = z\frac{\partial^2\psi}{\partial y\partial z}, \qquad \sigma_{zx} = z\frac{\partial^2\psi}{\partial z\partial x} \tag{14}$$

and $\psi$ must satisfy

$$\frac{\partial\psi}{\partial z} = V, \tag{15}$$

from (5), (12). Notice that expressions (13), (14) are identical to those defining solution A of Green and Zerna in [3, Table 19.1] except for a multiplying factor of $z$. It follows from the same Table that the corresponding expressions in cylindrical polar coordinates $(r, \theta, z)$ are

$$\sigma_{rr} = z\frac{\partial^2\psi}{\partial r^2}, \qquad \sigma_{\theta\theta} = \frac{z}{r}\frac{\partial\psi}{\partial r} + \frac{z}{r^2}\frac{\partial^2\psi}{\partial\theta^2}, \qquad \sigma_{zz} = z\frac{\partial^2\psi}{\partial z^2}, \tag{16}$$

$$\sigma_{r\theta} = \frac{z}{r}\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{z}{r^2}\frac{\partial\psi}{\partial\theta}, \qquad \sigma_{\theta z} = \frac{z}{r}\frac{\partial^2\psi}{\partial\theta\partial z}, \qquad \sigma_{zr} = z\frac{\partial^2\psi}{\partial z\partial r}. \tag{17}$$

The title problem therefore reduces to the determination of a solution of (2) for the body force potential $V$ followed by a harmonic partial integral of (15) for $\psi$, after which the stress field is obtained by substituting the resulting expression for $\psi$ into (13), (14) or (16), (17).

In fact, some of the stress components can be obtained without the necessity for the determination of a partial integral. We have

$$\sigma_{yz} = z\frac{\partial V}{\partial y}, \qquad \sigma_{zx} = z\frac{\partial V}{\partial x}, \qquad \sigma_{zz} = z\frac{\partial V}{\partial z},$$

$$\sigma_{\theta z} = \frac{z}{r}\frac{\partial V}{\partial\theta}, \qquad \sigma_{zr} = z\frac{\partial V}{\partial r},$$

from (13)–(17) and also

$$\begin{aligned}\sigma_{xx} + \sigma_{yy} = \sigma_{rr} + \sigma_{\theta\theta} &= z\frac{\partial^2\psi}{\partial r^2} + \frac{z}{r}\frac{\partial\psi}{\partial r} + \frac{z}{r^2}\frac{\partial^2\psi}{\partial\theta^2} \\ &= z\nabla^2\psi - z\frac{\partial^2\psi}{\partial z^2} = -z\frac{\partial V}{\partial z},\end{aligned}$$

where we have used the result $\nabla^2\psi = 0$.

## 5. Example – the Spherical Asteroid

As an example, we consider the case where the centre of a spherical body of density $\rho$ and radius $a$ is located at the point $(0, -h)$ in cylindrical polar coordinates $(r, z)$. Assuming $h > a$, the particular solution for $V$ in $z > 0$ is the same as that due to a particle of mass $4\pi\rho a^3/3$ located at $(0, -h)$ and is therefore given by

$$V = -\frac{4\pi G\rho\rho_0 a^3}{3\sqrt{r^2 + (z+h)^2}},$$

from (1). A harmonic partial integral of this expression satisfying (15) is

$$\psi = -\frac{4G\pi\rho\rho_0 a^3}{3}\ln\left(\sqrt{r^2 + (z+h)^2} + z + h\right).$$

The resulting stress field is therefore obtained immediately from equations (16), (17) as

$$\begin{aligned}
\sigma_{rr} &= \frac{4G\pi\rho\rho_0 a^3 z[\sqrt{r^2+(z+h)^2}z - r^2]}{3(r^2+(z+h)^2)^{3/2}(\sqrt{r^2+(z+h)^2}+z+h)},\\
\sigma_{\theta\theta} &= -\frac{4G\pi\rho\rho_0 a^3 z}{3(\sqrt{r^2+(z+h)^2}+z+h)\sqrt{r^2+(z+h)^2}},\\
\sigma_{zz} &= \frac{4G\pi\rho\rho_0 a^3 z(z+h)}{3(r^2+(z+h)^2)^{3/2}},\\
\sigma_{rz} &= \frac{4G\pi\rho\rho_0 a^3 rz}{3(r^2+(z+h)^2)^{3/2}},\\
\sigma_{r\theta} &= \sigma_{\theta z} = 0.
\end{aligned}$$

Results for asteroids of other shapes, including ellipsoids and rectangular parrallepipeds can be obtained using the many classical solutions of the gravitational potential problem [1, 2].

## References

1. O.D. Kellog, *Foundations of Potential Theory*. Dover, New York (1954).
2. W.D. MacMillan, *The Theory of the Potential*. Dover, New York (1958).
3. J.R. Barber, *Elasticity*, 2nd edn. Kluwer, Dordrecht (2002).
4. C.B. Brown and L.E. Goodman, Gravitational stresses in accreted bodies. *Proc. Roy. Soc. London Ser. A* **276** (1963) 571–576.
5. J. Kadish, J.R. Barber and P.D. Washabaugh, Stresses in rotating spheres grown by accretion. *Internat. J. Solids Struct.*, under review.
6. A.E. Green and W. Zerna, *Theoretical Elasticity*. Clarendon Press, Oxford (1954) pp. 174–178.