International Journal of Computers for Mathematical Learning (2005) 10: 31–47
DOI 10.1007/s10758-004-6999-x

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VALIDATION OF SOLUTIONS OF CONSTRUCTION PROBLEMS IN DYNAMIC GEOMETRY ENVIRONMENTS[1]

ABSTRACT. This paper discusses issues concerning the validation of solutions of construction problems in Dynamic Geometry Environments (DGEs) as compared to classic paper-and-pencil Euclidean geometry settings. We begin by comparing the validation criteria usually associated with solutions of construction problems in the two geometry worlds – the 'drag test' in DGEs and the use of only straightedge and compass in classic Euclidean geometry. We then demonstrate that the *drag test criterion* may permit constructions created using measurement tools to be considered valid; however, these constructions prove inconsistent with classical geometry. This inconsistency raises the question of whether dragging is an adequate test of validity, and the issue of measurement versus straightedge-and-compass. Without claiming that the inconsistency between what counts as valid solution of a construction problem in the two geometry worlds is necessarily problematic, we examine what would constitute the analogue of the *straightedge-and-compass criterion* in the domain of DGEs. Discovery of this analogue would enrich our understanding of DGEs with a mathematical idea that has been the distinguishing feature of Euclidean geometry since its genesis. To advance our goal, we introduce the *compatibility criterion*, a new but not necessarily superior criterion to the drag test criterion of validation of solutions of construction problems in DGEs. The discussion of the two criteria anatomizes the complexity characteristic of the relationship between DGEs and the paper-and-pencil Euclidean geometry environment, advances our understanding of the notion of geometrical constructions in DGEs, and raises the issue of validation practice maintaining the pace of ever-changing software.

KEY WORDS: drag test, Dynamic Geometry Environments (DGEs), Euclidean geometry, geometrical constructions, proof, validation of construction problems

INTRODUCTION

Despite their fundamental theoretical value, geometrical constructions seem to have lost their centrality in the school geometry curriculum (Mariotti, 2001). At the same time, the learning and teaching of these constructions has often been dissociated from meaningful mathematical activity (see, e.g., Schoenfeld, 1988). However, the appearance of computer geometry software packages seems to be spurring a new interest in geometrical constructions and supports the possibility of using construction tasks in Dynamic Geometry

Environments (DGEs) "as *a key to accessing* the theoretical world of geometry" (Mariotti, 2001, p. 279; italics in original). This potential depends to great extent on the relation between the notion of validation of solutions of such problems and the corresponding notion in the paper-and-pencil Euclidean geometry. In this paper we examine different aspects of this relationship.

In both geometry worlds, *geometrical constructions* are generally defined as valid solutions of construction problems. In the classic paper-and-pencil Euclidean geometry environment, the validation criterion for a solution of a construction problem requires that a solution is valid if and only if it has been produced using only straightedge and compass. In DGEs, the validation criterion most often used to date has been that a solution of a construction problem is valid if and only if it passes the 'drag test' (Jones, 2000; Mariotti, 2001). As we demonstrate in this paper, acceptance of the drag test as a necessary and sufficient indicator of validity for solutions of construction problems supports an inconsistency in the set of constructible figures between the geometry embodied in DGEs and the classic Euclidean geometry tradition. Specifically, the drag test may permit the validity of constructions created using measurement tools (such as angle measures, calculations, and rotations using numerically-specified angles). Such constructions, however, are inconsistent with the classical geometry, in which the only tools permitted are straightedge and compass. These observations raise issues of whether dragging is an adequate test of validity, and of measurement versus straightedge-and-compass.

Without claiming that the inconsistency between what counts as a valid solution of a construction problem in the two geometry words is necessarily problematic, we examine what in the domain of DGEs would constitute the analogue of the *straightedge-and-compass criterion*. An important reason for doing so is to enrich our understanding of DGEs with a mathematical idea that has been the distinguishing feature of Euclidean geometry since its genesis. To advance our goal, we introduce the *compatibility criterion*, a new criterion for the validation of solutions of construction problems in DGEs, and we discuss it in relation to the *drag test criterion*. This discussion aims to use the two criteria as a means to advance our thinking about issues of validity in DGEs, rather than to evaluate their relative worth.

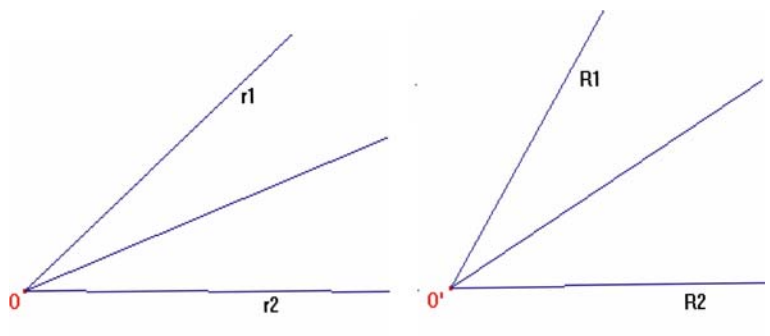
The paper is structured in three sections. In the first section, we examine whether the drag test is a viable criterion of validation for solutions of construction problems in DGEs by considering two alternative – and robust (under dragging) – constructions of an angle bisector. Also, we use the same constructions to motivate the development of the compatibility criterion. In the second section, we elaborate this new criterion of validation and discuss it in relation to the drag test criterion. In the third section, we propose two implications of considering the two criteria; the first relates to the notion of *geometrical constructions* in DGEs and the second concerns the issue of validation practice maintaining the pace of ever-changing software.

THE DRAG TEST AS CRITERION OF VALIDATION FOR SOLUTIONS OF CONSTRUCTION PROBLEMS IN DGEs

The two drawings in Figure 1, produced in the dynamic environment of *Cabri Geometry II* (Laborde and Bellemain, 1998), represent the bisectors of two arbitrary angles. Both of these constructions pass the drag test; when one moves (drags) any of the sides of the given angles, the rays that represent the angle bisectors change accordingly to preserve their geometrical properties.

Given the invariance of the two constructions under dragging, both can be characterized as *robust constructions* (as opposed to *non-robust constructions*). Also, both can be considered as *valid* solutions of the angle bisector problem, based on the *drag test criterion* described in the two excerpts below:

> The DGE ... introduces a specific criterion of validation for the solution of a construction problem: a solution is *valid* if and only if it is *not* possible to "mess it up" by dragging (to use the expression adopted by Healy et al., 1994, see also Noss



Construction A Construction B

Figure 1. The two angle bisectors.

> et al., 1994), or, in other words, that there is "robustness of a figure under drag" (as used by Balacheff and Sutherland, 1994, p. 147). This criterion of validation does not depend on the perceptive appearance of the product of the construction as this appearance can be modified using the drag facility. To pass this 'drag test' the figure has to be constructed in such a way that it is consistent with geometrical theory. (Jones, 2000, p. 58; italics in original)

> The presence of the dragging mode introduces a specific criterion of validation for the solution of construction problems: a solution is valid if and only if the figure on the screen is stable under the dragging test. (Mariotti, 2001, p. 260)

Although both constructions fulfill the drag test criterion, their construction processes fundamentally differ (see Table I in relation to Figure 2). Construction A has been created by using only tools from the 'creation' menu ('circle,' 'ray,' and 'point'), whereas Construction B has been created by using measurement tools supplemental to these. In particular, Construction B has been produced by using the 'ray' and 'point' tools from the 'creation' menu, supplemented by the 'measure angle,' 'calculate,' and 'rotate' tools from the 'metric' and 'transformations' menus. The different processes followed to obtain the two constructions raise the question of whether we want to consider both constructions as valid solutions of the angle bisector problem. If the answer to this question is in the affirmative, the drag test seems to be an appropriate criterion of validation. If, however, there are cases in which

TABLE I

Descriptions of the two construction processes for the angle bisectors

Construction process A	Construction process B
1. Draw an arbitrary circle C_1 with center O.	1. Take two arbitrary points on the rays R_1 and R_2, say A′ and B′ respectively, and mark the angle ∠ A′O′B′.
2. Label the points of intersection between the circle C_1 and the rays r_1 and r_2, say A and B respectively.	2. Measure the angle ∠ A′O′B′. (In the particular case presented in Figure 2, this angle appears to be equal to 54.6°.)
3. Draw circles C_2 and C_3 with centers A and B and radii the segments OA and OB, respectively.	3. Insert as a variable the measure of the angle ∠ A′O′B′ into the software calculator and divide it by 2.
4. Label the point of intersection between circles C_2 and C_3, say D.	4. Drag the above result into the screen; this equals 1/2 · ∠ A′O′B′.
5. Draw the ray from O through D.	5. Rotate point B′ around point O′ using the angle 1/2 · ∠ A′O′B′.
	6. Label the point obtained, say D′.
	7. Draw the ray from O′ through D′.

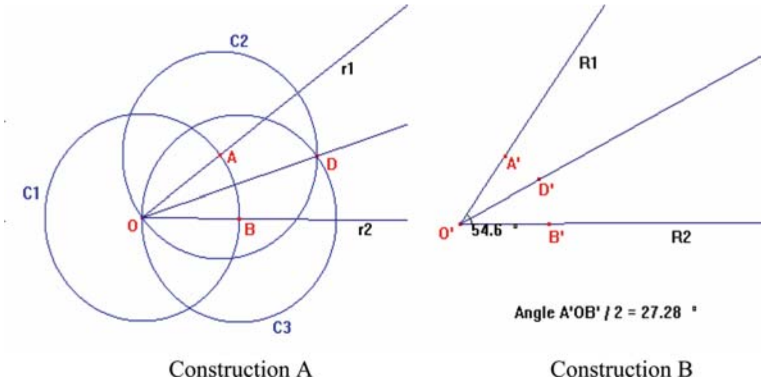


Figure 2. The two construction processes for the angle bisectors.

we would like to consider only Construction A as a valid solution (because, for example, of its correspondence to the straightedge-and-compass construction of an angle bisector), we must then reconsider the drag test as the only criterion of validation for solutions of construction problems in DGEs. The drag test does not distinguish between solutions involving measurement and solutions involving only those software tools analogous to straightedge-and-compass. In sum, the comparison raises the issue of measurement versus straightedge-and-compass, and suggests the need for an alternative criterion of validation for solutions of construction problems in DGEs.[2]

TWO CRITERIA OF VALIDATION FOR SOLUTIONS OF CONSTRUCTION PROBLEMS IN DGEs

In this section, we consider two criteria of validation for solutions of construction problems in DGEs. Our ultimate goal is to further illuminate issues of validity in DGEs. The section is organized in four parts. In the first, we present and discuss the two criteria. In the second, we examine how each of the two criteria represents a different conceptualization of the relation between the geometry embodied in DGEs and the classic paper-and-pencil Euclidean geometry environment. In the third, we use the context of a teaching experiment to explore further the relation between the two criteria. In the fourth, we investigate what it takes to apply, in practical terms, each of the two criteria to evaluate the validity of solutions of construction problems in DGEs.

The Two Criteria

Below we present the two criteria of validation for solutions of construction problems in DGEs. The first criterion is based on the remarks of Jones (2000) and Mariotti (2001), quoted previously. The second is an alternative criterion we introduce in this paper.

Drag Test Criterion: A solution of a construction problem carried out in a DGE is valid if and only if the final construction retains its geometrical properties under dragging.

Compatibility Criterion: A solution of a construction problem carried out in a DGE is valid if and only if the final construction retains its geometrical properties under dragging and its construction process does not violate the *DGE-construction restrictions*, that is, constraints in the use of software operations equivalent to those imposed on straightedge-and-compass constructions.

The notion of *DGE-construction restrictions* used in the formulation of the compatibility criterion has a contextual character; the set of constraints is not fixed but can change according to prior constructions. For example, in order not to violate the DGE-construction restrictions when dealing for the first time with the angle bisector problem, only primitive tools may be used to solve the problem (see Table I, construction process A); use of the 'angle bisector' macro of the software would certainly constitute a violation of the restrictions. However, once the angle bisector problem is solved, it can become a 'theorem' and the 'angle bisector' macro will subsequently become available for use in the solution of new construction problems.

Contrary to the drag test criterion, the compatibility criterion considers robustness under dragging a necessary, but not sufficient, condition for the validity of a given solution of a construction problem. To exemplify the difference between the two criteria, we take a closer look at the example about the angle bisectors. Both constructions are valid under the drag test criterion, for both are robust under dragging. However, only Construction A is valid under the compatibility criterion; Construction B uses software commands whose operations would not be permissible in straightedge-and-compass constructions. For example, the use of the software tools 'measure angle' and 'rotate' corresponds to the use of protractor to produce the equivalent operations in the paper-and-pencil environment. It is also important to note that, with only slight modifications, one could use the same software operations as in Construction B to provide a solution to the 'angle trisection problem' that would be valid under the drag test criterion. Yet, it has been proved impossible to solve this famous geometrical problem of antiquity with only straightedge and compass. Put differently, there is no solution of this problem that is valid under the straightedge-and-compass criterion. The solution of this problem in DGEs that would be valid under the drag test criterion would clearly be invalid under the compatibility criterion.

Different Conceptualizations of the Relation Between Two Geometry Worlds

The two criteria for what constitutes a valid solution of a construction problem in DGEs represent, in the specific domain of geometrical constructions, different conceptualizations of the relation between the geometry embodied in DGEs and the classic paper-and-pencil Euclidean geometry. In this part, we elaborate these conceptualizations.

The drag test criterion allows for an incompatibility between the two geometry worlds; this disjunction is primarily reflected in the fact that the sets of constructible figures in the two environments differ, as indicated by the angle trisection example mentioned previously. This discrepancy between the two worlds, however, is not necessarily considered problematic. If one believes that the notion of validation of solutions of construction problems in DGEs does not have to follow the norms of the paper-and-pencil Euclidean geometry tradition, then the drag test criterion is appropriate. The belief associated with the antecedent of the above conditional statement might be grounded in that DGEs create a new reality for engagement in construction problems from both practical and conceptual considerations (Straesser, 2001); therefore, these new environments do not have to be constrained by restrictions analogous to those imposed on the classic paper-and-pencil Euclidean geometry. Indeed, the two geometry worlds already differ in many respects. For example, they use different technologies (computer software versus paper and pencil) and allow different levels of manipulation of constructed figures (direct manipulation by use of the drag facility in DGEs versus static representations in paper and pencil). Furthermore, one might argue – along lines parallel to the following statement by Hersh (1993) about rigorous proof – that our inherited notion of validation of solutions of construction problems is not carved in stone; if it is advantageous to modify this notion to adapt better to the new reality created by DGEs, then we should do so.

> Our inherited notion of "rigorous proof" is not carved in marble. People will modify that notion, will allow machine computation, numerical evidence, probabilistic algorithms, if they find it advantageous to do so. (Hersh, 1993, pp. 395–396)

The compatibility criterion aims to address the discrepancy between the two geometry worlds by requiring consistency between the restrictions imposed on the operations allowed in the production of valid constructions. Achieving this kind of consistency is feasible, because, despite the de facto differences between the two worlds, it is possible to configure the available tools and menus of Cabri so that the engagement with construction problems in DGEs becomes analogous to the straightedge-and-compass experience in paper and pencil. A possible argument in favour of establishing a firm correspondence between DGE constructions and paper-and-pencil Euclidean geometry constructions is that one primary motivation for the development of DGEs has been the opportunity they offer for the creation of a dynamic *Euclidean* geometry environment. Such an environment can expand our capacity for figure manipulation and address some of the practical limitations of paper and pencil while retaining the basic characteristics of the geometry represented. This argument becomes more pertinent in cases where DGEs are used as substitutes for the paper-and-pencil environment in the teaching and learning of Euclidean geometry constructions.

The Relation Between the Two Criteria: An Example From a Teaching Experiment

Contrary to what the angle trisection example might suggest, the two criteria do not necessarily oppose one another. In this part, we use the context offered by Mariotti's (2000, 2001) teaching experiment with high-school students to provide an example of how the two criteria might interrelate, and to demonstrate that one criterion is not in general more correct than or superior to the other.

Mariotti's experiment drew heavily on the fact that Cabri not only provides the user with rich opportunities to engage in geometrical experiences not (readily) available in the paper-and-pencil environment, but also allows for the creation of different DGEs with features that can accommodate various needs and objectives. For example, the user can configure the available menus to suppress the tools that could produce operations incompatible with the operations produced by the use of only straightedge and compass in the paper-and-pencil environment.

The set of software tools initially available to the students in the experiment corresponded to the straightedge and compass tools in the paper-and-pencil environment. As the students built different geometrical constructions (such as the angle bisector), the Cabri menu was enlarged to include new macros (such as the 'angle bisector' macro) which then became 'theorems' available for use in subsequent constructions. This progressive enlargement of the Cabri menu paralleled the enlargement of the theoretical system; the new theorems were subsequently added to the theory.

In the context of this teaching experiment, the close correspondence between the world of Cabri and the world of Euclidean geometry was highly desirable. Building this correspondence was both a *goal* of the experiment (the students were expected to engage in constructing a parallel between the two worlds) and the *means* to achieve several other interrelated educational objectives (one of them being the teaching and learning of Euclidean geometry constructions). The compatibility criterion is particularly relevant here, for it is grounded in the same principle as the one suggested by the design of the teaching experiment – namely, establishing consistency in norms of valid construction in both worlds. Also, the progressive enlargement of the Cabri menu in the experiment reflects the continuous relaxation of the set of restrictions imposed on the tools available to the students as they advanced in building the theoretical system; this relates to the contextual character of the notion of DGE-construction restrictions.

The drag test criterion, however, would be equally applicable in this experiment, for the compatibility criterion would be reduced in this context to the drag test criterion. The configuration of the software menu ensured that, at any given time, no software tools with the potential to violate the concurrent notion of DGE-construction restrictions were available to the students. In particular, as Mariotti (2000) notes, it was

> possible to interpret the control 'by dragging' as corresponding to theoretical control – 'by proof and definition' – within the system of Euclidean Geometry. In

> other words, it [was] possible to state a correspondence between the world of Cabri constructions and the theoretical world of Euclidean Geometry. (pp. 27–28)

To conclude, in this specific example, the two criteria coexisted: the spirit of the compatibility criterion was reflected in the development of the students' activities, while the drag test criterion was used as a practical device for examining validity.

Applying the Two Criteria to Evaluate the Validity of Solutions of Construction Problems in DGEs

The final part of our discussion of the two criteria centers on the practical question of how the two criteria can be applied to evaluate the validity of solutions of construction problems in DGEs. The application of the drag test criterion goes as follows: If a Cabri construction, presented as a solution of a construction problem, is robust under dragging, then the solution is valid.[3]

Things are often more complicated when applying the compatibility criterion. In cases with an appropriate configuration of the available Cabri tools and menus, application of the compatibility criterion reduces to application of the drag test criterion. In other words, we can conveniently examine the validity of solutions of construction problems by using the drag facility. What happens, however, in cases where the entire Cabri menu is available and the examiner of the validity of solutions of DGE construction problems is committed to the compatibility criterion? In these cases, invariance under dragging cannot serve as a sufficient indicator of validity. It can still serve, though, as a necessary indicator of validity, thereby offering the examiner an important preliminary means of checking the validity of a solution with respect to the compatibility criterion. If the construction fails the drag test, then the solution is invalid. If, however, the construction passes the drag test, then the examiner must check whether or not the construction process violated the DGE-construction restrictions. This further requirement can be investigated in multiple ways. Below we describe three of these ways.

The first way is to have the solvers describe the construction process they followed. The second way is to use the 'hide/show' tool to make 'visible' most of the intermediate steps of the construction process, thus enabling the examiner to understand better the procedure the solver followed.[4] For example, after using this tool in the angle bisector example (see Figure 2 in relation to Figure 1), the examiner could say that Construction A most probably did not violate the DGE-construction restrictions whereas Construction B most probably did. The latter observation would alert the examiner to investigate further the process followed for Construction B. The third way is to use the 'replay construction' tool to observe, in order of execution, the essential steps in the construction process. By retracing the steps, the examiner is better prepared to make inferences about the validity of a given construction. Turning back to the angle bisector example, the 'replay construction' tool would provide the examiner with a clear sense of the process followed for Construction A but not of the process followed for Construction B. The use of the software calculator is not recorded in the 'history' of this process, thus leaving a gap in tracing the procedure followed for the creation of Construction B. Nevertheless, the steps tracked in the history of the creation of Construction B would be enough to alert the examiner to investigate further this construction process.

The several possible ways we discussed previously for examining whether a robust construction has violated the DGE-construction restrictions are particularly important when alignment of a given construction with (classical) geometrical theory is essential for validity. In cases where this kind of alignment is not necessary and the only criterion of validity is robustness under dragging, it would be acceptable to modify the Cabri environment so that violation of the DGE-construction restrictions becomes common practice. This can happen, for example, when the Cabri menu is configured to emphasize the use of 'transformation' tools.

IMPLICATIONS OF THE TWO CRITERIA: GEOMETRICAL CONSTRUCTIONS AND VALIDATION PRACTICE IN CHANGING DGE SOFTWARE

The concept of *geometrical constructions* as valid solutions of construction problems is inextricably related to the adopted criterion of validation for solutions of construction problems. In the classic paper-and-pencil Euclidean geometry environment, in which the straightedge-and-compass criterion is well-established, geometrical constructions have become synonymous with constructions created with only straightedge and compass. The situation is less clear in DGEs, given that they allow the consideration of (at least) two different criteria of validation. If one adopts the drag test criterion, then geometrical constructions become equivalent to robust constructions. If, however, one adopts the compatibility criterion, this equivalence breaks down – geometrical constructions are necessarily robust but the converse is not always true. Construction B presents a case where the converse fails (see Figures 1 and 2). This construction and others of a similar kind, whereby measurement plays a role in their creation, can form a new family of constructions, what we name *measurement constructions*. This class of constructions, which is a proper subset of robust constructions, exemplifies the difference between the two validation criteria – measurement constructions are valid under the drag test criterion but invalid under the compatibility criterion Figure 3 summarizes different kinds of constructions in DGEs and their relations to the two validation criteria.

The remarks above highlight the complexity of issues of validity in DGEs and suggest that geometrical constructions in these environments may never achieve a commonly-accepted meaning, for the

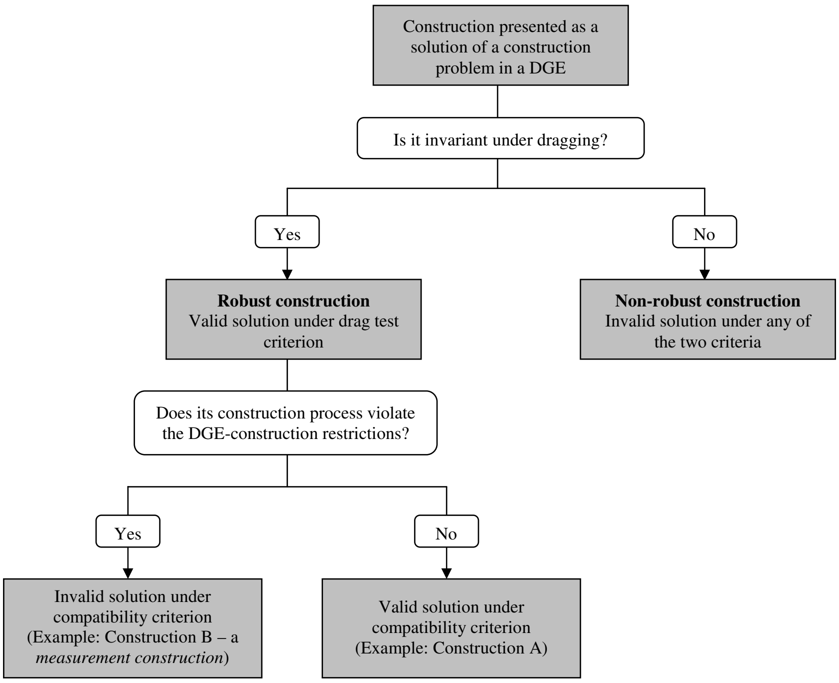


Figure 3. Different kinds of constructions in DGEs and their relations to the drag test and compatibility criteria.

establishment of such meaning requires universal agreement on a single criterion of validation. Things become more complicated if one also takes into consideration the challenge of validation practice in DGEs maintaining the pace of ever-changing software. The advancement from *Cabri Geometry I* to *Cabri Geometry II* provides a context in which we elaborate this point.

The tools offered by Cabri I resembled more closely those associated with straightedge-and-compass Euclidean geometry than those tools available in the more recent version of the software. For example, Cabri I did not provide the measurement tools necessary for creating constructions like Construction B; the construction process associated with this particular construction (see Table I) became possible only with the development of Cabri II. Therefore, the drag test – or the robustness of a construction under dragging – could be linked to different construction processes in the two versions of the software. In Cabri I, robustness of a given construction under dragging would be a fairly viable test of whether or not its construction process violated the DGE-construction restrictions; in this sense, the compatibility criterion could almost always be reduced to the drag test criterion. Violation of the DGE-construction restrictions would still be possible in the earlier version of Cabri in cases, for example, where construction macros were used before they were derived as theorems. The compatibility criterion is more meaningful (and, perhaps, essential) in Cabri II than in Cabri I, because the measurement tools in this later version have expanded tremendously the opportunities for creating constructions whose construction processes do not resemble the use of only straightedge and compass, thereby violating the DGE-construction restrictions.

The previous discussion suggests that the advancement of DGEs not only makes possible different kinds of geometry, but also introduces the need for the development of new validation criteria for examining compatibility with the straightedge-and-compass Euclidean geometry. More layers of complexity exacerbate this state of affairs in cases where one is not only interested in the correspondence between the geometry embodied in DGEs and the classic Euclidean geometry. If designers or users are interested in examining compatibility with other kinds of geometrical theories, the set of validation criteria in DGEs requires even further expansion. These criteria can, in turn, serve as a bridge to connect different geometry worlds.

CONCLUSION

Hersh (1986) noted that the most straightforward answer to the question of what is mathematics is that "mathematics deals with ideas. Not pencil marks or chalk marks, not physical triangles or physical sets, but ideas (which may be represented or suggested by physical objects)" (p. 22). The straightedge-and-compass criterion for determining the validity of solutions of construction problems produced in the paper-and-pencil Euclidean geometry environment represents a mathematical idea that has been in place for more than 2000 years. The recent development of DGEs has created the need for the formulation of a criterion of validation for solutions of construction problems in these new environments. The criterion most often used for this purpose thus far has been the drag test. As we have shown in this paper, however, acceptance of this criterion reveals an inconsistency between the geometry embodied in DGEs and the paper-and-pencil Euclidean geometry. This inconsistency is primarily reflected in the fact that the set of constructible figures in the latter is only a proper subset of the corresponding set in the former. Therefore, the drag test criterion in DGEs does not constitute the analogue of the straightedge-and-compass criterion in the paper-and-pencil Euclidean geometry environment.[5] Based on this remark and following Hersh's notion of mathematics described above, we may consider the drag test criterion as a *new* mathematical idea created in DGEs while the straightedge-and-compass criterion as an *existing* mathematical idea not yet explicitly transferred to DGEs. The new criterion of validation we introduced in this paper – the compatibility criterion – fills this 'gap' in the existing set of conceptualizations of validation in DGEs and aims to make possible, when desired, the establishment of a firm correspondence between geometrical constructions in DGEs and the paper-and-pencil Euclidean geometry environment.

The discussion of the drag test and compatibility criteria in this paper has developed our conceptualization of the relation between the world of DGEs and that of paper-and-pencil Euclidean geometry, and has anatomized the complexity that characterizes the validity of solutions of construction problems in these two worlds. Also, it has advanced our understanding of issues surrounding validity in DGEs in at least two ways. First, it improved our grasp of the notion of geometrical constructions in these environments, and, second, it increased our awareness of the issue of validation practice maintaining the pace of software evolution.

The discussion also raises pedagogical questions for further consideration. For example, what are the advantages and disadvantages of configuring software menus so that certain solutions of construction problems are privileged while others are made impossible? For what populations would dragging be pedagogically sufficient? In cases where the compatibility criterion is adopted as indicator of validity, how important is it to ensure that none of the DGE-construction restrictions is violated? For example, would it be appropriate for learners to use the 'angle bisector' macro in solving a more complicated construction problem before they had worked out how to construct angle bisectors for themselves? Some of these questions have previously appeared in various discussions related to other software. For instance, discussions about the purity of Logo have raised the issue of providing arc before or after learners had worked out how to do it for themselves. These pedagogical issues are important and complex, and they deserve research attention. In general, we believe that decisions about how to handle them should not be made universally. Instead, decisions should be made in accordance with the specific parameters of given contexts by *well-informed* users.

Finally, our discussion of the notion of validation of construction problems in DGEs and its connection to the corresponding notion in paper-and-pencil Euclidean geometry may be seen in the broader context of the relation between the mathematics embodied in computer environments and the mathematics outside these environments. In studying this relation, our field often tends to focus on the developments introduced to mathematical practice by the growing use of computers. One example relates to the major role of computers in the introduction of the relatively new concepts of zero-knowledge and holographic proofs (Cipra, 1993; Goldreich, 2002), and the development of graphic-oriented fields, such as the chaos and dynamical systems theories (Ott, 1993). These innovative types of proof and the reviving interest in experimental methods have changed our views of proving and thinking about mathematics (Dubinsky and Tall, 1991; Hanna, 1995). Yet, it is equally important, we argue, to pay attention to some significant mathematical ideas that do not seem to have found a firm place in computer environments. The case of the straightedge-and-compass criterion of validation for solutions of construction problems in DGEs constitutes one such example. What might be some other examples and how could they be transferred to the mathematics embodied in computer environments? How might this process of transfer affect the way we understand the mathematics embodied in these environments and its relation to the mathematics outside them?

ACKNOWLEDGEMENTS

We wish to thank the editor, Richard Noss, and two anonymous reviewers for their helpful comments on an earlier version of this paper.

NOTES

[1] The two authors have contributed equally in the preparation of this paper, some ideas of which have been presented at the 10th International Congress on Mathematical Education, Copenhagen, Denmark (July 2004).

[2] It is possible that the validation criterion suggested by Mariotti (2001) was formulated specifically for *Cabri Geometry I*; the software used in the teaching experiment she conducted was Cabri I, but by the time she reported this experiment Cabri II was also available. The Cabri I tools resembled more closely those associated with straightedge-and-compass Euclidean geometry than the tools available in the more recent version of Cabri; the angle bisection method associated with Construction B would not be possible in Cabri I. Therefore, the drag test – or the robustness of a construction under dragging – could be linked to different construction processes in the two versions of the software. This observation does not affect our argument regarding the tradition that considers the drag test as a necessary and sufficient criterion of validation for solutions of construction problems in DGEs, primarily because no alternative criterion seems to have been proposed to cope with the expanded set of tools introduced in the newer version of Cabri. Rather, the observation raises the interesting issue of how validation practice in DGEs may maintain the pace of ever-changing software. We take up this issue later on in the paper.

[3] We acknowledge the case of constructions in which dragging appears to confirm invariance but in fact the invariance is 'approximate.' Although this is an important issue, we do not discuss it further as it goes beyond the scope of this paper.

[4] The 'hide/show' tool cannot provide conclusive evidence for the use or not of tools like 'rotate.' For this reason, the examiner cannot be sure about the exact process followed by the solver.

[5] With this remark we do not mean to suggest that the drag test criterion should have been, or that it was intended to be, the analogue of the straightedge-and-compass criterion.

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