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THE PERMUTATION PRINCIPLE
IN QUANTIFICATIONAL LOGIC[1]

The story goes back to 1940, with the publication of Quine's *Mathematical Logic* [5]. He there presents a system of quantificational logic in which only sentences or closed formulas are theorems. In that system, the sole rule of inference is Modus Ponens and the axioms are the universal closures of the following formulas:

(1) ϕ, for ϕ tautologous;

(2) $\forall\alpha(\phi\supset\psi) \supset (\forall\alpha\phi\supset\forall\alpha\psi)$;

(3) $\phi \supset \forall\alpha\phi$, for α not free in ϕ;

(4) $\forall\alpha\phi(\alpha) \supset \phi(\beta)$, with usual restrictions;

(5) $\forall\alpha\forall\beta\phi \supset \forall\beta\forall\alpha\phi$.

Suppose the variables, in alphabetic order, are $x_1, x_2, \dots$. Then Quine takes the *universal closure* of a formula ϕ to be the result of prefixing the quantifiers $\forall\alpha$, for α free in ϕ, to that formula in their reverse of anti-alphabetic order.

The presence of the Permutation Principle (5) somewhat mars the elegance of the system; but Berry [1] was able to show how to do without it. Alter the definition of closure: prefix the universal quantifiers in their alphabetic, not their anti-alphabetic, order. The change is small and seemingly insignificant. But taking as axioms the alphabetic closures of (1)–(4) suffices to prove all theorems of the original system, including Permutation.

This change was incorporated into the second edition of *Mathematical Logic* [6]. But the question remained as to whether it was essential. Could axiom-scheme (5) also be dropped from the original system without any loss of theorems?

The answer is no. Let us define a certain transform $(\phi)^t$ of a formula ϕ. Call the universal formula $\forall\alpha\psi$ *non-vacuous* if α occurs free in ψ. Given a subformula occurrence ψ of ϕ, say that ψ is *bound in* ϕ if a free variable α of ψ is bound by a quantifier $\forall\alpha$ outside of ψ but in ϕ. Now call the subformula occurrence ψ of ϕ *replaceable* if (i) it is a non-vacuous formula of the form $\forall x_1 \chi$ (note the specific variable), (ii) it is bound in ϕ, and (iii) it is

not a proper subformula of any other formula satisfying (i) and (ii). Then we may define the *transform* $(\phi)^t$ *of* ϕ to be the result of replacing all replaceable subformula occurrences in ϕ with $\perp$.

Let Q be Quine's original system and let Q^- be the result of dropping axiom-scheme (5) from Q. Then it may be shown by an induction on proofs that:

LEMMA 1. For any theorem ϕ of Q^-, the transform $(\phi)^t$ is classically valid.

Consider now the formula $\phi = \forall x_1 \forall x_2 Fx_1 x_2 \supset \forall x_2 \forall x_1 Fx_1 x_2$. Then its transform $(\phi)^t$ is $\forall x_1 \forall x_2 Fx_1 x_2 \supset \forall x_2 \perp$, which is not classically valid. It therefore follows from Lemma 1 that

THEOREM 2. The formula $\forall x_1 \forall x_2 Fx_1 x_2 \supset \forall x_2 \forall x_1 Fx_1 x_2$ is not a theorem of Q^-, and hence the axiom-scheme (5) is not redundant in the original system Q.

It should be clear that the non-derivability of many other instances of the scheme could be established in the same way.

Although the above formula ϕ is not derivable within Q^-, the formula $\forall x_2 \forall x_3 Fx_2 x_3 \supset \forall x_3 \forall x_2 Fx_2 x_3$ is. Indeed, given any formula ϕ, let ϕ^n, for $n \geqslant 0$, be the result of replacing each variable x_i in ϕ with x_{n+i}. Then it may be shown by an easy induction that if ϕ is a theorem of Q then ϕ^n is a theorem of Q^- for some nonnegative integer n; though whether ϕ^n is derivable for a particular n would seem to depend upon the availability of variables alphabetically earlier to those appearing in ϕ in a way that is hard precisely to determine.

This observation has a curious consequence in regard to the completeness of Q^-. The original system Q is complete; each valid sentence is a theorem. The defective system Q^- is not complete, since instances of Permutation are not derivable. However, Q^- is complete up to relettering of variables. Given any valid sentence ϕ, ϕ is derivable within Q and hence an alphabetic variant ϕ^n of ϕ is derivable within Q^-.

The story now goes to 1963, with the publication of papers by Kripke [2] and Lambert [3]. Kripke was concerned to block the derivation of the Barcan formula or its converse within a quantified version of the modal logic S5. He was able to do this by requiring, as in Quine [6], that only closed formulas be theorems. However, because he wished to dispense with

the rule of necessitation and because he also wished to allow for the empty
domain, he did not quite take Quine's revised system as the quantificational
basis for his modal logic. Instead, he made two changes. First, he loosened
up the definition of universal closure. He did not require of a universal
closure $\forall\alpha_1 \forall\alpha_2 \ldots \forall\alpha_n \varphi$ of φ that the quantifiers $\forall\alpha_1$, $\forall\alpha_2$, ..., $\forall\alpha_n$
be prefixed in any particular order or that their variables $\alpha_1, \alpha_2, \ldots, \alpha_n$ be
free; he only required that the resulting formula $\forall\alpha_1 \forall\alpha_2 \ldots \forall\alpha_n \varphi$ be
closed. Second, he took as his axioms the closures, in this sense, of (1), (2)
and (3), and the following variant of (4):

   (4)' $\forall\beta(\forall\alpha\varphi(\alpha) \supset \varphi(\beta))$, with usual restrictions.

Quite independently, Lambert developed a similar system. Like Kripke,
he was concerned to allow for the empty domain; but he also wished to
allow for theorems with free variables. He therefore proposed, in effect, that
the formulas (1), (2), (3) and (4), themselves be axioms (not their closures)
and that both Modus Ponens and Generalization be taken as the rules of
inference. To these axioms and rules, he then considered the result of
adding the standard classical axioms for identity.

As later became clear, Lambert's full system (with identity) is complete
for its intended interpretation. But it was then generally assumed that this
system without its identity axioms and the corresponding quantificational
part of Kripke's system (which had not been formulated with identity in the
first place) were also complete. Indeed, in their paper [4] of 1970, Leblanc
and Meyer gave a metalogical investigation of the Lambert fragment in
which it was presupposed that Permutation and related principles were
derivable; and, in [2], Kripke claimed completeness for his full modal
system, which would have entailed completeness for its quantificational
fragment. But then, Lambert pointed out, in a letter to Meyer of around
1968–9, the difficulty of deriving Permutation within the identity-free part
of his system; and independently, in his paper of 1970 ([7], p. 286, fn. 6),
Trew pointed to the related difficulty of deriving Permutation within
Kripke's system. The problem of deriving the principle became open and, at
least within the world of free logicians, achieved some notoriety.

It now appears that Permutation is not derivable within these systems.
To see this, we construct a somewhat devious semantics for quantificational
logic. Assume, for simplicity, that dyadic F is the sole predicate in our
language. Let a *sensitive model M* for that language be a quadruple

(A, R, f, g), where A (domain) is a non-empty set, R (extension of F) is a subset of $A \times A$, and both f and g are functions from A into $P(A)$, satisfying the conditions that $a \in f(a)$ and $a \in g(a)$ for all $a \in A$. The pair (A, R) may be regarded as a classical model. What f and g add to this model are relationships of dependence in the assignment of values. If, in the formula $F\alpha\beta$, a has been assigned to α, then only b from $f(a)$ can be assigned to β (for the formula $F\alpha\beta$ to be satisfied by the resulting assignment); and likewise, if b has been assigned to β, only a form $g(b)$ can be assigned to α.

We make this idea precise as follows. Let $M = (A, R, f, g)$ be a sensitive model. Add all objects of A to the language as names of themselves. Let the *standard substitution* ϕ^a/α *of a for α in ϕ*, with $a \in A$, be the result of replacing all free occurrences of α in ϕ with a. Let the *deviant substitution* $\phi^a_{\not}/\alpha$ be obtained from the standard substitution φ^a/α in two complementary steps. First, put $\perp$ for all those occurrences of Fab in φ^a/α that come from $F\alpha b$ in φ upon replacing the free occurrence of α in $F\alpha b$ with a and for which $b \in A$ but $a \notin g(b)$; let the result be ψ. Now put $\perp$ for all those occurrences of Fba in ψ that come from $Fb\alpha$ in φ upon replacing the free occurrence of α in $Fb\alpha$ with a and for which $b \in A$ but $a \notin f(b)$.

Define *deviant truth* by the following clauses:

(i) $M \models Fab$ iff $\langle a, b \rangle \in R$;

(ii) $M \models -\phi$ iff not $M \models \phi$;

(iii) $M \models (\phi \vee \psi)$ iff $M \models \phi$ or $M \models \psi$;

(iv) $M \models \forall\alpha\phi$ iff $M \models \phi^a_{\not}/\alpha$ for all $a \in A$.

Say that a formula ϕ is *true in M* if the result of substituting any elements of A for the free variables of ϕ is true in M.

It may now be shown by a tedious induction on proofs that:

LEMMA 3. All the theorems of Lambert's system, and hence of Kripke's system too, are true in any sensitive model.

It is instructive to see why the proof breaks down for the systems with identity. Take the identity axiom $\varphi = \forall\alpha\forall\gamma\forall\beta(\alpha = \beta \supset (F\alpha\gamma \supset F\beta\gamma))$, for α, β and γ distinct variables. Then relative to the model M above, $\varphi \, {}^1\!/\alpha \, {}^0\!/\gamma \, {}^1\!/\beta = (1 = 1 \supset (F \, 1 \, 0 \supset \perp))$ and hence is false.

Consider now the following sensitive model $M = (A, R, f, g)$:

$$A = \{0, 1\}; \qquad R = A \times A; \qquad f(0) = f(1) = A;$$
$$g(0) = \{0\} \text{ and } g(1) = \{1\}.$$

Take distinct variables α and β. Then $M \models \forall\alpha\forall\beta F\alpha\beta$, since $F\alpha\beta\,^{a}\!/\alpha\,^{b}\!/\beta = \mathrm{F}ab$ for all $a, b \in A$. On the other hand, $M \not\models \forall\beta\forall\alpha F\alpha\beta$, since $F\alpha\beta\,^{0}\!/\beta\,^{1}\!/\alpha = \bot$.

From this and Lemma 3 follows the conclusion to our story:

THEOREM 4. The sentence $\forall\alpha\forall\beta F\alpha\beta \supset \forall\beta\forall\alpha F\alpha\beta$, for α distinct from β, is not derivable within the systems of Kripke and of Lambert.

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NOTE

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