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NATURAL DEDUCTION AND ARBITRARY OBJECTS

This paper is an abridged and simplified version of my monograph *Reasoning with Arbitrary Objects* [4]. It may be read by the diligent as a preparation for the longer work or by the indolent as a substitute for it. But the reader, in either case, may find it helpful to consult the paper, *A Defence of Arbitrary Objects* [3], for general philosophical orientation.

This paper deals with certain problems in understanding natural deduction and ordinary reasoning. As is well known, there exist in ordinary reasoning certain procedures for arguing to a universal conclusion and from an existential premiss. We may establish that all objects have a given property by showing that an arbitrary object has the property; and having shown that there exists an object with a given property, we feel entitled to give it a name and infer that it has the property. For example: we may establish that all triangles have interior angles summing to 180° by showing of an arbitrary triangle that its interior angles sum to 180°; and having established that there exists a bisector to an angle, we feel entitled to give it a name and infer that it is a bisector to the angle.

These informal procedures correspond to certain of the quantificational rules in systems of natural deduction. Corresponding to the first is the rule of universal generalisation, which allows us to infer $\forall x\varphi(x)$ from $\varphi(a)$ under suitable restrictions. Corresponding to the second is the rule of existential instantiation, which allows us to infer $\varphi(a)$ from $\exists x\varphi(x)$, again under suitable restrictions.

In these inferences, certain terms play a crucial role; and it is natural to ask how it should be understood. What role should be attributed to the term a in inferences from natural deduction? What is to be made of our talk of arbitrary triangles or indefinite bisectors in ordinary reasoning?

We here take seriously the idea that the critical terms in these inferences refer to arbitrary or representative objects. The term a in the inferences from natural deduction functions as a name of a suitable arbitrary object. And our talk of arbitrary triangles or indefinite triangles is to be taken at its face value as evincing reference to arbitrary objects.

Our principle concern will be to apply this hypothesis to two main systems of natural deduction: the one of Quine's *Methods of Logic* [14]; the

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other of Copi's *Symbolic Logic* [2], as reformed by Kalish [7]. We shall also have an incidental interest in Gentzen's original system of natural deduction. In the case of each of these systems, we shall put forward a rigorously formulated semantics in terms of arbitrary objects and then prove soundness with respect to that semantics.

However, the significance of the semantics does not merely rest with these particular results but lies as much with the general light it is able to shed on the methods of quantificational reasoning. The semantics provides a powerful heuristic for discovering new systems of natural deduction or rediscovering old ones. It serves to motivate, to make 'semantic sense' of restrictions on the rules, restrictions that would otherwise appear arbitrary; and it provides a semantic basis for the comparison and evaluation of the different systems that have been proposed. It yields a general method for constructing proofs of soundness, one that derives from very general considerations concerning arbitrary objects. And it provides an account of natural deduction that accords well with our ordinary understanding of quantificational practice. Indeed, it seems fair to say that once one is accustomed to thinking of these systems in terms of a theory of arbitrary objects, it is hard to think of them in any other way.

The present work is part of a larger project, one in which the theory of arbitrary objects is to be applied to the expression of generality in various different spheres of discourse. These further applications have been altogether ignored. But even some closely related topics have not been considered. These include the application of the semantics to: the general study of systems of natural deduction with a rule of existential instantiation; systems, such as Kalish's and Montague's [8], that do not permit unrestricted conditionalization; free logic systems; intuitionistic systems; the ϵ-calculus of Hilbert and the η-calculus of Hailperin [5]; systems without quantifiers but with 'definitions' of arbitrary objects in their place. However, all but the last two of these applications are treated in the monograph.

The paper divides into two parts. The first presents the general theory of arbitrary objects; the second gives its application to natural deduction. The first part presents the models (Section 1), defines the notions of truth and validity (Section 2), and embarks on a study of definitions (Section 3). The second part gives the application to Gentzen's system (Section 4), Quine's system (Section 5), and Copi's (Section 6). We conclude with a more detailed numeration of the advantages to be gained from the adoption of the generic semantics (Section 7).

1. THE MODELS

Let **L** be an arbitrary first-order language. To fix our ideas, we might suppose that **L** contains, for each nonnegative *n*, countably infinite predicate letters of degree *n*. But nothing turns on how exactly we stipulate **L** to be.

Let **M** be a classical model for **L**. So **M** is the form (I, . . .), where I is a non-empty set representing the domain of individuals and . . . is used to indicate the interpretation of the non-logical constants of **L**. So if they comprise a countable infinity of predicates in each degree, then . . . may well consist of a function that assigns to each predicate of degree *n* a set of *n*-tples from I.

We use the prefix '*I*' to indicate that an item is individual as opposed to variable, or classical as opposed to generic. So **M** itself is called an *I-model*, while I is called an *I-domain*, with *individuals* or *I-objects* as members.

A classical model **M** may be expanded to one that contains arbitrary objects. Any such model $\mathbf{M}^+$ is of the form (I, . . . , A, $\prec$, V), where:

(i) (I, . . .) is the model **M**;

(ii) A is a finite set of objects disjoint from I;

(iii) $\prec$ is a relation on A;

(iv) V is a non-empty set of partial functions from A into I, i.e., functions v whose domain Dm(v) is a subset of A and whose range Rg(v) is a subset of I.

Intuitively, the significance of the extra components is this. A is the set of arbitrary objects or variables. It is assumed that these are distinct from the individuals; it is merely from convenience that A is taken to be finite. $\prec$ is the relation of dependence between arbitrary objects. 'a $\prec$ b' indicates that the value of a (what is is) depends upon the value of b (what b is). V is the family of value assignments. Suppose that v belongs to V, with domain $\{a_1, a_2, \ldots, a_n\}$, and that $v(a_1) = i_1$, $v(a_2) = i_2, \ldots,$ $v(a_n) = i_n$. This may be pictured as follows:

$$v: \frac{a_1 a_2 \ldots a_n}{i_1 i_2 \ldots i_n}.$$

Then the presence of such a v in V indicates that $i_1, i_2, \ldots, i_n$ can simultaneously be assigned to $a_1, a_2, \ldots, a_n$. The members of V will be called the *admissable* value-assignments. The arbitrary partial functions from A into I might, by contrast, be called the *possible* value-assignments.

We use the prefix '*A*' for items that are arbitrary or generic as opposed to individual or classical. So $\mathbf{M}^+$ itself is called a *possible A-model*, while A is called an *A-domain*, with *A-objects* as members. We use 'i', 'j', 'k', ... as variables for *I*-objects, and 'a', 'b', 'c', ... as variables for *A*-objects.

I would prefer to follow traditional usage by calling the members of I *constants* and the members of *A variables.* But the modern practice of using these terms for symbols rather than objects is so well entrenched that a return to traditional usage could only invite confusion.

As an example of an *A*-model $\mathbf{M}^+ = (I, \ldots, A, \prec, V)$, we might take:

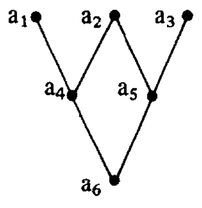
(i) $(I, \ldots) = \mathbf{N}$, the standard model for Peano arithmetic;

(ii) $A = \{a_1, a_2, \ldots, a_6\}$, for $a_1, \ldots, a_6$ distinct objects not in I;

(iii) $\prec = \{\langle a_4, a_1\rangle, \langle a_4, a_2\rangle, \langle a_5, a_2\rangle, \langle a_5, a_3\rangle, \langle a_6, a_4\rangle, \langle a_6, a_5\rangle\}$;

(iv) $V = \{v: v \subseteq \{\langle a_1, i_1\rangle, \langle a_2, i_2\rangle, \ldots, \langle a_6, i_6\rangle\}$, where $i_1, i_2, \ldots, i_6 \in \mathbf{I}$, i_1 and i_3 are even, $i_1 + i_2 = i_4$, $i_2 + i_3 = i_5$, and $i_4 \times i_5 = i_6\}$.

The diagram for $\prec$ is given below



The reader may find it helpful to think of a_4, a_5 and a_6 as the respective *A*-objects $a_1 + a_2$, $a_2 + a_3$ and $a_4 \times a_5$, although the result of applying the arithmetic operations to *A*-objects has not been officially defined.

We shall adopt the following terminology in connection with *A*-models. With $\mathbf{M}$ and $\mathbf{M}^+$ as above, we say that $\mathbf{M}^+$ is *based upon* $\mathbf{M}$ or that $\mathbf{M}$

underlies $\mathbf{M}^+$. We follow the convention that '$\mathbf{M}^+$' is always used to denote an A-model $(I, \ldots, A, \prec, V)$ based upon the classical model $\mathbf{M} = (I, \ldots)$.

For $a \in A$, the *value-range* $VR(a)$ of a is $\{v(a)\colon v \in V\}$. Thus the value-range of an A-object consists of all the values it can assume. If $VR(a) \neq I$, then a is said to be *value-restricted* and otherwise to be *value-unrestricted* or *universal.* In the special case in which $VR(a) = \Lambda$, we say a is null. An A-object $a \in A$ is *dependent* if $a \prec b$ for some b, and otherwise it is *independent.* An A-object is *restricted* if it is either value-restricted or dependent and otherwise it is *unrestricted.* In our previous example: $VR(a_1) = VR(a_3) = \{i \in I\colon i \text{ is even}\}$ and $VR(a_2) = I$; a_1 and a_3 are value-restricted and a_2 is value unrestricted; a_4, a_5 and a_6 are dependent and a_1, a_2 and a_3 are independent; a_2 is unrestricted and the other A-objects are restricted.

A subset B of A is said to be *closed* if $b \in B$ whenever $a \in B$ and $a \prec b$. We use $[B]$ for the closure of B, i.e., the smallest closed set to contain B; and we use $|B|$ for $[B] - B$. In case $B = \{a\}$, we write '$[B]$' and '$|B|$' as '$[a]$' and '$|a|$' respectively. In our previous example: the closed sets are $\{a_1\}$, $\{a_2\}$, $\{a_3\}$, $\{a_4, a_1, a_2\}$, $\{a_5, a_2, a_3\}$ and $\{a_1, a_2, a_3, a_4, a_5, a_6\}$; $[a_4]$ is $\{a_4, a_1, a_2\}$ and $|a_4|$ is $\{a_1, a_2\}$.

In order for the possible A-model $\mathbf{M}^+ = (I, \ldots, A, \prec, V)$ to be an *actual* A-model, it must be subject to some further conditions. These go as follows:

(v) (a) (Transitivity) $a \prec b$ & $b \prec c$ implies $a \propto c$;
(b) (Irreflexivity) There is no A-object a for which $a \prec a$.

(vi) (Restriction) V is closed under restriction, i.e., $v \in V$ and $B \subseteq A$ implies that $v \upharpoonright B \in V$ (where $v \upharpoonright B$ is the result of restricting the domain of v to the elements in B).

(vii) (Partial Extendibility) Any $v \in V$ can be extended to the closure of its domain, i.e., there exists a $v^+ \in V$ for which $v^+ \supseteq v$ and $Dm(v^+) \supseteq [Dm(v)]$;

(viii) (Piecing) Let v_1 and v_2 be two assignments in V. Suppose that (a) their domains are closed and (b) they do not differ on common arguments, i.e., for no a in both $Dm(v_1)$ and $Dm(v_2)$ does $v_1(a) \neq v_2(a)$. Then $v = v_1 \cup v_2 \in V$.

Transitivity, Irreflexivity and Restriction are very plausible under the proposed construal of $\prec$ and V. For first, if the value of a depends upon that of b and the value of b upon that of c, then the value of a depends (indirectly) upon that of c. Second, the value of an A-object can never depend upon its own value, but only the value of another A-object. Finally. Suppose that $v \in V$ assigns the individuals $i_1, i_2, \ldots, i_n$ to the A-objects $a_1, a_2, \ldots, a_n$ in its domain. This is represented by the diagram:

$$v: \frac{a_1 a_2 \ldots a_n}{i_1 i_2 \ldots i_n}.$$

Then we may allow that a selection $i_{j1}, i_{j2}, \ldots, i_{jk}$ of the individuals can be assigned to the corresponding selection $a_{j1}, a_{j2}, \ldots, a_{jk}$ of the A-objects, i.e., that there is a $v' \in V$ whose diagram is:

$$v': \frac{a_{j1}, a_{j2}, \ldots, a_{jk}}{i_{j1}, i_{j2}, \ldots, i_{jk}}.$$

Given (v), each A-object $a \in A$ can be assigned a *degree* or *level* $l(a)$ *of dependence.* If a is independent, then $l(a) = 0$. If a is dependent upon the A-objects $b_1, b_2, \ldots, b_m$, then the level $l(a)$ of a is 1 plus the maximum level of $b_1, b_2, \ldots, b_m$. So in our previous example, $l(a_1) = l(a_2) = l(a_3) = 0$, $l(a_4) = l(a_5) = 1$, and $l(a_6) = 2$.

The condition of Partial Extendibility can be seen as articulating an aspect of the sense of dependency. We so understand the relation of dependency that if a depends upon b then a cannot take a value without the support of a value for b. But this then has the consequence that an assignment of values cannot be made to $B \subseteq A$ except on the basis of an assignment that at least extends to the closure $[B]$ of B.

It is the condition of Piecing that is the most significant. Suppose that we have two value assignments v_1, v_2 in V whose diagrams are:

$$v_1: \frac{a_1, a_2, \ldots, a_k}{i_1, i_2, \ldots, i_k}; \qquad v_2: \frac{a_{k+1}, a_{k+2}, \ldots, a_l}{i_{k+1}, i_{k+2}, \ldots, i_l}.$$

Then Piecing tells us when we can put the two assignments v_1 and v_2 together to form the single admissable assignment $v = v_1 \cup v_2$, with diagram:

$$v: \frac{a_1, a_2, \ldots, a_k, a_{k+1}, a_{k+2}, \ldots, a_l}{i_1, i_2, \ldots, i_k, i_{k+1}, i_{k+2}, \ldots, i_l}.$$

The one proviso tells us that the two assignment should not assign distinct individuals to the same argument, i.e., there should not be a_p, with $1 \leqslant p \leqslant k$, and a_q, with $k + 1 \leqslant q \leqslant 1$, for which $a_p = a_q$ and $i_p \neq i_q$. This is what one might call a manifest obstacle to piecing. The second proviso says, in effect, that there must be no hidden obstacle to piecing. This concept may be illustrated by means of our previous example. We may assign 0 to a_4 (i.e., the assignment $v_1 = \{\langle a_4, 0\rangle\} \in V$), and we may assign 1 to a_5 (i.e., the assignment $v_2 = \{\langle a_5, 1\rangle\} \in V$). However, we cannot simultaneously assign 0 to a_4 and 1 to a_5 (i.e., $v = v_1 \cup v_2 \notin V$); for the assignment of 0 to a_4 requires that a_2 take the value 0, while the assignment of 1 to a_5 requires that a_2 take the value 1. This kind of hidden obstacle to piecing is excluded by requiring that the domains of the assignments to be pieced be closed.

2. TRUTH AND VALIDITY

The language **L** so far contains no means of referring to A-objects. We shall therefore suppose that it is endowed with a new stock of symbols – the A-letters $a, b, c, \ldots$ – that syntactically will behave just like individual names. The resulting language will be designated $\mathbf{L}^*$.

It is important to appreciate that the language $\mathbf{L}^*$ may contain three kinds of symbol in subject position: the variables $x, y, z, \ldots$; the constant symbols $m, n, p, \ldots$ (if they exist); and the A-letters $a, b, c, \ldots$. Only the constant symbols and A-letters will be thought to play a properly designatory role; the variables will merely serve as part of the apparatus of quantification.

Formulas from either **L** or $\mathbf{L}^*$ will now be allowed to contain free variables. Expressions with free variables that would normally be called formulas will instead be called *pseudo-formulas.* So Fa is a formula, but Fx is a pseudo-formula.

An A-model $\mathbf{M} = (I, \ldots, A, \prec, V)$ for **L** may be extended to $\mathbf{L}^*$ by adding a designation function. The resulting model $\mathbf{M}^*$ is then of the form $(I, \ldots, A, \prec, V, d)$, where d is a function taking each A-letter of $\mathbf{L}^*$ into an A-object of A. Often we shall simply use 'a' for the designation $d(a)$ of a.

Two concepts of truth for generic statements may be distinguished, one *relative* and the other *absolute.* Let $\varphi = \varphi(a_1, \ldots, a_n)$ be a formula whose A-letters are as displayed. Then φ is *true in* the A-model $\mathbf{M}^*$ *relative to* $v \in V$ – in symbols, $\mathbf{M}^* \models_v \varphi$ – if $a_1, \ldots, a_n \in \mathrm{Dm}(v)$ and $\mathbf{M} \models [v(a_1), \ldots,$

$v(a_n)$]. On the other hand, φ is *(absolutely) true in* $\mathbf{M}^*$ – in symbols, $\mathbf{M}^* \models \varphi$ – if $\mathbf{M}^* \models_v \varphi$ for any $v \in V$ for which $a_1, \ldots, a_n \in Dm(v)$. In other words, a statement concerning A-objects is true just in case it is true for all of their values. This stipulation is what we have previously called the *principle of generic attribution* [3].

It will be helpful to introduce some notation. Let $\mathbf{M}^*$ be an A-model for $\mathbf{L}^*$. Set $A_\varphi = \{a: a \text{ is an } A\text{-letter of } \varphi\}$. (This notation may be extended to sets of formulas or to items, like proofs, that are associated with sets of formulas.) Say that v is *defined on* B or *on* φ, with $A_\varphi = B$, if $Dm(v) = B$; and say that v is *defined over* B or *over* φ, with $A_\varphi = B$, if $Dm(v) \supseteq B$. Let us suppose that the language $\mathbf{L}$ contains each individual i of $\mathbf{M}$ as a name of itself and that v is defined over φ. Then we use $v(\varphi)$ for the result of substituting $v(a)$ for each occurrence of an A-letter a in φ. The principle of generic attribution now takes the following simple form:

$$\mathbf{M}^* \models \varphi \quad \text{iff} \quad \mathbf{M} \models v(\varphi) \quad \text{for each } v \in V \text{ defined over } \varphi.$$

The principle may be extended to a set Δ of formulas in the obvious way. Say that $v \in V$ is *defined over* Δ if $Dm(v) \supseteq A_\Delta$; and let $v(\Delta) = \{v(\varphi): \varphi \in \Delta\}$. Then we stipulate that:

$$\mathbf{M}^* \models \Delta \quad \text{iff} \quad \mathbf{M} \models v(\Delta) \quad \text{for each } v \in V \text{ defined over } \Delta.$$

If $\Delta = \{\varphi_1, \ldots, \varphi_n\}$, then the truth of Δ (in $\mathbf{M}^*$) is equivalent to the truth of $\varphi_1 \wedge \ldots \wedge \varphi_n$.

We turn now to the concept of validity. By a *sequent* or *inference* we mean an ordered pair (Δ, φ), where φ is a formula and Δ is a finite set of formulas (from $\mathbf{L}^*$). Where $\Delta = \{\varphi_1, \ldots, \varphi_n\}$, the sequent may be written more perspicuously in the form:

$$\varphi_1, \ldots, \varphi_n/\varphi;$$

or

$$\frac{\varphi_1, \ldots, \varphi_n}{\varphi}.$$

The *classical* validity of a sequent may be defined in the usual way. (The A-letters are treated as if they are the names of individuals.)

There are essentially two different ways of defining generic validity, one corresponding to the concept of absolute truth and the other to the concept of relative truth.

Let X be a set of A-models for $\mathbf{L}^*$. Then the inference (Δ, φ) is *truth-to-truth valid relative to X* if, for any model $\mathbf{M}^*$ of X, $\mathbf{M}^* \models \varphi$ whenever $\mathbf{M}^* \models \Delta$. In other words, an inference is valid if the conclusion is true whenever the premisses are true. The inference (Δ, φ) is *case-to-case valid relative to X* if, for any model $\mathbf{M}^*$ of X and any $v \in V$ defined over Δ and φ, $\mathbf{M} \models_v \varphi$ whenever $\mathbf{M} \models_v \Delta$. In other words, an inference is valid if the conclusion is true in any case in which the premisses are true. We say that an inference is *truth-to-truth* or *case-to-case valid* simpliciter if it is truth-to-truth or case-to-case valid relative to the class of all A-models for $\mathbf{L}^*$.

The difference in the two concepts may be appreciated by considering the inference $Fa/\forall xFx$. If a denotes a value-unrestricted A-object (in each of the A-models of X), then the inference is truth-to-truth valid. But the inference will not be case-to-case valid if F is true of some individuals yet not of others (in one of the models $\mathbf{M}^*$ of X). For we may then choose a $v \in V$ for which $v(Fa)$ is true and $v(\forall xFx) = \forall xFx$ is false in $\mathbf{M}$.

More informally, we may think of inferences evaluated under the concept of case-to-case validity as schematic inferences, representing a different specific inference for each choice of values for the A-objects. On the other hand, we may think of inferences evaluated under the concept of truth-to-truth validity as specific inferences concerning the A-objects themselves.

We are interested in the cases in which the concepts of classical and generic validity coincide. It is not true that each classically valid inference is truth-to-truth valid (relative to any X). For consider the following classically valid inference:

$$\frac{Fa \wedge Gb}{Fa}.$$

Is it truth-to-truth valid? Suppose, for reduction, that Fa is not true in the A-model $\mathbf{M}^*$. Then for some $v \in V$, $\mathbf{M} \not\models Fv(a)$. We now wish to define a $v \in V$ defined over both a and b: for then $\mathbf{M} \not\models Fv^+(a) \wedge Gv^+(b)$ and so $\mathbf{M}^* \not\models Fa \wedge Gb$. But we have, in general, no guarantee that such a v can be found. For example there could not be such a v if b were a null A-object.

It turns out that this is essentially the only obstacle to the coincidence and that we have the following general result:

THEOREM 1. Suppose that the inference (Δ, φ) is classically valid; and let X be a class of models $\mathbf{M}^*$ with the property that any $v \in V$ defined on A_φ is extendible to $v^+ \in V$ defined over A_Δ. Then (Δ, φ) is truth-to-truth valid relative to X.

Proof. Pick a $\mathbf{M}^*$ in X and suppose that not $\mathbf{M}^* \models \varphi$. Then for some $v \in V$ defined over φ, not $\mathbf{M} \models v(\varphi)$. By Restriction, we may suppose that v is defined on the *A*-letters of φ. Given the extendibility property, there is a $v^+ \supseteq v$ in V that is defined over Δ. Since (Δ, φ) is classically valid, not $\mathbf{M} \models v^+(\Delta)$. But then not $\mathbf{M}^* \models \Delta$ – and we are done.

Say that an *A*-model $\mathbf{M}^*$ is *extendible over* the set B of *A*-objects if every $v \in V$ defined on a subset of B can be extended to a $v^+ \in V$ defined over B. Then as an immediate consequence of the above theorem, we have:

COROLLARY 2. Let X be a class of *A*-models that is extendible over the set $B = \{b_1, \ldots, b_n\}$; and let (Δ, φ) be a classically valid inference whose *A*-letters are drawn from $B = \{b_1, \ldots, b_n\}$. Then the inference (Δ, φ) is truth-to-truth valid in X.

Another consequence of the theorem concerns the validity of a single formula. Say that the formula φ is *valid* if the inference (Λ, φ) is *valid.* This definition applies whether the validity is classical, truth-to-truth, or case-to-case; but for the generic case, the two concepts of validity coincide. We now have:

COROLLARY 3. Suppose the formula φ is classically valid. Then it is generically valid.

Proof. From Theorem 1, since the extendibility condition is trivially satisfied.

For case-to-case validity there is no impediment to classical reasoning.

THEOREM 4. Suppose the inference (Δ, φ) is classically valid. Then it is case-to-case valid.

Proof. Suppose otherwise. Then for some *A*-model $\mathbf{M}^*$ and $v \in V$, $\mathbf{M} \models_v \Delta$ and $\mathbf{M} \not\models_v \varphi$. But by letting the *A*-letters denote individuals in accord with v, it is then clear that there is a classical model in which φ is true and is false.

However, for the concept of case-to-case validity, the transitivity of implication may fail. For suppose that φ implies ψ and that ψ implies χ (i.e., that the inferences φ/ψ and ψ/χ are valid relative to X). We wish to show that φ implies χ. So let v be an assignment defined on both φ and χ. If we could show that v was extendible to the A-objects denoted by the A-letters in ψ, we would be done. But we have in general no guarantee that v can be so extended.

As in the previous case, the difficulty can be overcome by making the appropriate stipulation concerning extendibility:

THEOREM 5. (Cut for Case-to-case Validity.) Suppose that the inferences Δ/φ and $\varphi, \Gamma/\psi$ are case-to-case valid in X; and suppose that X is a class of models $\mathbf{M}^*$ with the property that any $v \in V$ defined on $A_\Delta \cup A_\Gamma \cup A_\psi$ is extendible to a $v^+ \in V$ defined over A_φ. Then $\Delta, \Gamma/\psi$ is case-to-case valid in X.

Proof. Straightforward.

3. DEFINITIONS

Any set Δ of pseudo-formulas in a single variable x can be regarded as a definition of an A-letter. Accordingly, we take a definition to be an ordered pair (a, Δ) where a is an A-letter and Δ is a set of pseudo-formulas in a single variable x. Given a definition (a, Δ), we take a to be the *defined term* and the A-letters in Δ to be the *given terms*; we call Δ itself the *defining condition.* We say that (a, Δ) *defines* the defined term a from the given terms. We allow that Δ may be empty or even contain an occurrence of a. In case $\Delta = \{\varphi\}$, we may write the definition (a, Δ) as (a, φ).

It is our view that definitions correspond to let-clauses in ordinary mathematical discourse. If, for example, we declare:

$$\text{Let } y = x^2,$$

then we may be taken to be fixing the A-object denoted by y in terms of the A-object denoted by x. Such a clause would be represented in our notation by the ordered pair $(a, x = b^2)$ (where 'y' has now been written as 'a' and 'x' as 'b').

Given a definition (a, Δ) and a term t, let $\Delta(t)$ be the result of substituting t for the free variable x of Δ. Then we say that the A-model

$\mathbf{M}^*$ *realizes* the definition (a, Δ) if the following two conditions are satisfied.

(i) $|a| = [A_\Delta]$, i.e., $|a| = \{c \in A: c$ is an A-letter of Δ or $b \prec c$ for b an A-letter of Δ;

(ii) for any $u \in V$ with domain $|a|$ and any $v = u \cup \{\langle a, i\rangle\}$, $v \in V$ iff $\mathbf{M} \models v(\Delta(a))$.

Thus the import of the definition is twofold. First, the object a denoted by the defined term a is to depend upon the objects $b_1, b_2, \ldots$ denoted by the given terms and whatever else they depend upon. Second, the values of a, for given values of $b_1, b_2, \ldots$ and their dependents, are to comprise all those individuals that then satisfy $\Delta(a)$.

Let us go back to our example of Section 1 and suppose that $\mathbf{M}^+$ is expanded to a model $\mathbf{M}^*$ in which $a_1, a_2, \ldots, a_6$ denote $a_1, a_2, \ldots, a_6$ respectively. Then we see that $\mathbf{M}^*$ realizes the definition $(a_4, x = a_1 + a_2)$; for $|a_4| = [\{a_1, a_2\}] = \{a_1, a_2\}$ and, whenever $u = \{\langle a_1, i\rangle, \langle a_2, j\rangle\} \in V$, then $v = u \cup \{\langle a_4, k\rangle\} \in V$ iff $\mathbf{M} \models k = i + j$, i.e., iff $k = i + j$.

Note that if the defining condition $\Delta = \Lambda$, then a model $\mathbf{M}^*$ will realize the definition (a, Δ) just in case a is unrestricted.

Our main interest will be in systems of definitions. A *definitional system* is simply a finite set of definitions. Its *defined terms* are the defined terms of the member definitions and its *given terms* are the given terms of the member definitions that are not also defined terms. We say that **S** *defines* a from b if some member definition defines a from b.

For a *system* of definitions **S**, we say that the model $\mathbf{M}^*$ *realizes* **S** if:

(i) $a \neq b$ for distinct defined terms a and b of **S**;

(ii) $\mathbf{M}^*$ realizes each member definition of **S**.

In reference to our previous example, let **S** be the system whose members are $(a_4, x = a_1 + a_2)$, $(a_5, x = a_2 + a_3)$, and $(a_6, x = a_4 \times a_5)$. Then it should be clear that $\mathbf{M}^*$ realizes this system.

The question now arises: when is a system of definitions realizable? This question may be answered by invoking two conditions on definitional systems. Say that a system **S** is *unequivocal* if no A-letter a is the defined term of two distinct definitions (a, Δ) and (a, Γ) of **S**. In an unequivocal system, no term gets defined twice; its meaning, if determined at all, is unequivocally determined. Now say that the system **S** is *non-circular* if there is no sequence of A-letters $a_1, a_2, \ldots, a_n$ such that $a_1 = a_n$ and for each $i = 1, 2, \ldots, n-1, a_i$ is defined from a_{i+1}. In a noncircular system that are no definitional circles, with one term ultimately being defined in terms of itself.

These two conditions may quite naturally be regarded as conditions of propriety on let-clauses. It is improper to say:

Let $y = x^2$.
Let $y = x^3$.

And it is improper to say:

Let $y = x^2$.
Let $z = y^3$.
Let $x = z^2$.

The first infringes unequivocality and the second infringes non-circularity.

We now have the following result:

THEOREM 6. Suppose that the system **S** of definitions is unequivocal and non-circular. Let **M** be any classical model. Then there is an A-model **M*** that realizes **S**.

Proof. Suppose **M** = (I, . . .). To define **M*** = (I, . . . , A,≺, V, d), it suffices to define A,≺, V and d. We let A be the set of A-letters appearing in **S** and d the identity function. Thus each A-letter will be taken to denote itself. We shall use 'a' or 'a' depending upon whether we are thinking of the item as a symbol or an object. For a, b ∈ A, we say that a ≺ b if there is a sequence of A-letters $a_1, a_2, \ldots, a_n$ such that $\mathrm{a}_1 = \mathrm{a}$, $\mathrm{a}_n = \mathrm{b}$ and, for $i = 1, 2, \ldots, n-1, a_i$ is defined from a_{i+1} in **S**.

It remains to define V. First, we define a subset V^- of V. $\mathrm{V}^- = \{$v: the domain B of v is closed and, for any definition (a, Δ) whose A-letters all occur in B, **M** ⊨ v($\Delta(a)$)$\}$. We then let V consist of all the restrictions of assignments in V^-.

We must now show that **M*** is an A-model. The verification of (i)–(vi) is trivial. Condition (v) (a) (Transitivity) follows from the definition of ≺; and (v) (b) (Irreflexivity) follows from the non-circularity of **S**. Conditions (vi) (Restriction) and (vii) (Partial Extendibility) are straightforward given the definition of V from V^-. This leaves (viii) (Piecing). First note that if $v^+ \in V^-$ and v is the restriction of v^+ to a closed domain then $v \in V^-$. Now take $v_1, v_2 \in V$ that have closed domains and agree on common arguments. Then v_1 and v_2 are the restriction of assignments v_1^+ and v_2^+ in V^- and, since their domains are closed, they also belong to V^-. Since v_1 and v_2 agree on common arguments, $v = v_1 \cup v_2$ is a function. Take any definition (a, Δ) of **S** whose A-letters all occur in the domain of v. Since $a \in v_1$ or $a \in v_2$, either all of the A-letters of the definition occur either in the domain of v_1 or in the domain of v_2. So $\mathbf{M} \models v(\Delta(a))$. But then $v \in V^- \subseteq V$.

Finally let us show that $\mathbf{M}^*$ realizes **S**. Since d is the identity function, it is clear that $a \neq b$ whenever $a \neq b$. Now take any definition (a, Δ) of **S**, any $u \in V$ with domain $|a|$, and any $v = u \cup \{\langle a, i\rangle\}$. We wish to show that $v \in V$ iff $\mathbf{M} \models v(\Delta(a))$. First suppose that $v \in V$. Since each A-letter of (a, Δ) appears in the domain of v, $\mathbf{M} \models v(\Delta(a))$. Now suppose that $\mathbf{M} \models v(\Delta(a))$. We wish to show that $v \in V$; and, since the domain of v is closed, it suffices to show that $v \in V^-$. So take any definition (b, Γ) of **S** all of whose A-letters appear in the domain of v. There are two cases. (1) $b = a$. Since S is unequivocal, $\Gamma = \Delta$ and $\mathbf{M} \models v(\Gamma(b))$ by hypothesis. (2) $b \neq a$. Then all the A-letters of (b, Γ) belong to $|a|$ and, since $u \in V$, $\mathbf{M} \models v(\Gamma(b))$.

In view of the connection with classical validity, we shall be interested in those cases in which a system of definitions defines an extendible set of A-objects. Say that a definition (a, Δ) is *total* in the classical model **M** if, for any function u from the set B of given terms of (a, Δ) into I, there is an $i \in I$ for which $\mathbf{M} \models u(\Delta(i))$. In a total definition, the object designated by the defined term takes a value whenever the objects designated by the given terms take a value. Now say that a system **S** of definitions is *complete* if each A-letter of **S** is a defined term of **S**. In a complete system, each given term of one definition is a defined term of another definition.

THEOREM 7. Let **S** be a complete set of total definitions in the model **M**. Then any model $\mathbf{M}^*$ that realizes **S** is extendible over the set B of objects designated by the A-letters in **S**.

Proof. Let v be any assignment in V defined on a subset B′ of B. We wish to show that v can be extended to an assignment in V that is defined on the whole of B. By Partial Extendibility, v can be extended to an assignment in V with closed domain; so we might as well suppose that v already has a closed domain.

Let $\mathrm{C} = \mathrm{B} - \mathrm{B}'$. Each A-object c in C may be assigned a level and so we may order the members $c_1, c_2, \ldots, c_n$ of C in such a way that the level of c_i, for $i = 1, 2, \ldots, n-1$, is never greater than the level of c_{i+1}. Let $\mathrm{C}_k \doteq \{c_1, \ldots, c_k\}$ (so that $\mathrm{C}_n = \mathrm{C}$). We then show by induction on k that v can be extended to a v_k in V with domain $\mathrm{B}' \cup \mathrm{C}_k$. Note that each of the domains $\mathrm{B}' \cup \mathrm{C}_k$ is closed.

$\underline{k = 0.}$ Then $\mathrm{B}' \cup \mathrm{C}_0 = \mathrm{B}'$; and we may let $v_k = v$.

$\underline{k > 0.}$ Since **S** is complete, there is a definition of the form (c_k, Δ) in **S**. Since $\mathbf{M}^*$ realizes **S**, we have, for each given term b of (c_k, Δ), that $c_k \prec b$ and hence that $b \in \mathrm{B}' \cup \mathrm{C}_{k-1}$. By the Inductive Hypothesis (IH), there is a v_{k-1} in V with domain $\mathrm{B}' \cup \mathrm{C}_{k-1}$. By **S** total in **M**, there is an $i \in \mathrm{I}$ for which $\mathbf{M} \models v_{k-1}(\Delta(i))$. Let $v_k = v_{k-1} \cup \{\langle c_k, i\rangle\}$. Then by the definition of realizability, $v_k \in \mathrm{V}$ – as required.

4. THE HILBERT AND GENTZEN SYSTEMS

We here apply our generic semantics to an axiomatic system in the style of Hilbert and to a natural deduction system in the style of Gentzen. The application in either of these cases is rather trivial and, if this were the only use of the generic semantics, it would be rather uninteresting. However, consideration of these cases serves to illustrate the more interesting applications that are to follow.

In formulating the various systems, we shall use the language $\mathbf{L}^*$ in the case in which it is endowed with a countable infinity of A-letters. In such a language, the A-letters take the role normally played by free variables. Our formulation is therefore very close to those in which a typographic distinction is made between free and bound variables (as in [6] or [9], for example).

It is well-known that the formulation of the restrictions on the axioms and rules becomes simpler once such a typographic distinction is drawn. For example, in the Axiom of Specification $\forall x \varphi(x) \supset \varphi(t)$, it must normally be required that t be free for x in $\varphi(x)$, but no such restriction is required with a typographic distinction between free and bound variables.

However, for us, the distinction is not merely one of syntactic convenience but is, at bottom, semantic. The letters $a, b, c, \ldots$ are not variables, but are names, albeit of a strange sort of object. On the other hand, the letters $x, y, z, \ldots$ have no designatory role but merely serve, in

Frege's phrase, as 'signs of generality'. To use an A-letter as a variable of quantification would be as bad, for us, as using an *individual* name as a variable of quantification.

Indeed, it is instructive in this regard to try to formulate a system in which individuals names can double up as variables. So if s is used for 'Socrates' and W for 'is wise', then a formula such as $Ws \wedge \exists s - Ws$ could be used to say that Socrates is wise but some man (or thing) is not wise. One will then find oneself in the very same difficulties as arise when no typographic distinction is made between so-called free and bound variables.

Let us now deal with each of the systems in turn.

The Hilbert System H. This has the following axioms and rules:

(1) All tautologous formulas

(2) $\forall x\varphi(x) \supset \varphi(t)$

(3) $\forall x(\varphi \supset \psi) \supset \forall x\varphi \supset \forall x\psi$

(4) $\varphi \supset \forall x\varphi$

MP $\varphi, \varphi \supset \psi/\psi$

Gen. $\varphi(a)/\forall x\varphi(x)$, for a not in $\varphi(x)$.

In (2), it is assumed that t is a term (containing no variables) and that $\varphi(t)$ comes from $\varphi(x)$ by replacing all free occurrences of x with t. In Gen., it is assumed that a is an A-letter and that $\varphi(a)$ comes from $\varphi(x)$ upon replacing all free occurrences of x with a.

The notions of *proof* and *theorem* for such a system are defined in the usual way.

We now wish to conceive of the A-letters as denoting suitable A-objects. What is required of these A-objects is given by the following definition. An A-model $\mathbf{M}^*$ for $\mathbf{L}^*$ is *suitable for* the proof P of H if:

(i) $\mathrm{a} \neq \mathrm{b}$ for distinct A-letters a and b of P;

(ii) each a, for a an A-letter of P, is unrestricted.

It will be recalled that an A-object is unrestricted if it is both independent and value-unrestricted.

The requirement of suitability may be reformulated in terms of the realizability of a system of definitions. Given a proof P within H, let the *associated* system **S** consist of all definitions (a, Λ) for a an A-letter of P. Then it is readily seen that an A-model $\mathbf{M}^*$ is suitable for P iff it realizes the system of definitions **S**.

Although we have formulated the concept of suitability relative to a proof, it should be clear that the concept also applies relative to any inference Δ/φ.

We wish to show that the proofs within the system H are correct when suitably interpreted by means of A-objects. There are two criteria of correctness that may be used. The first, which I call *Line Soundness*, requires that each line of the proof be correct. The second, which I call *Rule* or *Line-to-Line Soundness*, requires that each application of a rule of inference be correct. Under the first, it is the lines of proof that are to be correct: under the second, it is the methods by which the lines are obtained.

It is clear that, in principle, a system may have one kind of soundness and not the other. The axioms may not be true, even though the rules of inference preserve truth. On the other hand, the theorems may be true, even though the rules of inference do not in general preserve truth. This is the case, for example, with modal systems that contain a rule of necessitation.

First, we need:

LEMMA 8. Let $\mathbf{M}^*$ be a suitable model for the proof P of H. Then $\mathbf{M}^*$ is extendible over the set of A-objects designated by the A-letters in P.

Proof. Since $\mathbf{M}^*$ is suitable for P, M^* realizes the system S of definitions associated with P. Since each definition of **S** is of the form (a, Λ), it is trivial that **S** is a complete system of definitions that are total over **M**. Therefore, by Theorem 7, $\mathbf{M}^*$ is extendible over $\mathrm{A}_\mathbf{S} = \{\mathrm{a}\colon a \text{ in } P\}$.

Clearly, a similar result holds with an inference in place of P. We now obtain:

THEOREM 9. (Line-to-Line Soundness for H.) The inferences:

(a) $$\frac{\varphi \quad \varphi \supset \psi}{\psi}$$

(b) $$\frac{\varphi(a)}{\forall x\varphi(x)}\ (a \text{ not in } \varphi(x))$$

are truth-to-truth valid relative to the class X of A-models suitable for each of the respective inferences.

Proof. (a) The inference under (a) is classically valid. By Lemma 8, each A-model in X is extendible over the set of A-objects designated in the inference. So by Theorem 1, the inference is truth-to-truth-valid relative to X.

(b) Let the distinct A-letters in $\varphi(x)$ be $a_1, \ldots, a_n$, so that $\varphi(x)$ may be written in the form $\varphi(x, a_1, \ldots, a_n)$. Suppose, for reduction that $\forall x\varphi(x, a_1, \ldots, a_n)$ is not true in $\mathbf{M}^*$. Then for some assignment $\mathrm{v} \in \mathrm{V}$ defined over $\{\mathrm{a}_1, \ldots, \mathrm{a}_n\}$, $\mathrm{v}(\forall x\varphi(x))$ is false in $\mathbf{M}$. By Restriction, we may suppose that v is defined on $\{\mathrm{a}_1, \ldots, \mathrm{a}_n\}$. Given that v has the diagram:

$$\mathrm{v}: \frac{\mathrm{a}_1\mathrm{a}_2 \ldots \mathrm{a}_n}{\mathrm{i}_1\mathrm{i}_2 \ldots \mathrm{i}_n}$$

we then have that $\forall x\varphi(x, \mathrm{i}_1, \ldots, \mathrm{i}_n)$ is false in $\mathbf{M}$.

By the classical truth-condition for $\forall$, there is an $\mathrm{i}_0 \in \mathrm{I}$ for which $\varphi(\mathrm{i}_0, \mathrm{i}_1, \ldots, \mathrm{i}_n)$ is false in $\mathbf{M}$. We wish to show that the assignment v^+ with diagram:

$$\mathrm{v}^+: \frac{\mathrm{a}_0\mathrm{a}_1 \ldots \mathrm{a}_n}{\mathrm{i}_0\mathrm{i}_1 \ldots \mathrm{i}_n}$$

belongs to V.

Let u be the assignment represented by:

$$\mathrm{u}: \frac{\mathrm{a}}{\mathrm{i}_0}.$$

We show that $\mathrm{v}^+ \in \mathrm{V}$ by showing that the assignments v and u can be pieced together.

First note that $\mathrm{u} \in \mathrm{V}$. For by a value-unrestricted, $\mathrm{i}_0 \in \mathrm{VR(a)}$, i.e., $\mathrm{u}^+(\mathrm{a}) = \mathrm{i}_0$ for some $\mathrm{u}^+ \in \mathrm{V}$; and so by Restriction, $\mathrm{u} = \mathrm{u}^+ \restriction \{\mathrm{a}\} \in \mathrm{V}$.

Now note that the conditions for Piecing are satisfied. Since $\mathrm{a}, \mathrm{a}_1, \ldots, \mathrm{a}_k$ are all independent, the domains of u and v are closed. Also, u and v agree on common arguments, indeed have no common arguments: for by the syntactic restriction on the rule, a is distinct from each of $a_1, \ldots, a_n$; and so by clause (i) in the definition of suitability, the A-object a is distinct from each of $\mathrm{a}_1, \ldots, \mathrm{a}_n$.

We therefore have, by Piecing, that $v^+ = u \cup v \in V$. But $v^+(\varphi(a)) = \varphi(i_0, i_1, \ldots, i_n)$ is false in **M**; and so $\varphi(a)$ is not true in $\mathbf{M}^*$ – as required.

The proof for part (b) has been stated with some care in order to illustrate the rigorous use of the conditions on an A-model, the definition of suitability, and the restrictions on the rules. Note, in particular, how clauses (i) and (ii) in the definition of suitability are used and how it is significant both that a not occur in $\varphi(x)$ and that distinct A-letters designate distinct A-objects. In future, we shall not bother to formulate our proofs with such care.

From Theorem 9 may be derived:

THEOREM 10. (Line Soundness.) Let P be a proof of the theorem φ in H. Then φ is valid in the class X of suitable A-models for P.

Proof. Given the previous theorem, it suffices to show that each axiom of H is valid relative to X. But each axiom is classically valid and so, by Corollary 3, is valid relative to X.

These results on soundness lack interest unless it can be shown that each classical model **M** underlies a suitable A-model $\mathbf{M}^*$. We want to know, whatever the state of the real world (the model **M**), that suitable denotations for the A-letters can be found. This possibility is given by:

LEMMA 11. For any classical model **M** and proof P of H, there is a suitable A-model $\mathbf{M}^*$ for P.

Proof. Let **S** be the system of definitions associated with P. It should be clear that **S** is unequivocal and non-circular. Therefore, by Theorem 6, there exists an A-model $\mathbf{M}^*$ that realizes **S** and hence is suitable for P.

Call the system H *classically sound* if each theorem of H that is lacking in A-letters is classically valid. Note that nothing is said about theorems that contain A-letters. It is as if these play a purely instrumental role in proofs and all that matters is what theorems without A-letters can be proved.

THEOREM 12. The system H is classically sound.

Proof. Suppose that φ is a theorem of H, with proof P, but that φ is not classically valid. Then for some classical model **M**, $\mathbf{M} \not\models \varphi$. By Theorem 11, there is a suitable model $\mathbf{M}^*$ for P. By condition (iv) in the definition of

A-model, there is a $v \in V$. But then $\mathbf{M} \not\models v(\varphi) = \varphi$; and so $\mathbf{M}^* \not\models \varphi$ – in contradistinction to Theorem 11.

Note the essential use in the proof of V's being non-empty. Of course, a proof along classical lines can be given and, indeed, it may be shown that *any* theorem of H is classically valid. But for later purposes, it will be helpful to have a proof of the weaker results using generic methods.

The Gentzen System G. We are interested in a system of natural deduction of the sort proposed by Gentzen. Any reasonable formulation of the propositional rules will do: but it is important that the quantificational rules contain a principle of existential elimination rather than existential instantiation. To fix our ideas, we will suppose that we are working within the system of Lemmon's [9].

The quantificational rules, in schematic form, are as follows:

$$\forall E \frac{\forall x \varphi(x)}{\varphi(t)} \qquad \forall I \frac{\varphi(a)}{\forall x \varphi(x)}$$

$$\exists E \quad \frac{\exists x \varphi(x) \quad \begin{matrix}[\varphi(a)] \\ \vdots \\ \psi\end{matrix}}{\psi} \qquad \exists I \frac{\varphi(t)}{\exists x \varphi(x)}$$

In $\forall E$ and $\exists I$, it is supposed that t is a term (without free variables) and that $\varphi(t)$ comes from $\varphi(x)$ upon substituting the term t for the free occurrences of x. In $\forall I$, a must not occur in $\varphi(x)$ or in the assumptions upon which $\varphi(a)$ depends; and in $\exists E$, a must not occur in $\varphi(x)$ or ψ or any assumptions, other than $\varphi(a)$, upon which ψ depends. In the presentation of $\exists E$, we have followed the convention of Prawitz's [12], p. 20, for indicating the discharge of assumptions.

The distinctive feature of the present system, as opposed to the ones we shall later consider, is that there is no rule for inferring an instance of an existential statement from the existential statement itself. Instead, we have the rule $\exists E$, which licenses an inference from the existential when we have the inference from an instance.

We wish to extend the generic treatment of H to G. This may be done most simply by treating each line of a derivation as equivalent to a conditional. Recall that a sequent (Δ, φ) is *derivable in G* if there is a derivation whose last line has the formula φ as a conclusion and has only formulas of Δ as assumptions. If $\varphi_1, \ldots, \varphi_n$ are the distinct members of Δ, let $\Delta \supset \varphi$ be the formula $(\varphi_1 \wedge \ldots \wedge \varphi_n) \supset \varphi$. (The order in which we take the members of Δ will be immaterial.) We then have:

THEOREM 13. (Line-to-Line Soundness.) Let $(\Delta_1, \varphi_1), \ldots, (\Delta_n, \varphi_n)/(\Delta, \varphi)$ be an argument pattern corresponding to one of the rules of inference of G. Then the inference $\Delta_1 \supset \varphi_1, \ldots, \Delta_n \supset \varphi_n/\Delta \supset \varphi$ is truth-to-truth valid in its class of suitable models $\mathbf{M}^*$.

Proof. All of the rules, other than $\forall I$ and $\exists E$, correspond to classically valid inferences. Therefore their validation follows from the extendibility of $\mathbf{M}^*$. The treatment of $\forall I$ and $\exists E$ follows that of the rule of generalisation in the proof of Theorem 9, but with some slight adjustment to take care of the presence of suppositions.

We may now prove the other results – line soundness and classical soundness – in the same way as before.

Although this gives us the simplest way of unifying our treatment of the systems of Gentzen and Hilbert, it is not based upon a very plausible view of the role of suppositions in natural deduction reasoning. Suppositions are treated, in effect, as the implicit antecedents to conditionals. So the supposition of φ itself is taken as tantamount to the assertion of $\varphi \supset \varphi$, while the inference of $\varphi \cup \psi$ from the supposition φ is taken as tantamount to the inference of $\varphi \supset (\varphi \vee \psi)$ from $(\varphi \supset \varphi)$. Intuitively, though, we are inclined to think of the supposition of φ as not assertoric at all and the inference of $\varphi \vee \psi$ from the supposition φ as like the inference of $\varphi \vee \psi$ from the assertion φ, but without the commitment to the truth of φ.

The question of giving a satisfactory account of suppositional reasoning is much more difficult than has commonly been thought and is especially difficult on the generic view. For from the truth of Fa, with a unrestricted, follows the truth of $\forall xFx$. So why cannot $\forall xFx$ be derived from the supposition of Fa?

We can overcome this difficulty by operating with the concept of case-to-case rather than truth-to-truth validity. Let us follow Prawitz ([12], p. 23) in distinguishing between *proper* and *improper* inference rules. The proper rules are $\wedge I$, $\wedge E$, $\vee I$, $\supset E$, $-E$, $\forall E$ and $\exists I$; the improper rules are $\vee E$, $\supset I$, $-I$, $\forall I$ and $\exists E$. The proper rules will correspond to inferences that are straightforwardly valid; the improper rules will correspond to metalogical principles of reasoning. To be more exact:

THEOREM 14. Let $(\Delta_1, \varphi_1), \ldots, (\Delta_n, \varphi_n)/(\Delta, \varphi)$ be an argument pattern corresponding to one of the rules of inference of G and let X be the associated class of A-models. Then:

(a) if the rule is proper, the inference $\varphi_1, \ldots, \varphi_n/\varphi$ is case-to-case valid relative to X; and

(b) if the rule is improper, then (Δ/φ) is case-to-case valid relative to X wherever $\Delta_1/\varphi_1, \ldots, \Delta_n/\varphi_n$ are.

Proof. (a) If the rule is proper, the inference $\varphi_1, \ldots, \varphi_n/\varphi$ is classically valid and so, by Theorem 4, it is case-to-case valid.

(b) Along the lines of the relevant parts of the proof for Theorem 13.

If Line Soundness is now to be derived, it must be shown that the proper rules never lead us from valid to invalid sequents. It must be shown, for example, that from the fact that the sequents Δ/φ and $\varphi/\varphi \vee \psi$ are valid it follows that the sequent $\Delta/\varphi \vee \psi$ is valid. But this is essentially a matter of applying the Cut Rule for the concept of case-to-case validity (Theorem 5). Note that the proof of the rule is not completely straightforward in this case and calls for the satisfaction of an extendibility requirement.

We see then that there are two very different ways of providing a semantics for the system G. One treat $\forall I$ as a proper rule of inference but does not assign a plausible role to suppositions. The other treats $\forall I$ as an improper rule but assigns a more plausible role to suppositions. I am inclined to think that there is a compromise account that both treats $\forall I$ as more like a proper rule *and* assigns a plausible role to suppositions. But this is not a possibility I shall pursue here.

5. QUINE'S SYSTEM Q

We now embark on a study of systems that contain a rule of existential instantiation. In this section, we deal with the system proposed by Quine in

Methods of Logic [14]; and in the next section, we deal with the system that Copi tried to formulate in *Symbolic Logic* [2]. In both cases, our plan is more or less the same. We shall give a standard formulation of the system and then one in terms of what I call dependency diagrams; we shall present generic semantics for the system and prove soundness with respect to the semantics; we shall finally make some critical remarks on the system in the light of its semantics and proof theory.

First Formulation. We may suppose that the system Q has the standard propositional rules for a system of natural deduction, although this is not in keeping with Quine's own presentation. The quantification rules may be schematically represented as follows:

$$UI \quad \frac{\forall x\varphi(x)}{\varphi(t)} \qquad\qquad UG \quad \frac{\varphi(a)}{\forall x\varphi(x)}$$

$$EI \quad \frac{\exists x\varphi(x)}{\varphi(a)} \qquad\qquad EG \quad \frac{\varphi(t)}{\exists x\varphi(x)}$$

Here, as before, $\varphi(a)$ is the result of substituting a for all free occurrences of x in $\varphi(x)$.

The rules come in pairs for each quantifier; one is a rule of instantiation (I), and the other a rule of generalisation (G). Note that UI is the same as $\forall E$ and EG the same as $\exists I$. The difference in notation is used merely to indicate the difference in the kind of system from which the rule originates.

Without further restriction, the application of these rules quickly leads to the derivation of invalid sequents. For example, from $\exists x\varphi(x)$ we may obtain $\varphi(a)$ by EI; and from $\varphi(a)$ we may then obtain $\forall x\varphi(x)$ by UG. The system is therefore equipped with various restrictions that are designed to block the formation of these derivations. One is the same as for the Gentzen system G:

(LR) *Local Restriction.* The A-letter a does not occur in $\varphi(x)$ in any application of the rules UG or EI.

The other restrictions are peculiar to Quine's system. In contrast to LR, they are of a global rather than a local character; they concern not merely the immediate context in which the rule is applied but also its relation to the rest of the derivation.

To state the further restrictions, we shall need some terminology. Let $\varphi(a)/\forall x\varphi(x)$ or $\exists x\varphi(x)/\varphi(a)$ be an application of one of the rules UG or EI

respectively. If a does not actually occur in $\varphi(a)$, then the given application of the rule is said to be *vacuous*. In case the application is non-vacuous, a is said to be its *instantial term* and the A-letters of $\varphi(x)$ its *given terms*. For example, in the application $\exists xFxbc/Fabc$ of EI, a is the instantial term and b, c are the given terms.

Let D be a *potential* derivation, i.e., one in accordance with the propositional rules and the unrestricted version of the quantificational rules. Relative to such a derivation, we say that the A-letter *a immediately depends upon* b – in symbols, a $\ll$ b – if, in some non-vacuous application of UG or EI in D, a is the instantial term and b is a given term. Under the proposed semantics, a will actually denote an A-object that immediately depends upon the A-object denoted by b. But it should be noted that the present notion is purely syntactic.

The global restrictions on a potential derivation now take the following form:

(F) *Flagging.* No A-letter shall be an instantial term twice, i.e., to two applications of the same rule or of different rules.

(O) *Ordering.* It should be possible to order the distinct instantial terms in such a way $a_1, a_2, \ldots, a_n$ that, for each i from 1 to n, none of $a_{i+1}, \ldots, a_n$ immediately depend upon a_i.

The derivations of Q are then those potential derivations that conform to the restrictions LR, Flagging and Ordering. It should be noted that, in contrast to G, no restriction is placed on the assumptions in any application of UG.

A Reformulation. Small differences aside, the above restrictions are the same as those in Quine's [14], p. 164. However the ordering condition is not especially perspicuous and it is in fact possible to give it a much more perspicuous formulation.

Say that the A-letter a (*syntactically*) depends upon b – in symbols, $a < b$ – if there is a sequence of A-letters $a_1, \ldots, a_n, n > 1$, such that $a = a_1, b = a_n$ and, for each $i = 1, \ldots, n-1, a_i$ immediately depends upon a_{i+1}. So we see that the relation $<$ of dependence is the ancestral of the relation of immediate dependence. Under the generic semantics, the A-object denoted by a will depend upon the A-object denoted by b. We have called both the syntactic and semantic relations dependence, but no confusion should arise as to which is intended in any context.

It may now be shown that Ordering is equivalent to:

(AS) *Anti-symmetry.* The relation $<$ of dependence is anti-symmetric, i.e., never $a < b$ and $b < a$ for distinct A-letters a and b.

The proof of equivalence is straightfoward, but will not be given here.

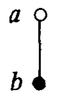
Further simplification may be obtained by lumping LR and Ordering together. The combined condition is then equivalent to:

(I) *Irreflexivity.* The relation $<$ of syntactic dependence is irreflexive, i.e., never $a < a$ for any A-letter a.

The advantage of the formulation (AS) over (O) or of (I) over (LR) & (O) is not merely one of perspicuity. The satisfaction of (O) is rather hard to determine, since it requires the discovery of an ordering and a verification that it conforms to the desired condition. The satisfaction of (AS) or of (I), on the other hand, is relatively easy to determine, since it requires only that we observe no cycles in the chains of immediate syntactic dependence. Indeed, once the restrictions are formulated in the form (F) & (I) or (LS) & (F) & (AS), it is rather easy to keep a running check on their satisfaction in terms of what I call *dependency diagrams.* The use of such diagrams is illustrated by the following examples.

First, we have a derivation of $\forall u\ \exists v\ Fuv$ from $\forall x\ \exists y\ Fxy$:

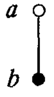
(1)	$\forall x\ \exists y\ Fxy$	Assumption
(2)	$\exists y\ Fay$	1, *UI*
(3)	Fab	2, *EI*
(4)	$\exists v\ Fav$	3, *EG*
(5)	$\forall u\ \exists v\ Fuv$	4, *UG.*



The diagram to the right of the derivation is constructed in stages. At line (3), the diagram:

is drawn. This indicates that b has been used as an instantial term that depends upon a.

At line (5), the diagram is expanded to:



The circular node indicates that a has been used as an instantial term that depends upon nothing else.

Second, we have an attempted derivation of $\exists y\, \forall x\, Fxy$ from $\forall x\, \exists y\, Fxy$:

(1)	$\forall x\, \exists y\, Fxy$	Ass
(2)	$\exists y\, Fay$	1, *UI*
(3)	Fab	2, *EI*
(4)	$\forall x\, Fxb$	3, *UG*

At line (3), the diagram

is drawn, since b has been used as an instantial term depending upon a. At line (4), we are obliged to expand the diagram to:

since a has been used as an instantial term depending upon b. The fact that we have been forced to label two nodes with the letter b then shows that the derivation is incorrect.

Next, we have an attempted derivation of $\forall x(Fx \supset \forall xFx)$

(1)	Fa	Ass		
(2)	$\forall xFx$	1, *UG*	a ○	a ○
(3)	$Fa \supset \;\; xFx$	1, 2, $\supset I$		
(4)	$\forall x(Fx \supset \forall xFx)$	3, *UG*.		

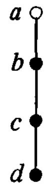
The two successive diagrams:

a ○ $\qquad\qquad$ a ○ $\quad$ a ○

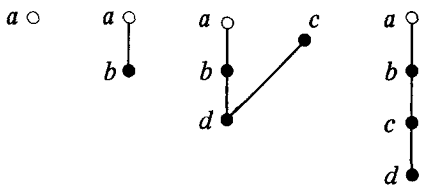
are completed at lines (2) and (4). The fact that a node has been labelled twice with a then indicates that the derivation is incorrect.

Finally, we have the more complicated example of a derivation of $\exists y\, \forall v\, \exists u\, \exists x\, Fxyuv$ from $\exists x\, \exists y\, \forall u\, \forall v\, Fsyuv$.

(1)	$\exists x\, \exists y\, \forall u\, \forall v\, Fxyuv$	Ass
(2)	$\exists y\, \forall u\, \forall v\, Fayuv$	1, *EI*
(3)	$\forall u\, \forall v\, Fabuv$	2, *EI*
(4)	$\forall v\, Fabdv$	3, *UI*
(5)	$Fabdc$	4, *UI*
(6)	$\exists x\, Fxbdc$	5, *EG*
(7)	$\forall u\, \exists x\, Fxbuc$	6, *UG*
(8)	$\forall v\, \forall u\, \exists x\, Fxbuv$	7, *UG*
(9)	$\exists y\, \forall v\, \forall u\, \exists x\, Fxyuv$	8, *EG*



In this example, the four successive diagrams:



are completed at lines (2), (3), (7) and (8). Note that the nodes need to be re-aligned at line (8).

By drawing a dependency diagram as we proceed with a derivation, it is possible to keep a running check on its correctness. The procedure by which the diagrams are constructed has not been stated with formal precision; but it should be reasonably clear from the examples how it is to go.

The implementation of the procedure can sometimes be rather clumsy. Lines can criss-cross; and when nodes are repositioned, whole parts of the diagram may need to be erased and duplicated elsewhere. But these difficulties are nothing to the problems that arise when there is no diagram for checking correctness. Conformity to Flagging may be checked by Quine's device of flagging variables or by a fairly simple inspection of the derivation. But when it comes to Irreflexivity or Ordering, there would seem to be nothing better but to go through the whole derivation, isolate the applications of *UG* and *EI*, work out what the relationships of immediate dependence are, and then see whether its ancestral is irreflexive. Quine has suggested as a 'rule of thumb' ([14], p. 164) that you 'pick your letters so that each flagged variable is alphabetically later than all other free variables of the line it flags'. Conformity to Ordering is then assured. But the difficulty now takes another form. Variables (or *A*-letters) may be introduced into a derivation prior to their use as an instantial term. So one must choose them that subsequent conformity to the alphabetic requirement can be maintained. For example, in the previous derivation, it is necessary that the A-letter introduced at line (4) be alphabetically later than the A-letter introduced at line (5). If one blunders, then the whole derivation must be re-written when it comes to the point at which conformity to the alphabetic condition is required. If, for example, one had (quite naturally) in the previous derivation introduced c at line (4) and d at line (5) then, when it came to line (7), the derivation would have to be rewritten, with c interchanged with d.

It seems, then, that dependency diagrams constitute one of the most effective ways of checking the correctness of derivations. Indeed, one may think of the diagram as embodying just that information from the rest of the derivation as is required to check the correctness of the current line. So a complete survey of the whole derivation may, in this way, be replaced by a simple check on its diagram.

In the light of these advantages, the dependency diagrams would appear to hold out great promise as a way of presenting derivations of Q within the class room. They provide a simple, natural, and unified method for checking correctness, one that is readily mastered and relatively easily implemented. When combined with the semantics that is about to be given, they impose upon a derivation a structure that motivates its development and gives it sense.

Semantics. We now wish to provide a generic interpretation for the system Q, one that will render each line of a derivation valid and each application of a rule correct.

Let us first provide an informal motivation for the semantics. The problematic rules are EI and UG; and so let us deal with each in turn. With EI, we pass from $\exists x\varphi(x)$ to $\varphi(a)$. Consider first the case in which we argue from the *supposition* $\exists x\varphi(x)$ to $\varphi(a)$, with no A-letters other than a occurring in $\varphi(a)$. Since we want later to be able to conditionalize, we want so to interpret $\underline{a}$ that $\exists x\varphi(x) \supset \varphi(a)$ is true. So suppose that $\exists x\varphi(x)$ is true. Then this means that the value of $\underline{a}$ must be confined to the individuals that satisfy $\varphi(x)$. So, in general, $\underline{a}$ will be a restricted A-object. Now, in principle, $\underline{a}$ could take any set of individual φ-ers as its values. But there is no good reason to countenance one individual φ-er as a value rather than any other. Moreover, $\underline{a}$ must take some individuals as values; for otherwise $\underline{a}$ will be a null A-object and the Extendibility requirement will not be satisfied. Therefore the only natural choice for a is the A-object whose values are all of the individual φ-ers', what one might call the arbitrary φ-er.

Suppose now that $\exists x\varphi(x)$ is false. Then it does not matter what a is, since there is no danger of the conditional failing to be true. There is no good reason for a to take one value rather than another; and, for the same reasons as before, a should take some value. So the only natural choice for a is an A-object which takes all individuals as values, what we have termed a universal or value-unrestricted A-object.

It is important to note that it is only in the case that $\exists x\varphi(x)$ is true that a turns out to be the arbitrary φ-er. If we wanted to give a compendious description of a, one that covers both cases, we might call it the *putative* arbitrary φ-er. It is the A-object that φ s if anything does.

Consider now the case in which $\exists x\varphi(x)$ contains other A-letters, say the A-letters b and c. $\varphi(x)$ may then be written in the form $\varphi(x, b, c)$. Since again we want to be able to conditionalize, we must so interpret the A-letter a that $\exists x\varphi(x, b, c) \supset \varphi(a, b, c)$ is true. In this case it is natural to treat a as an object dependent upon b and c; for what values a can take will be constrained by what values b and c take. It is also clear that, for given values of b and c, the values of a should be constrained in the same way as before. So we see that, in this case, a is most naturally taken to be an A-object dependent upon b and c (and whatever b and c depend upon) and such that, for given values j, k of b, c, a will take all values i for which $\exists x\varphi(x, \mathrm{j}, \mathrm{k}) \supset$ (i, j, k) is true, i.e., all values i for which $\varphi(\mathrm{i}, \mathrm{j}, \mathrm{k})$ is true should $\exists x(x, \mathrm{j}, \mathrm{k})$ be true and all values whatever should $\exists x\varphi(x, \mathrm{j}, \mathrm{k})$ be false.

Of course in general, $\varphi(a)$ will be derived, not from the *supposition* $\exists x\varphi(x)$, but from a conclusion $\exists x\varphi(x)$ that itself depends upon other suppositions. But it seems natural to suppose that the interpretation of a should not depend upon the status of the premiss $\exists x\varphi(x)$ as supposition or conclusion. And so, in this case, the interpretation of a may be determined in the same way as before.

This brings us to applications of the rule *UG*. As before, first consider a case in which we argue from the supposition $\varphi(a)$ to $\forall x\varphi(x)$, with no A-letter other than a occurring in $\varphi(a)$. One might suppose that one is here arguing from the truth of $\varphi(a)$, for a and unrestricted A-object, to the truth of $\forall x\varphi(x)$, in partial analogy to the Gentzen system G. But the principles of the present system give the lie to this interpretation. For in Q there is no requirement that, in generalizing, the instantial term not occur in the suppositions upon which the premisses depend. So we may pass to $\forall x\varphi(x)$ from the *supposition* of $\varphi(a)$; and so, by conditionalizing, we may then obtain $\varphi(a) \supset \forall x\varphi(x)$. But this last formula is not in general true for a a value-unrestricted A-object.

So let us ask: what must a be for $\varphi(a) \supset \forall x\varphi(x)$ to be true? If $\forall x\varphi(x)$ is false (in the underlying model), then φ(i) must be false for any value i of a; while if $\forall x\emptyset(x)$ is true, the value of a can be anything. It follows, by the same considerations as before, that the only natural choice for a is an A-object that has all non-φ-ers as its values in case $\forall x\varphi(x)$ is false and that has all individuals whatever as its values in case $\forall x\varphi(x)$ is true. So we see that far from being a universal A-object, a is best thought of as an arbitrary (putative) counterexample to the formula $\forall x\varphi(x)$. The intuitive justification for the rule *UG* in Quine's system is not that everything must φ if the arbitrary individual φ's, but that everything must φ if even the putative arbitrary counterexample to the generalization $\forall x\varphi(x)\varphi$'s.

The extension of the interpretation to the other cases proceeds in the same way as before. If $\varphi(a)$ contains other A-letters, then a is still a putative counterexample to $\forall x\varphi(x)$, but dependent upon the other A-objects mentioned in $\forall x\varphi(x)$; and if $\varphi(a)$ is not a supposition, then a must be treated in the same way as it would be if $\varphi(a)$ were a supposition.

After these informal remarks, let us now give a more rigorous formulation of the semantics for the system. An A-model M^* is said to be *suitable for* a derivation D if:

(i) a $\neq$ b for a and b distinct A-letters of D;

(ii) a is unrestricted for each non-instantial letter a in D;

(iii) if the inference $\exists x\varphi(x, b_1, \ldots, b_n)/\varphi(a, b_1, \ldots, b_n)$ occurs in D, then a is an A-object for which:

(a) $|\mathrm{a}| = [\mathrm{b}_1, \ldots, \mathrm{b}_n]$, i.e., $\mathrm{a} \prec \mathrm{c}$ iff $\mathrm{b}_1 = \mathrm{c}$ or $\mathrm{b}_i \prec \mathrm{c}$ for $i = 1, 2, \ldots, n$; and

(b) for any v with domain $|\mathrm{a}|$, $\mathrm{w} = \mathrm{v} \cup \{\langle \mathrm{a}, \mathrm{i}\rangle\} \in \mathrm{V}$ iff $\mathbf{M} \models \mathrm{w}\ (\exists x\varphi(x, b_1, \ldots, b_n) \supset \varphi(a, b_1, \ldots, b_n))$;

(iv) if the inference $\varphi(a, b_1, \ldots, b_n)/\forall x\varphi(x, b_1, \ldots, b_n)$ occurs in D, then a is an A-object for which:

(a) $|a| = [b_1, \ldots, b_n]$, and

(b) for any v with domain $|a|$, $w = v \cup \{\langle a, i \rangle\} \in V$ iff $\mathbf{M} \models w(\varphi(a, b_1, \ldots, b_n) \supset \forall x \varphi(x, b_1, \ldots, b_n))$.

As before, the notation of suitability may also be explained in terms of definitional realizability. With each derivation D we may associate a system of definitions **S**. Its members are:

(i) all pairs $(a, \exists x \varphi(x) \supset \varphi(x))$, where $\exists x \varphi(x)/\varphi(a)$ is an application of *EI* in D;

(ii) all pairs $(a, \varphi(x) \supset \forall x \varphi(x))$, where $\varphi(a)/\forall x \varphi(x)$ is an application of *UG* in D;

(iii) all pairs (a, Λ), where a is an A-letter of D that is not an instantial term.

The derivation provides us, in effect, with a definition of each of the A-letters that it uses. It is then a trivial matter to show that an A-model $\mathbf{M}^*$ is suitable for D iff it realizes the system of definitions **S** associated with D.

It should also be noted that there is a close connection between the dependency relation for a suitable model $\mathbf{M}^*$ and the dependency relation for its derivation D. The A-letter a will syntactically depend upon b in D just in case the A-object a objectually depends upon b in $\mathbf{M}^*$. Thus the dependency diagram for a derivation will actually have semantic significance as a graph for the objectual dependency relation.

Soundness. We wish to establish results on both the line and the line-to-line soundness of Q.

But first we need a result on the extendibility of suitable models.

LEMMA 15. Let $\mathbf{M}^*$ be a suitable model for the derivation D. Then $\mathbf{M}^*$ is extendible over the class of A-objects designated by the A-letters in D.

Proof. By Theorem 7, it suffices to show that each definition in the system **S** of definitions associated with D is total in **M**. For definitions of the form (a, Λ), this is obvious; and for definitions of the form $(a, \exists x \varphi(x) \supset \varphi(x))$ and $(a, \varphi(x) \supset \forall x \varphi(x))$, it follows from the logical truth of $\exists x(\exists x \varphi(x) \supset \varphi(x))$ and $\exists x(\varphi(x) \supset \forall x \varphi(x))$.

We may now show:

THEOREM 16. (Line-to-Line Souddness for Q.) Let D be a derivation in Q, and $(\Delta_1, \varphi_1), \ldots, (\Delta_n, \varphi_n)/(\Delta, \varphi)$ an argument pattern corresponding to an application of one of the rules in D. Then the inference $\Delta_1 \supset \varphi_1, \ldots, \Delta_n \supset \varphi_n/\Delta \supset \varphi$ is truth-to-truth validated relative to the class X of suitable models $\mathbf{M}^*$ for D.

Proof. There are three cases:

(1) The inference rule is classical,,i.e., not *UG* or *EI*. Then the inference $(\Delta_1, \varphi_1), \ldots, (\Delta_n, \varphi_n)/(\Delta, \varphi)$ is classically valid. By Lemma 15, each suitable model $\mathbf{M}^*$ is extendible over the set of A-objects mentioned in the inference. So by Corollary 2, the inference is truth-to-truth validated in X.

(2) The inference rule is *UG*. So it must be shown that $\Delta \supset \varphi(a)/\Delta \supset \forall x\varphi(x)$ is truth-to-truth valid relative to X. Now the inference $\Delta \supset \varphi(a)$, $\varphi(a) \supset \forall x\varphi(x)/\Delta \supset \forall x\varphi(x)$ is classically valid. So by Lemma 15 and Corollery 2, the inference $\Delta \supset \varphi(a), \varphi(a) \supset \forall x\varphi(x)/\Delta \supset \forall x\varphi(x)$ is truth-to-truth valid relative to X. But by clause (iv) in the definition of 'suitable', $\varphi(a) \supset \forall x\varphi(x)$ is truth-to-truth valid relative to X. It therefore follows that $\Delta \supset \varphi(a)/\Delta \supset \forall x\varphi(x)$ is truth-to-truth valid in X.

(3) The inference is *EI*. Similar to case (2).

Just as in the case of G, we may give a proof of line-to-line soundness that revolves around the concept of case-to-case validity. The verbal formulation of the result is exactly the same as for Theorem 14. But in striking contrast to the case of G, we may now classify *UG* (and *EI*) as proper rules of inference: for it is immediate from the definition of 'suitable' that each application $\varphi(a)/\forall x\varphi(x)$ of *UG* (or application $\exists x\varphi(x)/\varphi(a)$ of *EI*) is case-to-case valid in the class of suitable models. So we may secure the propriety of the inference $\varphi(a)/\forall x\varphi(x)$, not by insisting upon a truth-to-truth concept of validity, but by appropriately modifying the interpretation of the instantial terms.

From Theorem 16, we may obtain:

THEOREM 17. (Line Soundness for Q.) Let D be a derivation of the sequent (Δ, φ) in Q. Then $\Delta \supset \varphi$ is generically valid in the class X of suitable models.

Proof. For each uninferred sequent (Γ, ψ) of D, $\Gamma \supset \psi$ is truth-to-truth valid relative to X since $\psi \in \Gamma$. The property of being truth-to-truth valid relative to X is preserved by the inferences of Theorem 16 that correspond to the applications of the rules in D. Therefore (Δ, φ) is truth-to-truth valid in X.

As before, the interest of our soundness results depends upon the existence of suitable models. But this may be established on the basis of our theory of definition.

LEMMA 18. For any classical model **M** and derivation D from Q, there is an A-model $\mathbf{M}^*$ that is based upon **M** and suitable for D.

Proof. Let **S** be the system of definitions associated with the derivation D, and recall that a model $\mathbf{M}^*$ is suitable for D just in case it realizes the system **S**. So, given a classical model **M**, there will be an A-model $\mathbf{M}^*$ suitable for D just in case there is an A-model $\mathbf{M}^*$ that realizes the system **S**. Now, by Theorem 6, there will be an A-model $\mathbf{M}^*$ that realizes the system as long as the system satisfies the joint requirements of univocality and noncircularity. But it may immediately be checked that **S** conforms to univocality just in case D conforms to Flagging and that **S** conforms to noncircularity just in case D conforms to Irreflexivity. So since any derivation D of Q conforms to Flagging and Irreflexivity, there will be a suitable A-model $\mathbf{M}^*$ for D.

Given this existence result, classical soundness may then be proved in exactly the same way as for Theorem 12.

The above proof enables us to see the restrictions on the rules in a new light. For suppose we think of a derivation as implicitly defining the A-letters that it uses. Then we may see Flagging as merely the requirement that the same term shall not be defined twice and Ordering (or Irreflexivity) as merely the requirement that there be no definitional circles. Thus the restrictions turn out to be the counterparts to general desiderata upon a system of definitions.

Critical Remarks. Quine's system has some features that are peculiar and do not tie in well with ordinary reasoning. The main source of these pecularities is the rule *UG*. As we have already noted, this rule allows us to infer $\forall xFx$ from the *supposition* Fa and then discharge the supposition to obtain $Fa \supset \forall xFx$. This corresponds to nothing in ordinary reasoning. We may say: let n be an arbitrary number and suppose that it is even. We cannot then go on to conclude that every number is even, let alone assert, unconditionally, that if n is even then every number is even.

This defect would not be so bad if every ordinary piece of reasoning could be represented within Quine's system; for then the extra inferences could be regarded as a 'rounding out' of ordinary practise. But there are, in fact, many ordinary cases of quantificational reasoning that cannot be represented within Quine's system. An example is the argument from $\forall x(Fx \wedge Gx)$ to $\forall xFx \wedge \forall xGx$. This could ordinarily be represented as follows:

(1)	$\forall x(Fx \wedge Gx)$	Ass.
(2)	$Fa \wedge Ga$	1, UI
(3)	Fa	2, $\wedge E$
(4)	Ga	3, $\wedge E$
(5)	$\forall xFx$	3, UG
(6)	$\forall xGx$	4, UG
(7)	$\forall xFx \wedge \quad xGx$	5, 6, $\wedge I$

However, such a derivation cannot go through in Quine's system since the Flagging restriction is violated: a is an instantial term to two applications of UG. Instead, the derivation is forced to take the following devious route:

(1)	$\forall x(Fx \wedge Gx)$	Ass.
(2)	$Fa \wedge Ga$	1, UI
(3)	Fa	2, $\wedge E$
(4)	$Fb \wedge Gb$	1, UI
(5)	Gb	4, $\wedge E$
(6)	$\forall xFx$	3, UG
(7)	$\forall xGx$	6, UG
(8)	$\forall xFx \wedge \quad xGx$	6, 7, $\wedge I$

So we see that Quine's system cannot be regarded as a rounding out of ordinary practice. Indeed, the two are strictly incomparable, with each containing inferences that correspond to nothing in the other.

This lack of deductive fit has an underlying semantical explanation. In ordinary reasoning, when we go from $\varphi(a)$ to $\forall x\varphi(x)$, a is meant to be an unrestricted A-object. We cannot therefore assert $\varphi(a) \supset \forall x\varphi(x)$, since that would have the force of the general claim $\forall x(\varphi(x) \supset \forall x\varphi(x))$. We can, on the other hand, generalise upon a as many times as we like. In Quine's system, when we go from $\varphi(a)$ to $\forall x\varphi(x)$, a is meant to be a putative counterexample to $\forall x\varphi(x)$. We can therefore assert $\varphi(a) \supset \forall x\varphi(x)$, for if even the putative counterexample to $\forall x\varphi(x)$ φ's then everything must φ. On the other hand, it is not permissable to generalise upon *a* more than once, since the interpretation of *a* is tied to the particular application of *UG*.

This unorthodox interpretation of the *A*-letters leads to peculiarities of its own. It is a natural requirement on a derivation containing *A*-names, or any other names, that we know what those names denote as soon as they are introduced; their interpretation should not depend upon what subsequently happens in the derivation. Now our ordinary practice, when construed in generic terms, seems to conform to this ban on retrospective interpretation. It is always clear, upon the introduction of an *A*-name, what *A*-object we are talking about. However, Quine's system, when construed generically, goes against the ban. Suppose that, at a given state of a derivation, we have reached the conclusion $\varphi(a)$, with *a* unflagged. We might then go on to infer $\forall x\varphi(x)$; or we might go on to infer $\varphi(a) \vee \psi(a)$ and, from that, $\forall x(\varphi(x) \vee \psi(x))$. In the first case, a is interpreted as a putative counterexample to $\forall x\varphi(x)$ and, in the second case, as a putative counterexample to $\forall x(\varphi(x) \cup \psi(x))$. So what *a* means at a given stage of the derivation depends upon how the derivation is continued.

This semantical peculiarity is related to the difficulty in constructing dependency diagrams for derivations within Quine's system; and this, in its turn, is related to the more general difficulty in checking that derivations are correct. We have noted that the subsequent course of a derivation may require one not merely to extend the dependency diagram but to make bodily adjustments to it; new nodes are not merely added and suitably related to old nodes, but the old nodes may themselves be re-aligned. This is because the construction of a dependency diagram forces one to be explicit about the links among the different nodes and therefore about the dependency relations among the *A*-objects associated with them. Since we may not be sure, at a given stage of a derivation, exactly what *A*-objects are are talking about, the subsequent course of the derivation may compel us to revise our initial assay of the links.

In the light of these peculiarities, it is hard to regard Quine's system either as a very satisfactory system in its own right or as a faithful representation of ordinary reasoning.

6. THE COPI–KALISH SYSTEM

In this section we shall deal with the system that Copi attempted to formulate in [2] and that was correctly formulated by Kalish [7]. We shall present a standard formulation of the system and a reformulation in terms of dependency diagrams; we shall prove soundness with respect to the generic semantics; and finally we shall make a critical comparison between this system and Quine's.

The System. The system C has the usual propositional rules. The quantificational rules may be schematically represented in the same way as for Quine's system Q and with the same understanding concerning the relationship between the formulas $\varphi(x)$, $\varphi(t)$ and $\varphi(a)$:

$$UI \quad \frac{\forall x\varphi(x)}{\varphi(t)} \qquad\qquad UG \quad \frac{\varphi(a)}{\forall x\varphi(x)}$$

$$EI \quad \frac{\exists x\varphi(x)}{\varphi(a)} \qquad\qquad EG \quad \frac{\varphi(t)}{\exists x\varphi(x)}$$

Insofar as it is separately stated, the local restriction on the rules is the same as for Quine's system. However, the global restrictions are different. For EI, there is one such restriction:

Novelty. In any application $\exists x\varphi(x)/\varphi(a)$ of EI, a is to be an A-letter that has not previously occurred in the derivation.

For UG, there are two global restrictions. The first is:

Weak Flagging. No instantial term to an application of UG can previously have been used as an instantial term to an application of EI.

Call an instantial term that comes from an application of EI an $\exists$-*instantial* term and one that comes from an application of UG an $\forall$-*instantial* term. Then Weak Flagging and Novelty have the consequence that no $\exists$-instantial term is also an $\forall$-instantial term.

The second restriction is somewhat harder to state. Let us redefine the notion of immediate dependence for A-letters in a derivation by now saying that *a immediately depends* upon b – $a \ll b$ – if, in some application of EI, a is the instantial term and b is a given term. Note that the rule UG is no longer a source of dependency relations. We may call the present notion of dependence '$\exists$-dependence' if it needs to be distinguished from the previous notion. As before, take *dependence* ($<$) to be the ancestral of immediate dependence. Then the second restriction on UG is:

Independence. In any application of *UG*, no *A*-letter occurring in either the conclusion or the suppositions to the inference can be identical to or depend upon the instantial term.

Dependency Diagrams. As with Quine's system, it is helpful to keep a running check on the correctness of derivations with the aid of dependency diagrams. The use of such diagrams, and also some differences from Quine's system, are illustrated by the following derivations.

First, we have a derivation of $\forall u\, \exists v\, Fuv$ from $\forall x\, \exists y\, Fxy$:

(1)	$\forall x\, \exists y\, Fxy$	Ass.
(2)	$\exists y\, Fay$	1, *UI*
(3)	Fab	2, *EI*
(4)	$\exists v\, Fav$	3, *EG*
(5)	$\forall u\, \exists v\, Fuv$	4, *UG*.

The diagram to the right is drawn at line (3). At that line, we check that *b* is new to the derivation. At line (5), we check that *a* does not occur in the premiss or suppositions to the inference and we check, through the use of the diagram, that no *A*-letter dependent upon *a* occurs in the premiss or suppositions.

Second, we have a derivation of $\forall y\, \exists x\, Fxy$ from $\exists x\, \forall y\, Fxy$:

(1)	$\exists x\, \forall y\, Fxy$	Ass.
(2)	$\forall y\, Fay$	1, *EI*
(3)	Fab	2, *UI*
(4)	$\exists x\, Fxb$	3, *EG*
(5)	$\forall y\, \exists x\, Fxy$	4, *UG*

a ○

The diagram with circular node is drawn at line (3). The circular node indicates that *a* is an $\exists$-instantial term depending upon no other terms.

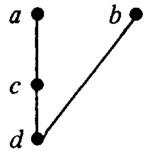
Next, we have an attempted derivation of $\exists y\, \forall x\, Fxy$ from $\forall x\, \exists y\, Fxy$:

(1)	$\forall x\, \exists y\, Fxy$	Ass.
(2)	$\exists y\, Fay$	1, *UI*
(3)	Fab	2, *EI*
(4)	$\forall x\, Fxb$	3, *UG*

The diagram is drawn at line (3). We see, by consulting the diagram, that the derivation breaks down at line (4), since the dependent *A*-letter *b* appears in the conclusion $\forall x\, Fxb$.

Finally, we have a derivation of $\forall x\, \exists s\, \forall y\, \exists t\, Fxsyt$ from $\forall x\, \exists u\, \forall y\, \exists v$ $Fxuyv$;

(1)	$\forall x\, \exists u\, \forall y\, \exists v\, Fxuyv$	Ass.
(2)	$\exists u\, \forall y\, \exists v\, Fauyv$	1, *UI*
(3)	$\forall y\, \exists v\, Facyv$	2, *EI*
(4)	$\exists v\, Facbv$	3, *UI*
(5)	$Facbd$	4, *EI*
(6)	$\exists t\, Facbt$	5, *EG*
(7)	$\forall y\, \exists t\, Facyt$	6, *UG*
(8)	$\exists x\, \forall y\, \exists t\, Fasyt$	7, *EG*
(9)	$\forall x\, \exists s\, \forall y\, \exists t\, Fxsyt$	8, *UG*



At line (3), the diagram

is drawn. At line (5), it gets extended to the diagram at the right of the derivation. At each of these lines it must be checked that the instantial terms are new to the derivation. At line (6), it must be checked that *a* and *c* do not depend upon *b*.

The formulation of Copi's system may be further simplified by making a typographic distinction between $\forall$-instantial and $\exists$-instantial terms (cf. [1], p. 88). Let us use *a*, *b*, *c* with variants for the former, and *e*, *f*, *g* with variants for the latter. This gives possibly *four* types of subject-term: variables of quantification; *UG*-terms; *EI*-terms; and individual terms. Intuitively, the letters *a*, *b*, *c*, . . . are to be the names of unrestricted *A*-objects and *e*, *f*, *g*, . . . the names of potentially restricted *A*-objects. Thus the difference between *UG*- and *EI*-terms marks not a difference in semantic role or even in the category of objects denoted, but merely a difference in how the denotation is determined.

Armed with this typographic distinction, we may simplify the formulation of the rules and the construction of the dependency diagram. In the application $\varphi(a)/\forall x\varphi(x)$ of *UG*, we now require that *a* be a *UG*-term; and in the application $\exists x\varphi(x)/\varphi(e)$ of *EI*, we require that *e* be an *EI*-term. The Weak Flagging restriction may then be dropped. Similarly, the distinction between point and circular nodes in the construction of dependency diagrams may also be dropped. What were formerly circular nodes will now be labelled by an *EI*-term and, by this fact alone, will be inadmissable as nodes for ∀-instantial terms. Furthermore, if it is required that no *EI*-term occur in a derivation until it has been introduced as an *E*-instantial term, then the satisfaction of Novelty can be determined from the diagram alone. For in the case of any occurrence of an *EI*-term in a derivation, we may perform the following check: if the occurrence is non-instantial, then the term must already occur in the diagram; if the occurrence is instantial, then the term must not already occur in the diagram.

The dependency diagrams, especially when employed in connection with a typographic distinction between *UG* and *EI*-instantial terms, provide a highly effective way of checking the correctness of derivations within *C*. It is certainly far superior to having no systematic method at all, i.e., to just surveying the derivation for dependencies at each application of *UG*. The only reasonable alternative I know of is to index the ∃-instantial terms with the ∀-instantial terms upon which they depend. This means, in effect, that the rule *EI* takes the form:

From $\exists x\varphi(x)$ infer $\varphi(a_{b_1, \ldots, b_n})$, where a is new and $b_1, \ldots, b_n$ are all of the ∀-instantial terms in $\exists x\varphi(x)$.

The rule *UG* may then take the form:

From $\varphi(a)$ infer $\forall x\varphi(x)$, where a does not occur in $\forall x\varphi(x)$ or any supposition.

Thus we obtain a system of the sort proposed by Suppes [17] and Borkowski–Słupecki [1].

This formulation has the advantage of reducing the rule *UG* to something familiar. But the method of letting the ∃-instantial terms carry the ∀-instantial terms upon which they depend as subscripts is very cumbersome. The same information is conveyed in a much more structured and compact form by means of dependency diagrams.

In the light of these advantages and in the light of their semantic significance, we see that diagrams should have great value as a way of presenting the derivations of *C* within the classroom.

Semantics. In view of the parallel treatment of Quine's system, we may, in this and the next section, be brief.

The basic idea behind the semantics is this: the ∀-instantial terms shall denote unrestricted *A*-objects; the ∃-instantial terms from inferences $\exists x\varphi(x)/\varphi(a)$ shall denote putative φ-ers, dependent upon the objects denoted by the given terms to the inference. Accordingly, let us say that an A-model $\mathbf{M}^*$ is *suitable for* the derivation D in C if:

(i) a $\neq$ b for distinct A-letters a and b of D;

(ii) a is unrestricted for each A-letter a of D not $\exists$-instantial in D;

(iii) if the inference $\exists x\varphi(x, b_1, \ldots, b_n)/\varphi(a, b_1, \ldots, b_n)$ occurs in D, then a is an A-object for which:

(a) $|a| = [b_1, \ldots, b_n]$, i.e., a $\prec$ c iff b_i = c or $b_i \prec$ c for $i = 1, 2, \ldots, n$, and

(b) for any v with domain $|a|$, w = v $\cup$ {⟨a, i⟩} $\in$ V iff $\mathbf{M} \models$ w $(\exists x\varphi(x, b_1, \ldots, b_n) \supset \varphi(a, b_1, \ldots, b_n))$.

With each derivation D may be associated a system of definitions $\mathbf{S}$ consisting of $(a, \exists x\varphi(x) \supset \varphi(x))$ for each non-vacuous application $\exists x\varphi(x)/\varphi(a)$ of EI in D and of (a, Λ) for each a in D that is not $\exists$-instantial. It is then a simple matter to show that an A-model is suitable for a system of definitions iff it realizes the associated system of definitions. If $\mathbf{M}^*$ is suitable for D and a and b are A-letters of D, then a will depend upon b in $\mathbf{M}^*$ iff a syntactically depends upon b in D. As before, the dependency diagram for D will actually mirror the relevant portion of the dependency relation in $\mathbf{M}^*$.

Soundness. We wish to establish line and line-to-line soundness for C.

First, we need a result on the extendibility of suitable models:

LEMMA 19. Let $\mathbf{M}^*$ be a suitable model for the derivation D. Then $\mathbf{M}^*$ is extendible over the class of A-objects designated by the A-letters in D.

Proof. By Theorem 7, it suffices to show that each definition in the system $\mathbf{S}$ of definitions associated with D is total in $\mathbf{M}$. For definitions of the form (a, Λ), this is obvious; and for definitions of the form $(a, \exists x\varphi(x) \supset \varphi(x))$, it follows from the logical truth of $\exists x(\exists x\varphi(x) \supset \varphi(x))$.

THEOREM 20. (Line-to-Line Soundness for C.) Let D be a derivation in C, and $(\Delta_1, \varphi_1), \ldots, (\Delta_n, \varphi_n)/(\Delta, \varphi)$ an argument pattern corresponding to an application of one of the rules in D. Then the inference $\Delta_1 \supset \varphi_1, \ldots, \Delta_n \supset \varphi_n/\Delta \supset \varphi$ is truth-to-truth validated relative to the class X of suitable models for D.

Proof. The extendibility property of suitable models takes care of the rules other than *EI* and *UG*. The rule *EI* is dealt with as before. This leaves *UG*.

So let $\varphi(a, a_1, \ldots, a_n)/\forall x\varphi(x, a_1, \ldots, a_n)$ be an application of *UG* in *D* with suppositions $\Delta(a_1, \ldots, a_n)$. Let $\mathbf{M}^*$ be a suitable model for *D*; and suppose, for reductio that $\Delta(a_1, \ldots, a_n) \supset \forall x\varphi(x, a_1, \ldots, a_n)$ is not true in $\mathbf{M}^*$. Then for some v′ in V defined over $\{a_1, \ldots, a_n\}$, $v'(\varphi(a_1, \ldots, a_n) \supset \forall x\varphi(x, a_1, \ldots, a_n))$ is false in **M**. By Restriction we may suppose that v′ is defined on $\{a_1, \ldots, a_n\}$. Given that v′ has the diagram:

$$v': \frac{a_1 a_2 \ldots a_n}{i_1 i_2 \ldots i_n}$$

we then have that $\varphi(i_1, \ldots, i_n) \supset \forall x\varphi(x, i_1, \ldots, i_n)$ is false in **M**.

Let $a_1, a_2, \ldots, a_n, \ldots, a_p, p \geqslant n$, be the closure $[a_1, a_2, \ldots, a_n]$ of $\{a_1, a_2, \ldots, a_n\}$. By Partial Extendibility and Restriction, there is an assignment with diagram:

$$v: \frac{a_1 a_2 \ldots a_n \ldots a_p}{i_1 i_2 \ldots i_n \ldots i_p}$$

in V.

By the classical truth-condition for ∀, there is an $i_0 \in I$ for which $\Delta(i_1, \ldots, i_n) \supset \varphi(i_0, i_1, \ldots, i_n)$ is false in **M**. We wish to show that the assignment v^+ with diagram:

$$v^+: \frac{a_0, a_1, \ldots, a_p}{i_0, i_1, \ldots, i_p}$$

belongs to V.

Let *u* be the assignment with diagram:

$$u: \frac{a}{i_0}.$$

We show that $v^+ \in V$ by showing that the assignments v and u can be pieced together.

Note first that $u \in V$. For by a value-unrestricted, $i_0 \in VR(a)$; and so by Restriction, $u = \{\langle a, i_0\rangle\} \in V$.

Now note that the conditions for Piecing are satisfied. The domain $\{a\}$ of u is closed since a is independent; and the domain $\{a_1, \ldots, a_p\}$ of v is closed by stipulation. Also, u and v agree on common arguments, indeed have no common arguments: for by the syntactic restrictions, $a_i < a$ and $a_i = a$ fail for each $i = 1, 2, \ldots, n$; and so by the definition of suitability, $\mathrm{a}_i \prec \mathrm{a}$ and $\mathrm{a}_i = \mathrm{a}$ fail for each $i = 1, 2, \ldots, n$.

We therefore have, by Piecing, that $\mathrm{v}^+ = \mathrm{u} \cup \mathrm{v} \in V$; and the failure of the truth of $\Delta(a_1, \ldots, a_n) \supset \varphi(a, a_1, \ldots, a_n)$ in $\mathbf{M}^*$ follows.

The reader may find it instructive to compare the proofs of Theorem 20 and Theorem 9 for points of similarity and dissimilarity.

Again, we have:

LEMMA 21. For any classical model **M** and derivation D from C, there is an A-model $\mathbf{M}^*$ that is based upon **M** and suitable for D.

Proof. By Theorem 7, it suffices to check that the system of definitions associated with D is unequivocal and non-circular. But unequivocality follows from Novelty and Weak Flagging; and non-circularity follows from Novelty alone.

Note that, again, we may see some of the restrictions on the rules having their origin in general desiderata upon a system of definitions.

From this theorem, classical soundness may be obtained in the usual way.

Critical Remarks. Which of the two system Q and C is to be preferred? There are various grounds for preference – closeness to ordinary reasoning, naturalness, ease of operation. But on all of them, it seems to me, the advantage lies with the system C.

First, the system Q embodies the restriction that no A-letter shall be generalised upon more than once. This is artificial and has no counterpart in ordinary reasoning. The system C, on the other hand, embodies no such restriction and, to that extent, is less artificial and closer to ordinary reasoning.

Secondly, the semantics of Quine's system requires that one interpret $\forall$-instantial terms in an artificial way as denoting some such thing as a putative counter-example to a generalisation. This is not in accord with our ordinary understanding and makes it hard to attach an intuitive significance to the derivations. On the other hand, the system C allows one to interpret $\forall$-instantial terms more naturally as denoting unrestricted A-objects. This is in accord with our ordinary understanding and makes it much easier to attach an intuitive significance to the derivations.

Moreover, the system *Q* cannot be interpreted in accord with the maxim that the designation of a term be fixed upon its introduction into the derivation; the interpretation of the ∀-instantial terms is retrospectively determined. In the system *C* the interpretation of all terms is fixed on introduction.

Finally, checks on the correctness of derivations are clumsy for the system *Q*. If we think of the dependency diagram as embodying that information from the rest of the derivation that is required to check the current line for correctness, then this difficulty is reflected in the fact that the diagram is not merely extended from one line to the next but may need to be substantially over-hauled. For the system *C*, on the other hand, no such difficulty arises.

Given the undeniable advantage of the Copi–Kalish system over Quine's, it is unfortunate that not more use of it has been made in the classroom. Of the many logic textbooks that have appeared over the years, several have been devoted to Quine's system, but none to the system of Copi and Kalish. Copi had the chance in the third edition of *Symbolic Logic*; but at the last moment, he got cold feet and opted for a system in the style of Gentzen. One can only hope that others will be more adventurous and that the system *C* will get the exposure at the introductory level that it deserves.

7. WHY GO GENERIC?

We here enumerate in more detail some of the advantages to be gained from the adoption of generic semantics for natural deduction. It needs to be emphasized that the application to natural deduction is only one of many applications of the theory of arbitrary objects that is to be made and that therefore part of the advantage of the given application rests upon its absorption within a wider theory. We do not see the application to natural deduction as an isolated case but as part of a more pervasive adoption of a generic apparatus for dealing with matters of generality. But this larger issue is not one that we shall go into here.

1. *A Method of Discovery.* The generic semantics constitutes a powerful heuristic device for discovering or rediscovering systems of natural deduction with a rule of existential instantiation.

It should be emphasized that the problem of finding a suitable set of restrictions on the rules for such systems is far from trivial. There is no obviously correct solution and there is no obvious procedure for finding the correct solution. Indeed, the history of attempts in this direction is studded with failures, with even such distinguished logicians as Quine [13] and Prawitz [12] coming up with faulty solutions.

However, with the advent of the generic semantics, these difficulties disappear. The correct restrictions may be found by seeing what would be required by a reasonable interpretation of the A-letters in terms of arbitrary objects. Suppose, for example, that we had started out with the idea that the instantial term a in applications $\exists x\varphi(x)/\varphi(a)$ of EI was to denote a putative φ-er, dependent upon the A-objects denoted by the given terms, and that the other A-letters were to denote unrestricted A-objects. Then the required restrictions on the rules in the system C could have been almost immediately forthcoming. Weak Flagging would have followed from the requirement that there should not be two different specifications of the same A-object; and Independence would have followed from an informal version of the reasoning in the proof of Line-to-line Soundness. We do not quite get Novelty; but we do get the condition that the relation of $\exists$-dependence be irreflexive, since otherwise one of the designated A-objects would depend upon itself. In fact, the system with the irreflexivity condition in place of Novelty is classically sound; and the stronger condition of Novelty may be thought to result from the imposition of the additional requirement that A-letters be interpreted upon introduction.

Or again, we might have started off with the idea that the instantial term a in applications of $\exists x\varphi(x)/\varphi(a)$ of EI was to denote a putative φ-er dependent upon the given A-objects, as before, but that the instantial term a in applications $\varphi(a)/\forall x\varphi(x)$ of UG was to denote a putative counter-example to $\forall x\varphi(x)$, dependent upon the other A-objects mentioned in $\varphi(x)$. One would then have been immediately led to the restrictions of Quine's system Q. Flagging would have followed from the requirement that there not be two different specifications of the same A-object; and Ordering or Irreflexivity would have followed from the requirement that the dependency relation among the A-objects be irreflexive.

2. *Semantic Significance.* With the generic semantics, syntactic features or distinctions possess a significance that they would otherwise appear to lack.

We already have an example in the restrictions on the rules *UG* and *EI*. Considered on their own account, it is not clear why they take the particular form that they do or what justifies them beyond the mere fact that "they work". With the generic semantics, the exact form of the restrictions is immediately explicable and their justification is immediately forthcoming.

The generic semantics also explains the syntactic discrepancies among systems. Why should the syntactic restrictions on the rule *UG* be so different for Quine's system than for Copi's? We see that there is an underlying semantical explanation in terms of the *UG*-instantial terms: in Quine's system, they denote putative counter-examples; in Copi's system, they denote unrestricted *A*-objects.

Finally, the generic semantics confers a meaning on the derivations themselves. We need no longer think of a derivation as taking us through a detour of meaningless steps in order to establish what we are interested in. Instead, each line of the derivation is endowed with a meaning and, through the relations of dependency, the overall structure of the derivation acquires a significance that it would otherwise lack.

3. *Proofs of Soundness.* The generic semantics enables one to provide simple proofs of classical soundness for systems of natural deduction with a rule of existential instantiation.

Again, it needs to be emphasized that it is by no means a straightforward matter to provide such proofs. In contrast to more orthodox systems, it is not possible to prove, by a straightforward induction on the construction of the derivation, that each line is classically valid, since some of the lines with *A*-letters will not be classically valid. In fact, most authors have in effect followed the devious strategy of first associating derivations in the given system with derivations in a more orthodox system and then showing how classical soundness for the orthodox system transfers to the given system.

With the generic semantics, these proofs become simpler and more direct. Indeed, they flow naturally from the particular semantics for the given system and from general results concerning *A*-objects. True, the general results must first be established and this involves at least as much work as any particular argument. But these results provide a general framework for dealing with the particular cases. Once the framework is set up, the proofs in particular cases become routine.

These proofs have a deeper philosophical significance. The classical ('non-inductive') proofs are suggestive of an instrumentalist conception of deduction. According to this conception, the justification of such rules as *UG* or *EI* depends not upon the validity of the inferences that they sanction but upon the fact that their application never leads to any harm: they never take us from meaningful premisses (ones without *A*-letters) to meaningful conclusions that are not classical consequences of them. Our own proofs, on the other hand, are suggestive of a more orthodox and, to my mind, more satisfactory conception of deduction. According to this conception, the justification of such rules as *UG* or *EI* rests upon the local character of the inferences that they sanction as valid and not upon the global character of the system of rules to which they belong.

4. *Accord with Ordinary Reasoning.* The generic semantics provides an interpretation of natural deduction that is more in accord with our understanding of ordinary quantificational reasoning than any of the familiar alternatives. The issue is a large one, and I hope to deal with it fully elsewhere. I shall therefore content myself with the briefest account here.

As has been pointed out, there exist in ordinary reasoning certain procedures for arguing to a universal conclusion and from an existential premiss. We may establish that all triangles have interior angles summing to 180° by showing of an arbitrary triangle that its interior angles sum to 180°; and having established that there exists a bisector to an angle, we feel entitled to give it a name and declare that it is a bisector to the angle.

Without prejudging the question of their correct formalisation, let us call the first of these procedures *informal UG* and the second *informal EI.* The question now arises as to the correct formalization of these procedures.

It will readily be conceded that informal *UG* is correctly formalised by *UG* (or $\forall I$), though there may be some doubt as to exactly what restrictions on the procedure are observed in our ordinary practice. The problem is with informal *EI.*

Let us revert to a typical application of this procedure:

> There exists a bisector to the angle α.
> Call it (let it be) *B*

There seem to be two main views on how such a piece of reasoning is to be formalised.

According to the first, the second clause 'Call it B' corresponds to an *assumption* that B is a bisector to the angle. So the only reason we are entitled to infer that B is a bisector to the angle is that we have already assumed that it is. On this view, the reasoning would most appropriately be formalised using the rule $\exists E$.

On the other view, the clause 'Call it B' is already taken to represent an inference from the premiss 'There exists a bisector to the angle α'. So there is no need for a further assumption. On this view, then, the reasoning is most appropriately formalised using the rule *EI*.

My own view is that both of these previous accounts are incorrect. I hold that the claim 'There exists a bisector to the angle α' introduces a certain arbitrary object into the discourse. The second clause then serves to give it a name, just as the ordinary English suggests.

However, this is already to buy very heavily into the theory of arbitrary objects. Since I wish to be as cautious as possible in the postulation of arbitrary objects, let us play along with the hypothesis that the premiss is a straightforward existential statement (of the form $\exists x$ (x is a bisector to the angle α)). And under that hypothesis, let us ask what is the most plausible formalisation of the reasoning.

It then seems to me that the advantages lie with *EI*. Since the status of the second clause 'Call it B' is in question, let us consider an application of informal *EI* that does not involve such a clause. Suppose we are arguing from the density of the reals to the conclusion that between any two reals there are two other reals. We might then proceed as follows:

(1) Take any two reals a and b.
(2) We may suppose $a < b$ (since the other case is similar).
(3) Since $a < b$, there is a real c with $a < c < b$.
(4) Since $c < b$, there is a real d with $c < d < b$.
(5) But $a < c < b$ (from (3)), $a < d < b$ (since $a < c < d < b$), and $c \neq d$ (since $c < d$).
(6) So the conclusion then follows.

Allowing for obvious compression, the formalisation using *EI* can represent this piece of reasoning pretty much as it stands. In step (3), we go from $a < b$ to $\exists c(a < c < b)$, and thence to $a < c < b$. In step (4), we likewise to go from $c < b$ to $\exists d(c < d < b)$, and thence to $c < d < b$.

However, the formalisation using $\exists E$ calls for a radical modification in the structure of the reasoning. In the transition between (3) and (4) it must be supposed that the assumption $a < c < b$ is tacitly made. Likewise, in the transition between (4) and (5) it must be supposed that the assumption $c < d < b$ is made. And when it comes to the conclusion (6), it must be supposed that the two assumptions are as quietly dropped as they were made.

This is all most implausible and, along with other difficulties, makes the account using $\exists E$ far inferior to the account using *EI*. If then informal *EI* corresponds to formal *EI*, what account is to be given of the instantial terms in the formal rule and, by extension, of the instantial terms in the informal rule?

Again, there are various hypotheses that have been proposed. One very common one is that, in the inference $\exists x\varphi(x)/\varphi(a)$, the term a is the name of a specific individual. Given the truth of $\exists x\varphi(x)$, we arbitrarily select one of the individuals that φ's and give it a name. But this proposal does not even get off the ground. For first, the statement $\exists x\varphi(x)$ might be false (it could, for example, be a false supposition) and so there might be no individual φ-er for a to name. But second, even when $\exists x\varphi(x)$ is true, we may be in no position to name any individual that φ's.

A more plausible proposal is the Skolemite hypothesis, according to which the instantial terms correspond to functional expressions. For example, the inference $\exists xFxbc/Fabc$ would more explicitly be represented as $\exists xFxbc/F\mathfrak{f}(b,c)bc$. But now the question arises as to the interpretation of the function symbol $\mathfrak{f}$. There would appear to be two main views: according to the first, $\mathfrak{f}$ stands for a specific function ($\mathfrak{f}$ is a functional constant); according to the second, $\mathfrak{f}$ "stands for" an indefinite function (it is a functional variable, existentially quantified from the outside). But the first view suffers from the problem that we may be in no position to denote an appropriate function, even though it exists. The second view, on the other hand, requires a radical alteration in the structure of our reasoning. The inference $\varphi(a), \psi(a)/\varphi(a) \wedge \psi(a)$ (where a is a 0-ary function variable) would most naturally be represented by $\exists x\varphi(x), \exists x\psi(x)/\exists x(\varphi(x) \wedge \psi(x))$. But this latter inference is valid (cf. Schagrin [16] and Lemmon [9]). So it must be supposed that the whole derivation takes place within the scope of outermost existential quantifiers for the functional variables. But this means that the lines up to a given stage of the derivation must all somehow be wrapped up into a single conjunction for the outermost quantifiers to apply to, and that subsequent inferences must be seen as operations on the conjuncts within the resulting quantified statement.

Perhaps the most plausible proposal of conventional sort is the one that treats the instantial terms as schematic or ambiguous names. The proposal might be formulated in the following way. Take my own semantics for a system of natural deduction and suppose that the individual values attach directly to the names. So instead of supposing that the A-letters $a_1, \ldots, a_n$ denote the A-objects $\mathrm{a}_1, \ldots, \mathrm{a}_n$ which can then simultaneously assume such values as $\mathrm{i}_1, \ldots, \mathrm{i}_n$, we suppose that the A-letters can directly assume the values $\mathrm{i}_1, \ldots, \mathrm{i}_n$. Subject to this modification, the definitions of validity and the proofs of Soundness may proceed as before.

We might even suppose that the ambiguous names are given some structure, so that it is clear from that structure what values they can simultaneously assume. To this end, we might appropriate the symbolism of the ϵ-calculus. In the inference $\exists x\varphi(x)/\varphi(a)$, the term a would then be taken to be of the form $\epsilon x\varphi(x)$ (as in Routley [15]) and the terms $s = \epsilon xPx$ and $t = \epsilon yRys$, let us say, would be taken to assume all values i and j for which $\exists xPx \supset P\mathrm{i}$ and $\exists yRy\mathrm{i} \supset R\mathrm{ji}$ were true.

Given that this proposal is satisfactory and that it does little injustice to the structure of derivations, why opt for the generic semantics? Why interpose the arbitrary objects between the names and the individuals? Why have an objectual rather than a nominal theory?

There are various advantages, it seems to me, in having an objectual theory. One has to do with our intuition that, in informal reasoning involving *UG* or *EI*, we are dealing with definite objects. The objectual theory supports this intuition; the nominal theory does not. In any case, it is useful to have the statements of such reasoning express definite propositions. For we can then suppose that in following through the reasoning the reasoner has those propositions in mind and that the propositions or the inferences concerning them are evaluated for truth or validity in the normal way.

The objectual theory is also conducive to conceptual clarity. For in its definition of A-model (with the conditions of Partial Extendibility, Piecing, etc.), it provides an analysis of that abstract structure that may be thought to underly the suitable assignment of values to terms. The structure may be realized linguistically: A-objects may be replaced by A-names, and objectual dependency by syntactic dependency. But that is only to obscure the abstract nature of the underlying structure.

In this respect, the objectual approach is somewhat akin to Kripke's treatment of possible worlds as unstructured elements as opposed to Carnap's treatment of them as state-descriptions. For many purposes, Kripke's analysis is just what one needs; Carnap's linguistic representation merely gets in the way. Similarly, for many purposes, the objectual approach is just what one needs; the linguistic representation of the A-objects or of the relation of dependency merely introduces irrelevancies.

But these considerations are marginal. The main advantage in adopting the generic semantics has to do with the faithful representation of ordinary reasoning. In regard to straightforward first-order reasoning, this is not clear; for the nominal approach seems to perform as well as the objectual. But it becomes especially clear in regard to extensions of first-order reasoning: for then the weakness of the nominal approach begin to show up.

Consider the following piece of ordinary mathematical reasoning:

Let $y = \mathsf{f}(x)$ be a continuous function. Take any real h. Then for some k, $\mathsf{f}(x + h) = y + k$. Now since f is continuous, $k \to 0$ as $h \to 0$. So . . . .
On the objectual approach, the statement '$k \to 0$ as $h \to 0$' is readily intelligible, for this just attributes a certain property to the set of all pairs of values assumed by the A-objects k and h. But on the nominal approach, the statement is hard to make sense of, for the A-objects k and h are not at hand and it is not simply an ambiguous statement about each of the pairs of values that k and h can assume. In general, it seems impossible to reconcile the nominal approach with a reasonable view of the role of variable signs in mathematics.

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