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TOTALLY DISCONNECTED SETS, JORDAN CURVES, AND CONFORMAL MAPS[1]

by

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Dedicated to the memory of ALFRÉD RÉNYI

Each bounded, closed, totally disconnected set M in the w-plane lies on some Jordan curve Γ (see R. L. MOORE and J. R. KLINE [3]). Let G denote the bounded domain determined by Γ, let D and C denote the unit disk and the unit circle in the z-plane, and let f be a mapping of $D \cup C$ onto $G \cup \Gamma$, holomorphic in D and continuous and univalent in $D \cup C$. Kikuji MATSUMOTO ([2], Theorem 4) showed that if we pinch the domain G in appropriate places, then the set $f^{-1}(M)$ has logarithmic capacity 0. In this note, we prove that we can not only make the set $f^{-1}(M)$ arbitrarily thin, but that we can require it to lie in any preassigned perfect subset of C.

THEOREM. *Let M be a bounded, closed, totally disconnected set in the w-plane, and let E be a perfect set on C. Then there exists a function f, holomorphic in D and continuous and univalent in $D \cup C$, such that $f^{-1}(M) \subset E$.*

Our proof is based on the construction of a certain tree T and a certain Jordan domain G_0 in the w-plane. The tree lies in G_0, and the derived set of its set of vertices is M. A simple analytic process allows us to replace the tree T with a subdomain G of G_0 such that one of the corresponding holomorphic and univalent functions f from $D \cup C$ onto $\bar{G}$ satisfies the condition $f(E) \supset M$.

The tree. Without loss of generality, we may assume that the set M lies in the open rectangle Q whose vertices are the points $w = \pm\sqrt{2}/2$ and $w = \pm\sqrt{2}/2 + i$. Since M is closed and totally disconnected, there exists a directed polygonal arc P that begins at the point 0, lies in $Q \setminus M$, and separates Q into two components Q_0 and Q_1, each of diameter less than $\sqrt{3}\,(4/5)$. Similarly, there exist two directed polygonal arcs P_0 and P_1 in $Q_0 \setminus M$ and $Q_1 \setminus M$, with a common initial point on P, and such that each of the four corresponding sets $Q_{00}, Q_{01}, Q_{10}, Q_{11}$ has diameter less than $\sqrt{3}\,(4/5)^2$. We continue the dissec-

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tion of Q indefinitely, in such a way that each polygonal arc of the n^{th} stage is divided into two parts by the common initial point of two arcs of the $(n+1)^{\text{st}}$ stage. The union of the anterior parts thus determined constitutes a tree T_0, and each vertex of T_0 (except the point $w = 0$) has degree 2 or 3 (see the heavily drawn portion of Figure 1). We may assume that the directions of two consecutive segments of T_0 always differ by less than $\pi/2$.

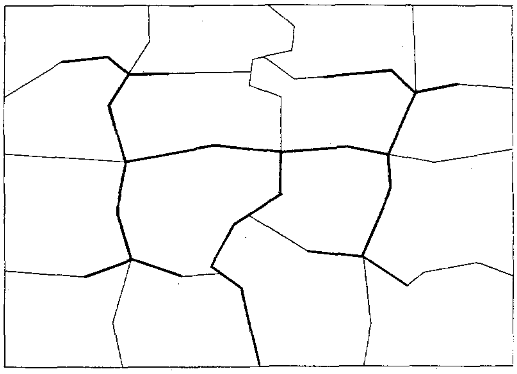

Fig. 1

Because the set M meets none of the polygonal arcs $P, P_0, P_1, P_{00}, P_{01}, \ldots$, each point of M is the limit point of exactly one simple path that begins at 0 and lies in T_0. The union of all simple paths beginning at 0, lying in T_0, and having a limit point in M constitutes our tree T.

The domain G_0. We arrange the segments of T into a sequence $\{S_m\}$ so that $m_0 < m$ whenever S_{m_0} precedes S_m in T, and so that $|m_0 - m| = 1$ whenever S_{m_0} and S_m have a common initial point. We then choose a sequence $\{\delta_m\}$ of positive numbers, and we denote by H_m the set of all points whose distance from S_m is less than δ_m. If $\delta_m \to 0$ rapidly enough, then the set $G_0 = \bigcup H_m$ is a Jordan domain, and for each index m the intersection of M with the closure of H_m is empty.

The analytic device. Barring an obvious geometric obstacle, the following lemma allows us to pass from any univalent function f in $|z| < r_0$ $(r_0 > 1)$ to a univalent function g such that the essential difference between the domains $f(D)$ and $g(D)$ is a narrow rod of prescribed base, length, and direction.

LEMMA (compare [1], pp. 43—44). *Suppose that the function f is holomorphic and univalent in some disk $|z| < r_0$ $(r_0 > 1)$. Let $\zeta = e^{i\theta}$, and let L be a complex number such that*

$$|\arg L - \arg \zeta f'(\zeta)| < \pi/2 \tag{1}$$

and such that the line segment S joining the points $f(\zeta)$ and $f(\zeta) + L$ meets the set $f(D \cup C)$ only at $f(\zeta)$. Corresponding to each real number ϱ ($\varrho < 1$), write

$$g_\varrho(z) = f(z) + L\frac{\log(1 - z/z_0)}{\log(1 - 1/\varrho)}, \tag{2}$$

where $z_0 = \varrho e^{i\theta}$. Then there exists a constant $\varrho_0 (\varrho_0 > 1)$ such that for $1 < \varrho < \varrho_0$ the function g_ϱ is univalent in some disk $|z| < r_1$ ($1 < r_1 < \varrho$).

To prove the lemma, we write $\varrho = 1 + \varepsilon$, we impose the preliminary restrictions $\varepsilon < 1/e$ and $\varepsilon < (r_0 - 1)/2$, and we observe that the univalence of f in $|z| < r_0$ implies the existence of a positive constant A_1 such that the inequality

$$|f(z_2) - f(z_1)| \geqq A_1 |z_2 - z_1| \tag{3}$$

holds for all z_1 and z_2 in $D \cup C$. We write $z/\zeta = \alpha + i\beta$ (a and β real), and we consider the function g_ϱ separately in the two overlapping regions

$$D_1 = \{z : |z| \leqq 1,\ \alpha \leqq 1 - K/|\log \varepsilon|\},$$

$$D_2 = \{z : |z| \leqq 1,\ \alpha \geqq 1 - (\log|\log \varepsilon|)^{-1}\}$$

(see Figure 2); here K denotes a positive number to be chosen below.

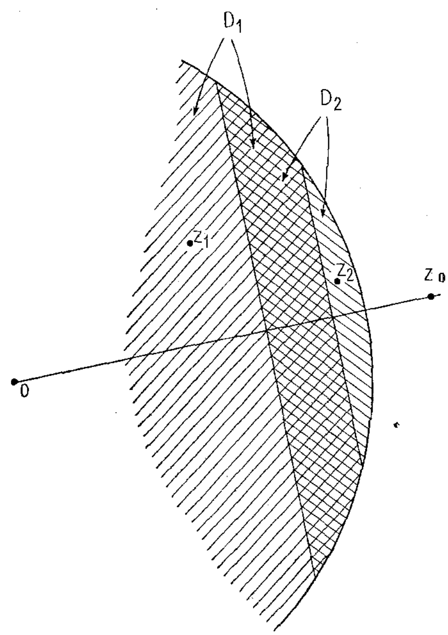

Fig. 2

Since

$$g_\varrho'(z) - f'(z) = \frac{L}{|\log \varepsilon/\varrho\,|\,(z - z_0)},$$

and since in D_1 the maximum modulus of the right-hand member is

$$\frac{|L|}{|\log \varepsilon/\varrho\,|\,(\varepsilon + K/|\log \varepsilon|)} < |L|\,/\,K,$$

the inequality

$$|\,g_\varrho(z_2) - g_\varrho(z_1)\,| \geqq |\,z_2 - z_1\,|\,(A_1 - |\,L\,|/K)$$

holds for all z_1 and z_2 in D_1. In particular, the choice $K = A_1/2\,|\,L\,|$ gives the inequality

$$|\,g_\varrho(z_2) - g_\varrho(z_1)\,| \geqq A_1\,|\,z_2 - z_1\,|/2,$$

and therefore g_ϱ is univalent in D_1.

To establish univalence in D_2, we examine the argument of the derivative

$$g_\varrho'(z) = \frac{1}{z_0}\left[z_0 f'(z) + \frac{L}{|\log \varepsilon/\varrho|}\cdot\frac{1}{(1 - z/z_0)}\right].$$

By the inequality (1), the argument of the first term in the brackets is restricted to some interval $[\arg L - \eta,\ \arg L + \eta]$, where $\eta < \pi/2$ if ε is sufficiently small. Because the argument of the second term is also restricted to such an interval, the theorem of K. NOSHIRO and S. E. WARSCHAWSKI implies that the function g_ϱ is univalent in D_2 (see [4], Theorem 12, p. 151; [5], Lemma 1, p. 312).

To conclude the proof of the lemma, we shall show that if $z_1 \in D_1 \setminus D_2$ and $z_2 \in D_2 \setminus D_1$, then $g_\varrho(z_1)$ lies at a greater distance from the segment S than $g_\varrho(z_2)$.

Our hypothesis on the line segment S implies the existence of a positive constant A_2 such that for each z in $D \cup C$ the distance between $f(z)$ and the segment S is at least $A_2\,|\,z - \zeta\,|$. Therefore the distance between $f(z_1)$ and the segment S is at least $A_2(\log|\log \varepsilon|)^{-1}$. Since the imaginary part of $\log(1 - z/z_0)$ is bounded by $\pi/2$, the distance between $g_\varrho(z_1)$ and S is at least

$$A_2(\log|\log \varepsilon|)^{-1} - 2\,|\,L\,|\cdot|\log \varepsilon|^{-1} > A_3(\log|\log \varepsilon|)^{-1}.$$

On the other hand, (2) implies that if A_4 denotes the maximum modulus of f' on C, then the distance between $g_\varrho(z_2)$ and S is less than

$$A_4\sqrt{2K/|\log \varepsilon|} + \frac{|L|\,\pi/2}{|\log \varepsilon/\varrho|} < A_5/\sqrt{|\log \varepsilon|}.$$

This shows that $g_\varrho(z_1) \neq g_\varrho(z_2)$, and the lemma is proved.

Construction of the domain G. We choose any point z_1 in the perfect set E, and we denote by L_1 the coordinate of the endpoint of the segment S_1 in the tree T. If $\varrho_1 - 1$ is small enough, then the function

$$f_1(z) = L_1 \cdot \frac{\log(1 - z/\varrho_1 z_1)}{\log(1 - z/\varrho_1)}$$

maps the set $D \cup C$ onto a region lying in H_1 and containing the segment S_1.

If L_1 is not a branch point of the tree T, we write $z_2 = z_1$, and we construct the function

$$f_2(z) = f_1(z) + L_2 \frac{\log(1 - z/\varrho_2 z_2)}{\log(1 - 1/\varrho_2)},$$

choosing L_2 so that f_2 maps z_2 onto the endpoint of S_2, and choosing ϱ_2 near enough to 1 so that $f(D \cup C) \subset H_1 \cup H_2$. If L_1 is a branch point of T, we choose two distinct points z_2 and z_3 of E near z_1 (this is possible, since E is perfect), and we construct the function f_3 so that

$$f_3(D \cup C) \subset H_1 \cup H_2 \cup H_3,$$

and so that $f_3(z_2)$ and $f_3(z_3)$ are near enough to the endpoints of S_2 and S_3 to allow the obvious continuation of the process.

Clearly, the function $f = \lim f_m$ is univalent and continuous in $D \cup C$, and $f(D) \subset G_0$. Since E is closed and each point of M is a limit point of the sequence $\{f(z_m)\}$, the set M lies in the set $f(E)$. This concludes the proof of the theorem.

If we drop the hypothesis that the set M is bounded, the theorem remains valid provided we interpret continuity in terms of the spherical metric.

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