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New Realizations of the Maximal Satake Compactifications of Riemannian Symmetric Spaces of Noncompact Type

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Abstract. We give new realizations of the maximal Satake compactifications of Riemannian symmetric spaces of noncompact type as orbit closures inside Grassmannians and orthogonal groups. Our constructions are partially motivated by Poisson geometry.

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1. Introduction and the Main Results

Let $X = G/K$ be a Riemannian symmetric space of noncompact type, where G is a connected real semi-simple Lie group with trivial center, and K is a maximal compact subgroup of G. The maximal Satake compactification $\overline{X}^S_{\max}$ of X was constructed by Satake [5] using any finite-dimensional faithful irreducible projective representation of G with generic highest weight (see Remark 5.3). Let $m = \dim K$, and let $\mathfrak{g}$ be the Lie algebra of G. In this note, we first show that $\overline{X}^S_{\max}$ can also be obtained as a G-orbit closure inside the Grassmannian $\mathrm{Gr}(m, \mathfrak{g})$ of m-dimensional subspaces of $\mathfrak{g}$. More precisely, G acts on $\mathrm{Gr}(m, \mathfrak{g})$ *via* the adjoint action of G on $\mathfrak{g}$. Let $\mathfrak{k}$ be the Lie algebra of K and regard $\mathfrak{k}$ as a point in $\mathrm{Gr}(m, \mathfrak{g})$. Then the map

$$\mu : X = G/K \longrightarrow \mathrm{Gr}(m, \mathfrak{g}) : gK \mapsto \mathrm{Ad}_g \mathfrak{k}$$

is a G-equivariant embedding of X into $\mathrm{Gr}(m, \mathfrak{g})$.

THEOREM 1.1. *The closure* $\overline{\mu(X)}$ *of* $\mu(X)$ *in* $\mathrm{Gr}(m, \mathfrak{g})$ *is G-isomorphic to the maximal Satake compactification* $\overline{X}^S_{\max}$ of X.

Theorem 1.1 gives rise to other realizations of $\overline{X}^S_{\max}$, three of which will be presented in this note. Consider first the complex symmetric variety $X_{\mathbb{C}} := \hat{G}/K_{\mathbb{C}}$, where $\hat{G}$ is the adjoint group of the complexification $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}$ of $\mathfrak{g}$ and $K_{\mathbb{C}}$ is the connected subgroup of $\hat{G}$ with Lie algebra $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \otimes \mathbb{C}$. In [1], De Concini and Procesi constructed a 'wonderful' compactification of $X_{\mathbb{C}}$ which is a smooth $\hat{G}$-variety. Embed X into $X_{\mathbb{C}}$ *via* the inclusion $G \hookrightarrow \hat{G}$. We have

COROLLARY 1.2. *The closure of X in the wonderful compactification of $X_{\mathbb{C}}$ with respect to the regular topology is G-isomorphic to the maximal Satake compactification $\overline{X}^S_{\max}$ of X.*

Let $n = \dim \mathfrak{g}$, and consider now the Grassmannian $\mathrm{Gr}(n, \hat{\mathfrak{g}})$ of n-dimensional real subspaces in $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}$. Then again G acts on $\mathrm{Gr}(n, \hat{\mathfrak{g}})$ via its adjoint action on $\hat{\mathfrak{g}}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. Then $\mathfrak{u} := \mathfrak{k} + i\mathfrak{p}$ is a compact real form of $\mathfrak{g}$. By regarding $\mathfrak{u}$ as a point in $\mathrm{Gr}(n, \mathfrak{g})$, we have a G-equivariant embedding

$$\hat{\mu} : X = G/K \longrightarrow \mathrm{Gr}(n, \hat{\mathfrak{g}}) : gK \mapsto \mathrm{Ad}_g \mathfrak{u}. \tag{1}$$

COROLLARY 1.3. *The closure $\overline{\hat{\mu}(X)}$ of $\hat{\mu}(X)$ in $\mathrm{Gr}(n, \hat{\mathfrak{g}})$ is G-isomorphic to the maximal Satake compactification $\overline{X}^S_{\max}$ of X.*

Let $\mathrm{O}(\mathfrak{u})$ be the orthogonal group of $\mathfrak{u}$ defined by the Killing form of $\mathfrak{u}$. We will regard $\mathrm{O}(\mathfrak{u})$ as a subgroup of $\mathrm{GL}(\hat{\mathfrak{g}})$ by complex linear extensions. For $g \in \hat{G}$ and $\phi \in \mathrm{O}(\mathfrak{u})$, let

$$\begin{aligned} g \cdot \phi := {} & i(\mathrm{Ad}_g(\phi + i) + \mathrm{Ad}_{\hat{\theta}(g)}(\phi - i))(\mathrm{Ad}_g(\phi + i) - \\ & - \mathrm{Ad}_{\hat{\theta}(g)}(\phi - i))^{-1} \in \mathrm{GL}(\hat{\mathfrak{g}}), \end{aligned} \tag{2}$$

where $\hat{\theta}$ denotes the complex conjugation on $\hat{\mathfrak{g}}$ defined by $\mathfrak{u}$ as well as its lifting to $\hat{G}$. In Proposition 5.1, we will show that (2) defines a left action of $\hat{G}$ on $\mathrm{O}(\mathfrak{u})$. Further study of the embedding $\hat{\mu} : X \to \mathrm{Gr}(n, \hat{\mathfrak{g}})$ shows that the image $\hat{\mu}(X)$ in fact lies in a certain $\hat{G}$-invariant closed subvariety $\mathcal{I}$ of $\mathrm{Gr}(n, \hat{\mathfrak{g}})$ which can be $\hat{G}$-equivariantly identified with $\mathrm{O}(\mathfrak{u})$. Consequently, we have a G-equivariant embedding of X into $\mathrm{O}(\mathfrak{u})$ given by

$$\nu : X = G/K \longrightarrow \mathrm{O}(\mathfrak{u}) : gK \mapsto \frac{i\mathrm{Ad}_{g\theta(g)^{-1}} + 1}{i + \mathrm{Ad}_{g\theta(g)^{-1}}},$$

where G, as a subgroup of $\hat{G}$, acts on $\mathrm{O}(\mathfrak{u})$ by (2).

COROLLARY 1.4. *The closure $\overline{\nu(X)}$ of $\nu(X)$ in $\mathrm{O}(\mathfrak{u})$ is G-isomorphic to the maximal Satake compactification $\overline{X}^S_{\max}$* of X.

The constructions of $\overline{X}^S_{\max}$ in this note all fit into the general framework as in the Satake and Furstenberg compactifications: embed X into a compact G-space equivariantly and take the closure of the embedding. The construction in Theorem 1.1 is similar to the intrinsic construction of $\overline{X}^S_{\max}$ in [4, Ch. IX]. The construction in Corollary 1.4 resembles Satake's original construction of $\overline{X}^S_{\max}$ in the sense that we obtain $\overline{X}^S_{\max}$ by first using the adjoint representation of G to embed X (as a totally geodesic submanifold) into the symmetric space $\mathrm{O}(\hat{\mathfrak{g}})/\mathrm{O}(\mathfrak{u})$ which is compactified by using the Cayley transform

$$S \mapsto \frac{iS+1}{i+S}.$$

Here $\mathrm{O}(\hat{\mathfrak{g}})$ is the (complex) orthogonal group of $\hat{\mathfrak{g}}$ defined by the Killing form of $\hat{\mathfrak{g}}$. (See Remark 5.3 for more detail.)

The closure $\overline{\hat{\mu}(X)}$ of $\hat{\mu}(X)$ in $\mathrm{Gr}(n, \hat{\mathfrak{g}})$ as in Corollary 1.3 appeared in [2] on our study of certain 'moduli space' of Poisson homogeneous spaces, which was in turn motivated by the theory of quantum groups. One can show (see [3, Section 3]) that there is a natural Poisson structure π on $X = G/K$ which extends to $\overline{\hat{\mu}(X)}$. Corollary 1.3 will enable us to use the structure theory of $\overline{X}^S_{\max}$ to study the boundary behavior of π on X. We will carry out this study in a future paper, and we refer to [3] and [2] for the related background on Poisson geometry. As is explained in [4], there are characterizations of $\overline{X}^S_{\max}$ from various points of view, such as that of Riemannian geometry, of the theory of random walks, and of harmonic analysis on X, each of which has its own advantage and sheds lights on the others. Our characterization of $\overline{X}^S_{\max}$ in Corollary 1.3 is suitable for the study of Poisson structures on X, and it is the first step in our work on establishing connections between Poisson geometry and harmonic analysis on X.

In the rest of the Letter, we give proofs for Theorem 1.1 and Corollaries 1.2, 1.3, and 1.4.

2. Proof of Theorem 1.1

We will use Satake's characterization of $\overline{X}^S_{\max}$ as stated in [4, Proposition 4.42]. We will mostly follow the notation used in [4].

Fix the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of $\mathfrak{g}$, and let θ be the corresponding Cartan involution. Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Let Σ be the set of roots of $\mathfrak{a}$ in $\mathfrak{g}$, and let Σ_+ be a choice of positive roots in Σ. Let

$$c(\mathfrak{a}^+) = \{\lambda \in \mathfrak{a} : \alpha(\lambda) \geqslant 0, \forall \alpha \in \Sigma_+\}$$

be the *closed* positive Weyl chamber defined by Σ_+, and let $c(A^+) = \exp c(a^+)$. Then we know from the Cartan Decomposition $G = Kc(A^+)K$ that for any topological G-compactification $\overline{X}$ of X, we have $\overline{X} = K \cdot \overline{c(A^+)}$, where $\cdot$ denotes the K-action on $\overline{X}$. Thus $\overline{X}$ is determined by the topology of the closure $\overline{c(A^+)}$ of $c(A^+)$ in $\overline{X}$ and the G-action on $\overline{X}$. A characterization of $\overline{X}^S_{\max}$ in these terms is given in [4, Proposition 4.42].

We first determine the topology of $\overline{\mu(c(A^+))}$, the closure of $\mu(c(A+))$ in $\mathrm{Gr}(m, \mathfrak{g})$. Let Δ be the set of all simple roots in Σ_+. For each subset $I \subset \Delta$, let

$$\mathfrak{a}_I = \{\lambda \in \mathfrak{a} : \alpha(\lambda) = 0, \forall \alpha \in I\},$$

and let $\mathfrak{a}^I$ be the orthogonal complement of $\mathfrak{a}_I$ in $\mathfrak{a}$ with respect to the Killing form of $\mathfrak{g}$. Let

$$c(\mathfrak{a}^{I,+}) = \{\lambda \in \mathfrak{a}^I : \alpha(\lambda) \geqslant 0\, \forall \alpha \in I\},$$

and let $c(A^{I,+}) = \exp(c(\mathfrak{a}^{I,+}))$. We will use $[I]$ to denote the set of roots that are linear combinations of elements in I. Let $\mathfrak{n}_I = \sum_{\alpha \in \Sigma_+ \setminus [I]} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the root space of α.

Let $\mathfrak{g}^I$ be the derived subalgebra of the centralizer of $\mathfrak{a}_I$ in $\mathfrak{g}$ [4, Proposition 2.10] and let $\mathfrak{k}^I = \mathfrak{g}^I \cap \mathfrak{k}$. Set $\mathfrak{d}^I = \mathfrak{m} + \mathfrak{k}^I + \mathfrak{n}_I$, where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. We will describe the space $\overline{\mu(c(A^+))}$ in terms of the $c(A^{I,+})$'s and the $\mathfrak{d}^I$'s.

Assume now that $\mathfrak{l} \in \overline{\mu(c(A^+))}$. Then there exists a sequence $\lambda_n \in c(\mathfrak{a}^+)$ such that

$$\mathfrak{l} = \lim_{n\to\infty} \mu(\exp(\lambda_n)) = \mathrm{Ad}_{\exp(\lambda_n)}\mathfrak{k}.$$

For each $\alpha \in \Sigma_+$, let $\mathfrak{k}_\alpha = \{X + \theta(X) : X \in \mathfrak{g}_\alpha\}$. Then we have $\mathfrak{k} = \mathfrak{m} + \sum_{\alpha\in\Sigma_+} \mathfrak{k}_\alpha$ as a direct sum. Since $\alpha(\lambda_n) \geqslant 0$ for all $\alpha \in \Sigma_+$ and all n, there exists a subsequence λ_n' such that $\{\alpha(\lambda_n')\}$ either converges or diverges to $+\infty$ for each simple root α. Let I be the set of all simple roots α such that $\{\alpha(\lambda_n')\}$ converges. Note that $\mathfrak{l}$ is in the boundary of $\overline{\mu(c(A^+))}$ if and only if $I = \Delta$. Let $\lambda_0 \in c(\mathfrak{a}^{I,+})$ be such that $\alpha(\lambda_0) = \lim_{n\to\infty} \alpha(\lambda_n')$ for all $\alpha \in I$. Now choose nonzero vectors $Y_\mathfrak{m} \in \wedge^{\dim \mathfrak{m}}\mathfrak{m}$ and $Y_\alpha \in \wedge^{\dim(\mathfrak{k}_\alpha)}\mathfrak{k}_\alpha$ for each $\alpha \in \Sigma_+$. Then,

$$v = \mathbb{R}\left(Y_\mathfrak{m} \wedge \bigwedge_{\alpha\in\Sigma_+} Y_\alpha\right) \in P(\wedge^m\mathfrak{g})$$

represents the point $\mathfrak{k} \in \mathrm{Gr}(m, \mathfrak{g})$ under the Plucker embedding of $\mathrm{Gr}(m, \mathfrak{g})$ into the projective space $P(\wedge^m\mathfrak{g})$. Since $\mathfrak{m}$ centralizes $\mathfrak{a}$, we have $\mathrm{Ad}_{\exp(\lambda_n')} Y_\mathfrak{m} = Y_\mathfrak{m}$ for all n. For $\alpha \in \Sigma_+$ and $X \in \mathfrak{g}_\alpha$, we have, for all n,

$$\mathrm{Ad}_{\exp(\lambda_n')}(X + \theta(X)) = \mathrm{e}^{\alpha(\lambda_n')}X + \mathrm{e}^{-\alpha(\lambda_n')}\theta(X) = \mathrm{e}^{\alpha(\lambda_n')}(X + \mathrm{e}^{-2\alpha(\lambda_n')}\theta(X)).$$

Since $\lim_{n\to\infty} \alpha(\lambda_n') = +\infty$ for $\alpha \in \Sigma_+\backslash[I]$ and since $\lim_{n\to\infty} \alpha(\lambda_n') = \alpha(\lambda_0)$ for $\alpha \in \Sigma_+ \cap [I]$, we see that the limit of $\mathrm{Ad}_{\exp(\lambda_n')}v$ in $P(\wedge^m\mathfrak{g})$ as $n \to \infty$ corresponds to

$$\mathfrak{l} = \mathfrak{m} + \mathrm{Ad}_{\exp(\lambda_0)}\left(\sum_{\alpha\in\Sigma_+\cap[I]} \mathfrak{k}_\alpha\right) + \mathfrak{n}_I = \mathrm{Ad}_{\exp(\lambda_0)}(\mathfrak{m} + \mathfrak{k}^I + \mathfrak{n}_I) = \mathrm{Ad}_{\exp(\lambda_0)}\mathfrak{d}^I \tag{3}$$

in $\mathrm{Gr}(m, \mathfrak{g})$ under the Plucker embedding. Using '$\cdot$' to denote the action of G on $\mathrm{Gr}(m, \mathfrak{g})$, we see that $\mathfrak{l} \in c(A^{I,+}) \cdot \mathfrak{d}^I$. Conversely, for any subset I of Δ, let $\lambda \in \mathfrak{a}$ be such that $\alpha(\lambda) = 0$ for all $\alpha \in I$ and $\alpha(\lambda) > 0$ for $\alpha \notin I$, where α is a simple root. Then it is easy to see that

$$\mathfrak{d}^I = \lim_{n\to\infty} \mathrm{Ad}_{\exp(n\lambda)}\mathfrak{k} \in \overline{\mu(c(A^+))}.$$

Thus we have

$$\overline{\mu(c(A^+))} = \bigcup_{I\subset\Delta} c(A^{I,+}) \cdot \mathfrak{d}^I. \tag{4}$$

It is easy to prove that (4) is a disjoint union, and $c(A^{I,+}) \cdot \mathfrak{d}^I \cong c(A^{I,+})$ for each I. Moreover, a computation similar to the one that leads to (3) shows that a sequence $\exp(\lambda_n) \cdot \mathfrak{d}^{I_1} \in c(A^{I_1,+}) \cdot \mathfrak{d}^{I_1}$ converges to $\exp(\lambda) \cdot \mathfrak{d}^{I_2} \in c(A^{I_2,+}) \cdot \mathfrak{d}^{I_2}$ if and only if $I_2 \subset I_1$, $\lim_{n\to\infty} \alpha(\lambda_n) = \alpha(\lambda)$ for all $\alpha \in I_2$ and $\lim_{n\to\infty} \alpha(\lambda_n) = +\infty$ for $\alpha \in I_1\backslash I_2$. Thus the closure $\overline{\mu(c(A^+))}$ of $\mu(c(A^+))$ in $\mathrm{Gr}(m, \mathfrak{g})$ is homeomorphic to the closure of $c(A^+)$ in $\overline{X}^S_{\max}$ (see [4, Proposition 4.42]).

It follows from (4) that the closure $\overline{\mu(X)}$ of $\mu(X)$ in $\mathrm{Gr}(m, \mathfrak{g})$ is the union $\bigcup_{I\in\Delta} G \cdot \mathfrak{d}^I$. By [4, Corollary 9.15], this is a disjoint union, and it follows from [4, Lemma 9.13] that each G-orbit $G \cdot \mathfrak{v}^I$ fibers over the flag manifold G/P^I whose fiber is isomorphic to the symmetric space X^I (see notation in [4, Ch. IX]). Thus we know by [4, Proposition 4.42] that $\overline{\mu(X)}$ is G-isomorphic to $\overline{X}^S_{\max}$. □

3. Proof of Corollary 1.2

Let $\mathrm{Gr}_{\mathbb{C}}(m, \hat{\mathfrak{g}})$ be the Grassmannian of complex m-dimensional subspaces of $\hat{\mathfrak{g}}$. Recall from [1, Section 6] that the map

$$\kappa\colon X_{\mathbb{C}} = \hat{G}/K_{\mathbb{C}} \longrightarrow \mathrm{Gr}_{\mathbb{C}}(m, \hat{\mathfrak{g}}) : \ gK_{\mathbb{C}} \mapsto \mathrm{Ad}_g \mathfrak{k}_{\mathbb{C}}$$

is an embedding and that the closure of $\kappa(X_{\mathbb{C}})$ in $\mathrm{Gr}_{\mathbb{C}}(m, \hat{\mathfrak{g}})$ is isomorphic to De Concini and Procesi's wonderful compactification of $X_{\mathbb{C}}$. By considering the G-equivariant embedding of $\mathrm{Gr}(m, \mathfrak{g})$ into $\mathrm{Gr}_{\mathbb{C}}(m, \hat{\mathfrak{g}})$ which maps $l \in \mathrm{Gr}(m, \mathfrak{g})$ to its complexification, we see that Corollary 1.2 follows immediately from Theorem 1.1. □

4. Proof of Corollary 1.3

Let $\imath\colon \hat{G} \to \mathrm{PSL}(V)$ be any faithful irreducible projective representation of $\hat{G}$ with generic highest weight. Then the restriction of $\imath$ to G is such a representation for G. Let U be the connected subgroup of $\hat{G}$ with Lie algebra $\mathfrak{u}$, and let $\hat{X} = \hat{G}/U$. Then we have the embedding $X \hookrightarrow \hat{X}$ induced from the inclusion $G \hookrightarrow \hat{G}$. It follows by Satake's definition of $\overline{X}^S_{\max}$ and $\overline{\hat{X}}^S_{\max}$ that $\overline{X}^S_{\max}$ is the closure of $X \hookrightarrow \hat{X}$ inside $\overline{\hat{X}}^S_{\max}$. Thus Theorem 1.3 follows from applying Theorem 1.1 to $\hat{X}$. □

5. Proof of Corollary 1.4

Let $\langle\, , \rangle$ be the imaginary part of the Killing form $\ll\, , \gg$ of $\hat{\mathfrak{g}}$. Denote by $\mathcal{I}$ the set of all maximal isotropic subspaces of $\hat{\mathfrak{g}}$ with respect to $\langle\, , \rangle$. By Witt's theorem, the dimensions of such subspaces are n, so $\mathcal{I}$ is an algebraic subvariety of $\mathrm{Gr}(n, \hat{\mathfrak{g}})$. It is clear that $\mathcal{I}$ is $\hat{G}$-invariant, and $\mathfrak{u} \in \mathcal{I}$. Thus the $\hat{\mu}(X) \subset \mathcal{I}$ and so we can regard $\hat{\mu}$ as an embedding of X into $\mathcal{I}$ and $\overline{X}^S_{\max}$ is then the closure of $\hat{\mu}(X)$ inside $\mathcal{I}$.

Recall that $\mathrm{O}(\mathfrak{u})$ is the orthogonal group of $\mathfrak{u}$ defined by the Killing form of $\mathfrak{u}$. We can regard $\mathrm{O}(\mathfrak{u})$ as a subgroup of $\mathrm{GL}(\hat{\mathfrak{g}})$ by complex linearly extending an element $\phi \in \mathrm{O}(\mathfrak{u})$ to a linear map from $\hat{\mathfrak{g}}$ to $\hat{\mathfrak{g}}$ using the decomposition $\hat{\mathfrak{g}} = \mathfrak{u} + i\mathfrak{u}$. Let $\hat{\theta}$ be the complex conjugate linear involution on $\hat{\mathfrak{g}}$ determined by $\mathfrak{u}$ as well as its lifting to $\hat{G}$. We will now describe an identification of $\mathrm{O}(\mathfrak{u})$ and $\mathcal{I}$.

PROPOSITION 5.1. *The map,*

$$\Phi\colon\ \mathrm{O}(\mathfrak{u}) \longrightarrow \mathcal{I}\colon \phi \mapsto l_\phi := \{(1+i)x + (1-i)\phi(x) : x \in \mathfrak{u}\} \tag{5}$$

is a diffeomorphism. Under Φ, *the action of* $\hat{G}$ *on* $\mathcal{I}$ *becomes the following action of* $\hat{G}$ *on* $\mathrm{O}(\mathfrak{u})$: *for* $g \in \hat{G}$ *and* $\phi \in \mathrm{O}(\mathfrak{u})$:

$$g \cdot \phi := i(\mathrm{Ad}_g(\phi + i) + \mathrm{Ad}_{\hat{\theta}(g)}(\phi - i))(\mathrm{Ad}_g(\phi + i) - \mathrm{Ad}_{\hat{\theta}(g)}(\phi - i))^{-1}. \tag{6}$$

In particular, $\Phi(1) = \mathfrak{u}$, *and*

$$g \cdot 1 = \frac{i\mathrm{Ad}_{g\hat{\theta}(g)^{-1}} + 1}{i + \mathrm{Ad}_{g\hat{\theta}(g)^{-1}}} \in \mathrm{O}(\mathfrak{u}), \quad \forall \mathfrak{g} \in \hat{G}. \tag{7}$$

Proof. It is easy to check that l_ϕ is in $\mathcal{I}$ for every $\phi \in \mathrm{O}(\mathfrak{u})$. Conversely, set $V_+ = (1-i)\mathfrak{u}$ and $V_- = (1+i)\mathfrak{u}$. Then $\langle , \rangle$ is respectively positive and negative definite on V_+ and V_-, and $\langle V_+, V_- \rangle = 0$. Thus if l is a maximal isotropic subspace of $\hat{\mathfrak{g}}$, we must have $l \cap V_+ = 0$ and $l \cap V_- = 0$. Hence, there exists $\phi \in \mathrm{GL}(\mathfrak{u})$ such that

$$l = \{(1+i)x + (1-i)\phi(x) : x \in \mathfrak{u}\}.$$

The fact that l is isotropic implies that $\phi \in \mathrm{O}(\mathfrak{u})$. Thus $\Phi : \mathrm{O}(\mathfrak{u}) \to \mathcal{I}$ is a bijection. Let $\mathrm{O}(\hat{\mathfrak{g}})$ be the complex orthogonal group of $\hat{\mathfrak{g}}$ defined by $\ll , \gg$. Then $\mathrm{O}(\hat{\mathfrak{g}})$ preserves $\langle , \rangle$, so $\mathrm{O}(\hat{\mathfrak{g}})$ acts on $\mathcal{I}$. It is straightforward to check that the action of $\mathrm{O}(\hat{\mathfrak{g}})$ on $\mathrm{O}(\mathfrak{u})$ obtained by the identification Φ is given by (6), with Ad_g replaced by any $T \in \mathrm{O}(\hat{\mathfrak{g}})$ and $\mathrm{Ad}_{\hat{\theta}(g)}$ by $\hat{\theta}T\hat{\theta}$. It then induces an action of $\hat{G}$ on $\mathrm{O}(\mathfrak{u})$ by the group homomorphism $\mathrm{Ad} : \hat{G} \to \mathrm{O}(\hat{\mathfrak{g}})$: $\mathfrak{g} \mapsto \mathrm{Ad}_g$. It is easy to check that action of $g \in \hat{G}$ on $1 \in \mathrm{O}(\mathfrak{u})$ is as given. □

Remark 5.2. For an integer $n \geqslant 2$, let $\ll , \gg$ be the symmetric inner product on $\mathbb{C}^n$ given by $\ll u, v \gg = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$, and let $\langle , \rangle$ be the imaginary part of $\ll , \gg$. Denote by $\mathcal{I}$ the set of all maximal isotropic subspaces of $\mathbb{C}^n$ with respect to $\langle , \rangle$. Then the complex orthogonal group $\mathrm{O}(n, \mathbb{C})$ acts on $\mathcal{I}$ since it preserves $\langle , \rangle$. On the other hand, we can identify $\mathcal{I}$ with $\mathrm{O}(n)$ as in Proposition 5.1, so we get an action of $\mathrm{O}(n, \mathbb{C})$ on $\mathrm{O}(n)$, which, one can easily check, is given by

$$\mathrm{O}(n, \mathbb{C}) \times \mathrm{O}(n) \longrightarrow \mathrm{O}(n) : (g, \phi) \mapsto \mathrm{Re}(g(\phi + i))(\mathrm{Im}(g(\phi + i)))^{-1}. \tag{8}$$

The action of $\hat{G}$ on $\mathrm{O}(\mathfrak{u})$ in (6) is then a special case of (8) if we identify $(\mathfrak{g}, \ll , \gg)$ with $(\mathbb{C}^n, \ll , \gg)$. We also remark that if P_n is the set of all matrices in $\mathrm{O}(n, \mathbb{C})$ that are Hermitian symmetric and positive definite so that $\mathrm{O}(n, \mathbb{C}) = \mathrm{O}(n) P_n$ is a Cartan decomposition of $\mathrm{O}(n, \mathbb{C})$, and if we let C be the following Cayley transform

$$C : P_n \longrightarrow \mathrm{O}(n) : C(S) = \frac{iS + 1}{i + S}, \tag{9}$$

then the action of $\mathrm{O}(n, \mathbb{C})$ on $\mathrm{O}(n)$ given by (8) is a continuous extension *via* the Cayley transform of the natural action of $\mathrm{O}(n, \mathbb{C})$ on P_n given by $(g, S) \to \mathfrak{g} \cdot S := \mathfrak{g} S \bar{\mathfrak{g}}^{-1}$ for $g \in \mathrm{O}(n, \mathbb{C})$ and $S \in P_n$.

Proof of Corollary 1.4. Corollary 1.4 follows immediately from Proposition 5.1 and Corollary 1.3. □

Remark 5.3. Recall [4, Ch. IV] that in the original definition of $\overline{X}^S_{\max}$ by Satake, one first compactifies the most basic symmetric space $\mathrm{SL}(m, \mathbb{C})/\mathrm{SU}(m)$ by embedding it into the projectivization of the space of all Hermitian symmetric m by m matrices and taking its closure therein. One then obtains $\overline{X}^S_{\max}$ by embedding X into $\mathrm{SL}(m, \mathbb{C})/\mathrm{SU}(m)$ *via* an m-dimensional projective representation of G with generic highest weight. The image of X into $\mathrm{SL}(m,\mathbb{C})/\mathrm{SU}(m)$ is a totally geodesic submanifold. Our construction of $\overline{X}^S_{\max}$ is similar. Namely, we first compactify the symmetric space $\mathrm{O}(n,\mathbb{C})/\mathrm{O}(n)$ by embedding it into $\mathrm{O}(n)$ *via* the Cayley transform and taking its closure in $\mathrm{O}(n)$. Then by identifying $\mathrm{O}(\hat{\mathfrak{g}})$ with $\mathrm{O}(n,\mathbb{C})$ as in Remark 5.2, the map

$$X = G/K \longrightarrow \mathrm{O}(n,\mathbb{C})/\mathrm{O}(n)\colon gK \mapsto \mathrm{Ad}_{g\theta(g)^{-1}}$$

is an embedding of X into $\mathrm{O}(n,\mathbb{C})/\mathrm{O}(n)$ as a totally geodesic submanifold. The compactification $\overline{X}^S_{\max}$ is then the closure of X inside the above compactification of $\mathrm{O}(n,\mathbb{C})/\mathrm{O}(n)$.

When $\mathfrak{g}$ has a complex structure, we can combine Theorem 1.1 and Proposition 5.1 to get the following characterization of $\overline{X}^S_{\max}$ using the Cayley transform and the adjoint representation of G on $\mathfrak{g}$ (without having to complexify $\mathfrak{g}$).

PROPOSITION 5.4. *Assume that* $\mathfrak{g}$ *is a* complex *semi-simple Lie algebra. Let G be the adjoint group of* $\mathfrak{g}$, *let K be a maximal compact subgroup of G, and let* θ *be the Cartan involution on G defined by K. Set*

$$v\colon X = G/K \longrightarrow \mathrm{GL}(\mathfrak{g})\colon gK \mapsto \frac{i\mathrm{Ad}_{g\theta(g)^{-1}} + 1}{i + \mathrm{Ad}_{g\theta(g)^{-1}}}.$$

Then the closure of $v(X)$ in $\mathrm{GL}(\mathfrak{g})$ *is a G-compactification of X that is G-isomorphic to the maximal Satake compactification* $\overline{X}^S_{\max}$ *of X.*

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