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Biaxial Yield for Nonlinearly Viscoelastic Materials with a Strain Clock

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Abstract. A constitutive equation with a dilatation dependent reduced time is used to model the mechanical response of solid amorphous polymers such as polycarbonate. Such constitutive equations have the property that stress relaxation occurs faster with increasing dilatation. In previous work, it has been shown that this constitutive equation can account for yield in materials undergoing uniaxial strain or stress control histories. In the present work, yield is discussed when materials described by this constitutive equation undergo homogeneous biaxial and triaxial strain histories. Four sets of conditions are considered: in-plane biaxial constant strain rate histories and in-plane biaxial constant stress rate histories, for both plane stress and plane strain states. Yield is defined in a manner analogous to that in the corresponding strain and stress control conditions in the uniaxial case.

Key words: biaxial yield, plane strain, plane stress, reduced time, stress relaxation

1. Introduction

There has been a great deal of interest in a particular class of constitutive equations for the non-linear viscoelastic response of amorphous polymers, such as polycarbonate. The dominant feature of this class of constitutive equations is a reduced time variable by means of which stress relaxation occurs faster with increasing strain. This variable defines a relation between a material time scale and the laboratory time scale, and is often referred to as a strain ‘clock’.

Substantial experimental and theoretical effort is being directed at the evaluation of constitutive equations based on the clock concept. Shay and Caruthers (1986) and Knauss and Emri (1981, 1987) investigated forms of the constitutive equation in which stress relaxation is accelerated by volumetric strain. McKenna and Zapas (1979) obtained results which suggest that shear deformations also cause stress relaxation to be accelerated.

There has been a parallel analytical effort in which the constitutive equation is used to study the interaction of the acceleration of stress relaxation with strain and the spatial variation of strain. This effort has been motivated by experiments involving non-homogeneous deformations as well as by structural applications in which polymeric materials may operate near yield. Most of the analytical work is restricted to conditions in which the strains and rotations are small while the material response is non-linear. Moran and Knauss (1992) studied the stresses near crack tips. Wineman and Waldron (1993) considered circumferential shear of a hollow cylinder. Wineman and Kolberg (1995, 1997) provided an extensive discussion of pure bending, Min (1995) extended the study to cantilever and simply supported beams under time dependent concentrated forces and Wineman and Min (1996a, 1996b) discussed spherical and cylindrical containers under internal pressure.

An important consequence of the 'strain clock' is that there may not be a monotonic increase in stress under constant strain rate conditions. Instead, the stress can reach a local maximum, decrease to a local minimum and then increase. In addition, under constant stress rate conditions, the strain will increase slowly at first, and then at a finite time, approach a very rapid rate of increase. In both cases, there is a time when the strain can begin to increase rapidly with respect to the stress. In the context of the response of polymers, this is referred to as yield. Shay and Caruthers (1986) and Knauss and Emri (1981, 1987) provided numerical simulations which showed that their constitutive equation can simulate yield in uniaxial constant strain rate tests. Wineman and Waldron (1993) presented a mathematical analysis which showed that a 'strain clock' based constitutive equation can describe yield under both uniaxial strain control and stress control histories.

Most of the studies with 'strain clock' based constitutive equations have been concerned with yield during uniaxial response. Shay and Caruthers (1987) used a 'clock' model to predict yield under multiaxial deformations, but presented very few details of either their calculations or definition of yield. Moreover, their work only discussed deformation control conditions. In the present work, the constitutive equation with a 'strain clock' used by Knauss and Emri (1981, 1987), Wineman and Waldron (1993), Wineman and Kolberg (1995, 1997) and Wineman and Min (1996a, 1996b) is used to study yield under a variety of homogeneous deformations involving triaxial extensional strain histories. In particular, yield is considered for four sets of conditions: in-plane biaxial strain control and in-plane biaxial stress control for both plane stress and plane strain states. Definitions of yield are given which are generalizations of those used in uniaxial strain control and uniaxial stress control tests. In each case, a mathematical analysis of the constitutive equations is presented which indicates the expected yield behavior, which is then illustrated with numerical simulations. The results show that yield strongly depends on which of the four sets of conditions is applied. The constitutive equation is outlined in Section 2. Section 3 contains a discussion of yield under biaxial strain control conditions for plane strain and plane stress states. Stress based yield criteria are discussed in Section 4. Section 5 contains a discussion of yield under biaxial stress control conditions for plane strain and plane stress states. Conclusions are presented in Section 6.

2. Constitutive Equation, Triaxial Deformations

The deformation is assumed to be sufficiently small that the linearized strain measure is valid. Let σ_{ij} and ε_{ij} denote components of the stress and strain tensor, respectively, with respect to a Cartesian coordinate system. The deviatoric part of the stress tensor is denoted by $\hat{\sigma}_{ij} = \sigma_{ij} - (\sigma_{kk}/3)\delta_{ij}$, δ_{ij} is the Kronecker delta, and $\sigma_{kk}/3$ denotes the hydrostatic part of the stress tensor. The deviatoric part of the strain tensor is denoted by $\hat{\varepsilon}_{ij} = \varepsilon_{ij} - (\varepsilon_{kk}/3)\delta_{ij}$, and $\varepsilon_{kk} = \theta$ is the volumetric strain or dilatation. It is assumed that $\varepsilon_{ij} = \sigma_{ij} = 0$ for times less than zero.

The material is assumed to be isotropic. $\mu(t)$ is the shear relaxation function and $K(t)$ is the bulk relaxation function for linear viscoelastic response. As in Knauss and Emri (1981, 1987), the constitutive equation used here has the form

$$\hat{\sigma}_{ij} = 2 \int_{0^-}^{t} \mu[\xi(t) - \xi(s)] \frac{\mathrm{d}\hat{\varepsilon}_{ij}(s)}{\mathrm{d}s} \,\mathrm{d}s, \tag{2.1}$$

$$\sigma_{kk} = 3 \int_{0^-}^{t} K[\xi(t) - \xi(s)] \frac{\mathrm{d}\theta(s)}{\mathrm{d}s} \,\mathrm{d}s. \tag{2.2}$$

$\xi(t)$ is defined in terms of the dilatation history $\theta(x)$, $0 \le x \le t$, by

$$\xi(t) = \int_{0}^{t} \frac{\mathrm{d}x}{a(\theta(x))} . \tag{2.3}$$

$\xi(t)$ is a new variable which introduces non-linear dependence on strain into the constitutive equation. The function $a(\theta)$ has properties similar to those of the time-temperature shift function, namely, it is a monotonically decreasing function of θ and $a(0) = 1$.

The variable $\xi(t)$ is called a material time (also referred to as pseudo, intrinsic, or reduced time) and is related by Equation (2.3) to the physical or laboratory time t. $\xi(t)$ is often described as the time according to a material clock, in this case a ‘strain clock’. The material clock can run at different speeds relative to the physical or laboratory clock in a manner which depends on the volumetric strain history. In deformation histories for which $\theta(s) = \varepsilon_{kk}(s)$ increases with time, $\xi(t)$ increases faster than physical time. This results in an acceleration of stress relaxation. Wineman and Waldron (1993) have shown that if the function $a(\theta)$ can become sufficiently small, the constitutive equation can describe yield phenomena observed in polymers in uniaxial response.

In this study, attention is restricted to strain histories in which there are only extensional strains along the coordinate axes. Two sets of conditions are considered:

(1) Biaxial strain control in which strain histories $\varepsilon_{11}(t), \varepsilon_{22}(t), t \geq 0$ are specified and stress histories $\sigma_{11}(t), \sigma_{22}(t), t \geq 0$ are studied;
(2) Biaxial stress control in which stress histories $\sigma_{11}(t)$, $\sigma_{22}(t)$, $t \geq 0$ are specified and strain histories $\varepsilon_{11}(t)$, $\varepsilon_{22}(t)$, $t \geq 0$ are studied.

In each case, $\varepsilon_{33}(t)$ and $\sigma_{33}(t)$, $t \geq 0$ are determined by specifying either plane strain or plane stress. There are no jump discontinuities in the stress or strain histories at $t = 0$.

For such deformation histories, the constitutive equation in (2.1) and (2.2) can be restated as

$$\sigma_{ii} = 2\int_0^t \mu[\xi(t) - \xi(s)]\dot{\varepsilon}_{ii}(s)\, \mathrm{d}s + \int_0^t \left(K - \frac{2}{3}\mu\right)[\xi(t) - \xi(s)]\dot{\theta}(s)\, \mathrm{d}s, \quad (2.4)$$

in which there is no summation over repeated indices and $ii = 11, 22, 33$. The notation for the coefficient of $\dot{\theta}(s)$ is defined by $(K - 2\mu/3)[x] = K[x] - 2\mu[x]/3$.

The discussion of yield in later sections will make use of an expression for the time derivative of the stress. This expression, obtained by applying Leibniz' rule Kaplan (1962)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t g(t, u)\, \mathrm{d}u = g(t, t) + \int_0^t \frac{\partial g(t, u)}{\partial t}\, \mathrm{d}u \quad (2.5)$$

to Equations (2.3) and (2.4), for $ii = 11, 22, 33$, is given by

$$\begin{aligned}\dot{\sigma}_{ii} &= 2\mu(0)\dot{\varepsilon}_{ii}(t) + \frac{2}{a(\theta(t))}\int_0^t \dot{\mu}[\xi(t) - \xi(s)]\dot{\varepsilon}_{ii}(s)\, \mathrm{d}s \\ &\quad + \left(K - \frac{2}{3}\mu\right)(0)\dot{\theta}(t) + \frac{1}{a(\theta(t))}\int_0^t \left(\dot{K} - \frac{2}{3}\dot{\mu}\right) \\ &\quad \times [\xi(t) - \xi(s)]\dot{\theta}(s)\, \mathrm{d}s. \end{aligned} \quad (2.6)$$

It is convenient for later use to rewrite Equation (2.6) in the form

$$\begin{aligned}\dot{\sigma}_{ii} &= 2\mu(0)\dot{\varepsilon}_{ii}(t) + \left[K(0) - \frac{2}{3}\mu(0)\right]\dot{\theta}(t) \\ &\quad + \frac{1}{a(\theta(t))}\left\{2\int_0^t \dot{\mu}[\xi(t) - \xi(s)]\dot{\varepsilon}_{ij}(s)\, \mathrm{d}s \right. \\ &\quad \left. + \int_0^t \left(\dot{K} - \frac{2}{3}\dot{\mu}\right)[\xi(t) - \xi(s)]\dot{\theta}(s)\, \mathrm{d}s\right\}. \end{aligned} \quad (2.7)$$

3. Yield Under Biaxial Strain Control

For uniaxial extension carried out under strain control at a constant strain rate, yield is associated with the occurrence of a local maximum in the stress-time plot, or equivalently, a local maximum in the stress-strain plot. In this section, we consider yield under biaxial strain control, for histories of the form $\varepsilon_{11}(t) = \alpha_1 t$ and $\varepsilon_{22}(t) = \alpha_2 t$, with $\alpha_1 > 0, \alpha_2 > 0$. A corresponding definition of yield would be to associate it with local maxima in both the $\sigma_{11} - t$ and $\sigma_{22} - t$ plots.

3.1. Plane Strain

For biaxial strain control under plane strain, $\varepsilon_{33}(t) = 0$ and $\theta(t) = (\alpha_1 + \alpha_2)t$. By Equation (2.4), the in-plane stresses are given by

$$\sigma_{11} = \alpha_1 \int_0^t \left(K + \frac{4}{3}\mu\right)[\xi(t) - \xi(s)]\,\mathrm{d}s + \alpha_2 \int_0^t \left(K - \frac{2}{3}\mu\right)[\xi(t) - \xi(s)]\,\mathrm{d}s \tag{3.1}$$

and

$$\sigma_{22} = \alpha_2 \int_0^t \left(K + \frac{4}{3}\mu\right)[\xi(t) - \xi(s)]\,\mathrm{d}s + \alpha_1 \int_0^t \left(K - \frac{2}{3}\mu\right)[\xi(t) - \xi(s)]\,\mathrm{d}s, \tag{3.2}$$

in which the notation $K + 4/3$ is defined by

$$\left(K + \frac{4}{3}\mu\right)[\xi(t) - \xi(s)] = K[\xi(t) - \xi(s)] + \frac{4}{3}\mu[\xi(t) - \xi(s)]. \tag{3.3}$$

The stress required to maintain the constraint $\varepsilon_{33}(t) = 0, t \geq 0$ is given by

$$\sigma_{33}(t) = (\alpha_1 + \alpha_2) \int_0^t \left(K - \frac{2}{3}\mu\right)[\xi(t) - \xi(s)\,\mathrm{d}s. \tag{3.4}$$

By Equation (2.7), the time derivatives of the in-plane stresses are

$$\dot{\sigma}_{11} = \alpha_1\left[K(0) + \frac{4}{3}\mu(0)\right] + \alpha_2\left[K(0) - \frac{2}{3}\mu(0)\right]$$

$$+ \frac{1}{a(\theta(t))} \left\{ \alpha_1 \int_0^t \left(\dot{K} + \frac{4}{3} \dot{\mu} \right) [\xi(t) - \xi(s)] \, \mathrm{d}s \right.$$

$$\left. + \alpha_2 \int_0^t \left(\dot{K} - \frac{2}{3} \dot{\mu} \right) [\xi(t) - \xi(s)] \, \mathrm{d}s \right\}, \tag{3.5}$$

$$\dot{\sigma}_{22} = \alpha_2 \left[K(0) + \frac{4}{3} \mu(0) \right] + \alpha_1 \left[K(0) - \frac{2}{3} \mu(0) \right]$$

$$+ \frac{1}{a(\theta(t))} \left\{ \alpha_2 \int_0^t \left(\dot{K} + \frac{4}{3} \dot{\mu} \right) [\xi(t) - \xi(s)] \, \mathrm{d}s \right.$$

$$\left. + \alpha_1 \int_0^t \left(\dot{K} - \frac{2}{3} \dot{\mu} \right) [\xi(t) - \xi(s)] \, \mathrm{d}s \right\}. \tag{3.6}$$

Consider first the case of equal biaxial strain histories, for which $\alpha_1 = \alpha_2$. Then, according to Equations (3.1) and (3.2),

$$\sigma_{11} = \sigma_{22} = 2\alpha_1 \int_0^t \left(K + \frac{1}{3} \mu \right) [\xi(t) - \xi(s)] \, \mathrm{d}s, \tag{3.7}$$

and by Equations (3.5) and (3.6),

$$\dot{\sigma}_{11} = \dot{\sigma}_{22}$$

$$= 2\alpha_1 \left[K(0) + \frac{1}{3} \mu(0) + \frac{1}{a(\theta(t))} \int_0^t \left(\dot{K} + \frac{1}{3} \dot{\mu} \right) [\xi(t) - \xi(s)] \, \mathrm{d}s \right]. \tag{3.8}$$

In Equation (3.8), the first two terms in the square brackets are positive. The integrand is negative since $\dot{\mu}(t) < 0$ and $\dot{K}(t) < 0$ due to shear and bulk stress relaxation. Moreover, as $\theta(t)$ increases, $a(\theta(t))$ decreases. Expressions for $a(\theta(t))$ used in ‘clock’ models (e.g. see (Knauss and Emri, 1981, 1997; Wineman and Waldron, 1993)) can decrease by several orders of magnitude. Consequently, it is possible that there is a time t^* such that $\dot{\sigma}_{11}(t^*) = 0$. As in uniaxial extension, this indicates the occurrence of yield in equal biaxial extension.

Next, consider unequal biaxial extension. Let $\alpha_2 = \alpha_1 + \hat{\alpha}$ and rewrite Equations (3.1) and (3.2) as

$$\sigma_{11} = 2\alpha_1 \int_0^t \left(K + \frac{1}{3} \mu \right) [\xi(t) - \xi(s)] \, \mathrm{d}s$$

$$+\hat{\alpha}\int_0^t \left(K - \frac{2}{3}\mu\right)[\xi(t) - \xi(s)]\,\mathrm{d}s \tag{3.9}$$

and

$$\sigma_{22} = 2\alpha_1 \int_0^t \left(K + \frac{1}{3}\mu\right)[\xi(t) - \xi(s)]\,\mathrm{d}s$$

$$+\hat{\alpha}\int_0^t \left(K + \frac{4}{3}\mu\right)[\xi(t) - \xi(s)]\,\mathrm{d}s. \tag{3.10}$$

Also, rewrite Equation (3.5) and (3.6) as

$$\dot{\sigma}_{11} = 2\alpha_1\left[K(0) + \frac{1}{3}\mu(0)\right] + \hat{\alpha}\left[K(0) - \frac{2}{3}\mu(0)\right]$$

$$+\frac{1}{a(\theta(t))}\left\{2\alpha_1\int_0^t\left(\dot{K} + \frac{1}{3}\dot{\mu}\right)[\xi(t) - \xi(s)]\,\mathrm{d}s\right.$$

$$\left.+\hat{\alpha}\int_0^t\left(\dot{K} - \frac{2}{3}\dot{\mu}\right)[\xi(t) - \xi(s)]\,\mathrm{d}s\right\} \tag{3.11}$$

$$\dot{\sigma}_{22} = 2\alpha_1\left[K(0) + \frac{1}{3}\mu(0)\right] + \hat{\alpha}\left[K(0) + \frac{4}{3}\mu(0)\right]$$

$$+\frac{1}{a(\theta(t))}\left\{2\alpha_1\int_0^t\left(\dot{K} + \frac{1}{3}\dot{\mu}\right)[\xi(t) - \xi(s)]\,\mathrm{d}s\right.$$

$$\left.+\hat{\alpha}\int_0^t\left(\dot{K} + \frac{4}{3}\dot{\mu}\right)[\xi(t) - \xi(s)]\,\mathrm{d}s\right\}. \tag{3.12}$$

Although $\dot{K} + \dot{\mu}/3 < 0$ and $\dot{K} + 4\dot{\mu}/3 < 0$, the sign of $\dot{K} - 2\dot{\mu}/3$ is uncertain. However, if $|\hat{\alpha}|$ is sufficiently small, a discussion similar to that applied to Equation (3.8) suggests that there are times t_1^* and t_2^* such that $\dot{\sigma}_{11}(t_1^*) = 0$ and $\dot{\sigma}_{22}(t_2^*) = 0$.

From Equations (3.11) and (3.12), it can be seen that

$$\dot{\sigma}_{22}(t) - \dot{\sigma}_{11}(t) = 2\hat{\alpha}\left[\mu(0) + \frac{1}{a(\theta(t))}\int_0^t \dot{\mu}[\xi(t) - \xi(s)]\,\mathrm{d}s\right]. \tag{3.13}$$

According to the above discussion the quantity in square brackets may equal zero at some time $\tilde{t}$. It is clear from Equations (3.11) and (3.12) that this time will be different from the times t_1^* and t_2^* when the individual stress rates vanish. It then follows that $\dot{\sigma}_{22} - \dot{\sigma}_{11} \neq 0$ at either t_1^* or t_2^* and hence that $\dot{\sigma}_{22}(t_1^*) \neq 0$ and $\dot{\sigma}_{11}(t_2^*) \neq 0$. In other words, a local maximum in the $\sigma_{11} - t$ plot occurs at a different time than a local maximum in the $\sigma_{22} - t$ plot.

An exception to this result can arise in the special case when the bulk relaxation function is proportional to the shear relaxation function, i.e. $K(t) = C_0\mu(t)$ for some constant C_0. Then $\dot{\sigma}_{11}(t)$ and $\dot{\sigma}_{22}(t)$ are each proportional to the expression

$$\mu(0) + \frac{1}{a(\theta(t))} \int_0^t \dot{\mu}[\xi(t) - \xi(s)]\,\mathrm{d}s. \tag{3.14}$$

If this expression vanishes at some time $\tilde{t}$, then $\dot{\sigma}_{22}(\tilde{t}) = \dot{\sigma}_{11}(\tilde{t}) = 0$.

3.2. Plane Stress

Now $\sigma_{33}(t) = 0$ and $\theta(t) = (\alpha_1 + \alpha_2)t + \varepsilon_{33}(t)$. The expressions for the in-plane stresses become

$$\begin{aligned}\sigma_{11} \;=\; & 2\alpha_1 \int_0^t \left(K + \frac{1}{3}\mu\right)[\xi(t) - \xi(s)]\,\mathrm{d}s \\ & + \hat{\alpha} \int_0^t \left(K - \frac{2}{3}\mu\right)[\xi(t) - \xi(s)]\,\mathrm{d}s \\ & + \int_0^t \left(K - \frac{2}{3}\mu\right)[\xi(t) - \xi(s)]\dot{\varepsilon}_{33}(s)\,\mathrm{d}s\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}\sigma_{22} \;=\; & 2\alpha_1 \int_0^t \left(K + \frac{1}{3}\mu\right)[\xi(t) - \xi(s)]\,\mathrm{d}s \\ & + \hat{\alpha} \int_0^t \left(K + \frac{4}{3}\mu\right)[\xi)t) - \xi(s)]\,\mathrm{d}s \\ & + \int_0^t \left(K - \frac{2}{3}\mu\right)[\xi(t) - \xi(s)]\dot{\varepsilon}_{33}(s)\,\mathrm{d}s.\end{aligned} \tag{3.16}$$

An equation for ε_{33} is obtained from Equation (2.4) with $ii = 33$ and $\sigma_{33}(t) = 0$:

$$0 = 2\int_0^t \mu[\xi(t) - \xi(s)]\dot{\varepsilon}_{33}(s)\,\mathrm{d}s + \int_0^t \left(K - \frac{2}{3}\mu\right)[\xi(t) - \xi(s)](\alpha_1 + \alpha_2 + \dot{\varepsilon}_{33}(s))\,\mathrm{d}s \tag{3.17}$$

or

$$0 = \int_0^t \left(K + \frac{4}{3}\mu\right)[\xi(t) - \xi(s)]\dot{\varepsilon}_{33}(s)\,\mathrm{d}s + (\alpha_1 + \alpha_2)\int_0^t \left(K - \frac{2}{3}\mu\right)[\xi(t) - \xi(s)]\,\mathrm{d}s. \tag{3.18}$$

It can be shown that $K(s) - 2\mu(s)/3 > 0$ for $s = 0$ and in the limit as $s \to \infty$. Let it be assumed that this inequality holds for all $0 \leq s$. Since $\alpha_1 + \alpha_2 > 0$, Equation (3.18) implies that $\dot{\varepsilon}_{33}(s) < 0, 0 \leq s$, which is consistent with physical intuition. It then follows from Equation (3.17) that $\dot{\theta}(s) = \alpha_1 + \alpha_2 + \dot{\varepsilon}_{33}(s) > 0, 0 \leq s$. This argument shows that the dilatation under plane stress increases slower than the dilatation under plane strain.

The expressions for the time derivatives of the in-plane stresses become

$$\begin{aligned}\dot{\sigma}_{11} = {} & 2\alpha_1\left[K(0) + \frac{1}{3}\mu(0)\right] + (\hat{\alpha} + \dot{\varepsilon}_{33}(t))\left[K(0) - \frac{2}{3}\mu(0)\right] \\ & + \frac{1}{a(\theta(t))}\left\{2\alpha_1\int_0^t \left(\dot{K} + \frac{1}{3}\dot{\mu}\right)[\xi(t) - \xi(s)]\,\mathrm{d}s\right. \\ & \left. + \int_0^t \left(\dot{K} - \frac{2}{3}\dot{\mu}\right)[\xi(t) - \xi(s)](\hat{\alpha} + \dot{\varepsilon}_{33}(s))\,\mathrm{d}s\right\},\end{aligned} \tag{3.19}$$

$$\begin{aligned}\dot{\sigma}_{22} = {} & 2\alpha_1\left[K(0) + \frac{1}{3}\mu(0)\right] \\ & + \hat{\alpha}\left[K(0) + \frac{4}{3}\mu(0)\right] + \dot{\varepsilon}_{33}(t)\left[K(0) - \frac{2}{3}\mu(0)\right] \\ & + \frac{1}{a(\theta(t))}\left\{2\alpha_1\int_0^t \left(\dot{K} + \frac{1}{3}\dot{\mu}\right)[\xi(t) - \xi(s)]\,\mathrm{d}s\right.\end{aligned}$$

$$+\hat{\alpha}\int_0^t\left(\dot{K}+\frac{4}{3}\dot{\mu}\right)[\xi(t)-\xi(s)]\,\mathrm{d}s$$

$$\left.+\int_0^t\left(\dot{K}-\frac{2}{3}\dot{\mu}\right)[\xi(t)-\xi(s)]\dot{\varepsilon}_{33}(s)\,\mathrm{d}s\right\}. \tag{3.20}$$

Suppose $|\hat{\alpha}|$ and $|\dot{\varepsilon}_{33}(s)|$ are sufficiently small. A discussion similar to that applied in the case of plane strain suggests that there are again times t_1' and t_2' such that $\dot{\sigma}_{11}(t_1') = 0$ and $\dot{\sigma}_{22}(t_2') = 0$. Moreover, Equation (3.13) holds for plane stress and it may again be concluded that a local maximum in the $\sigma_{11} - t$ plot occurs at a different time than a local maximum in the $\sigma_{22} - t$.

Numerical simulations were carried out to illustrate these qualitative conclusions using the following forms for the material properties:
Shear Relaxation Function

$$\mu(t) = \mu_0 + (\mu_\infty - \mu_0)\left(1 - e^{-t/a_\mu}\right), \tag{3.21}$$

Bulk Relaxation Function

$$K(t) = K_0 + (K_\infty - K_0)\left(1 - e^{-t/a_k}\right), \tag{3.22}$$

Shift Function

$$\log a = \frac{b}{2.303}\left(\frac{1}{f_0 + c\theta} - \frac{1}{f_0}\right). \tag{3.23}$$

The values of the parameters in Equations (3.21), (3.22) and (3.23) were taken from Knauss and Emri (1981, 1987). For the shear and bulk relaxation functions, $K_0 = 3350.0$ MPa, $K_\infty = 1340.0$ MPa, $\mu_0 = 635.0$ MPa, $\mu_\infty = 0.2$ MPa and $a_k/a_\mu = 20$. For the shift function $b = 0.16$, $c = 1.0$ and $f_0 = 0.01$. Note that setting $c = 0$ gives $a = 1$ and the constitutive equations reduce to those for linear viscoelasticity. Non-dimensional quantities are defined as follows: $\bar{\sigma}_{ii} = \sigma_{ii}/\mu_0, \tau = t/a_\mu, \alpha_1' = \alpha_1 a_\mu, \alpha_2' = \alpha_2 a_\mu$. The constant strain rate histories then become $\varepsilon_{11} = \alpha_1'\tau$ and $\varepsilon_{22} = \alpha_2'\tau$.

It is useful to briefly comment on the numerical method which was used to obtain the results presented below. For plane strain conditions, the stress components are directly evaluated using Equations (3.1) and (3.2). In the case of plane stress, ε_{33} is obtained by solving the non-linear Volterra integral equation (3.17) and the stress components are evaluated using Equations (3.1) and (3.2). The integrals in Equations (2.3), (3.1), (3.2) and (3.17) were approximated on a set of equally spaced times using the trapezoidal rule and the derivatives were approximated with simple finite differences. Equations (2.3), (3.1), (3.2) and (3.17) provided equations for the stresses and ε_{33} at each new time in terms of their values determined at previous times. The use of equal time increments and material properties with exponentials

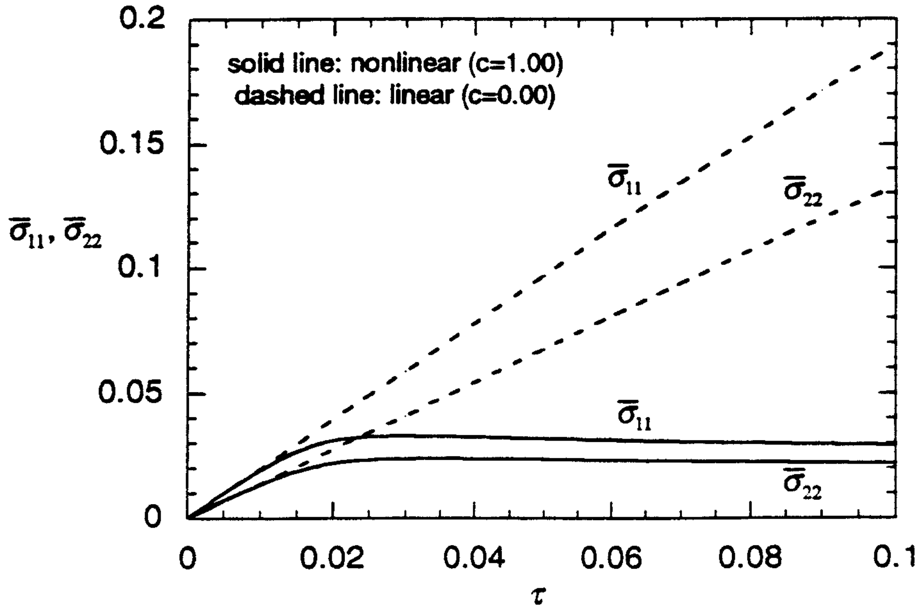

Figure 1. Histories of $\bar{\sigma}_{11}$ and $\bar{\sigma}_{22}$ corresponding to the biaxial strain history $\varepsilon_{11} = 0.5\tau$, $\varepsilon_{22} = 0.2\tau$ under plane stress conditions. Comparison for linear ($c = 0.0$) and non-linear ($c = 1.0$) viscoelastic response.

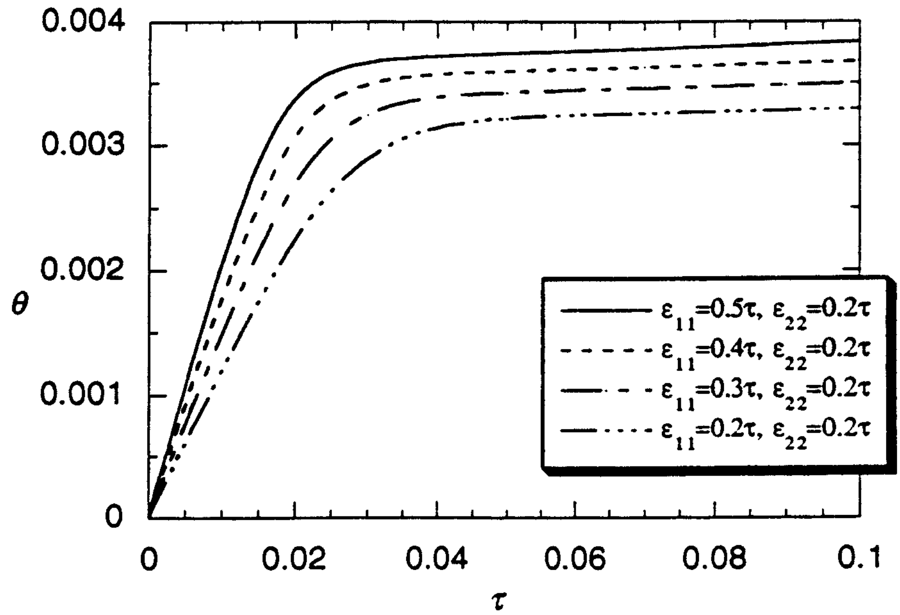

Figure 2. Dilatation histories corresponding to various biaxial constant strain rate histories, under plane stress conditions.

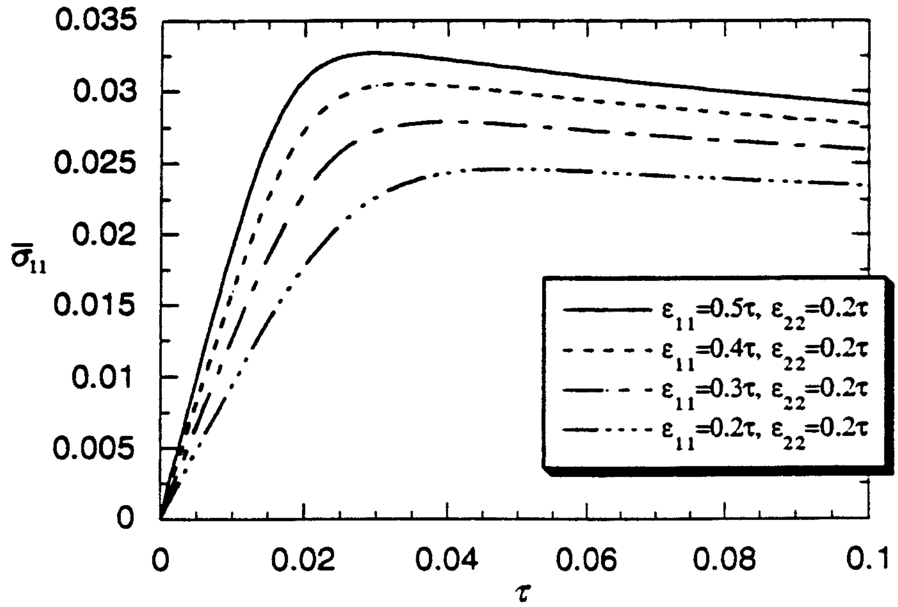

Figure 3. Histories of $\bar{\sigma}_{11}$ corresponding to various biaxial constant strain rate histories, under plane stress conditions.

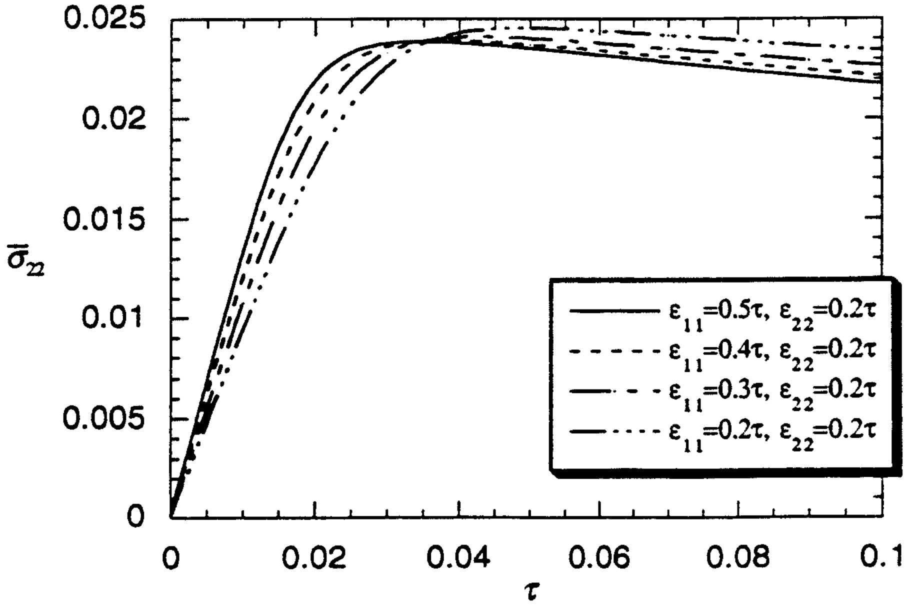

Figure 4. Histories of $\bar{\sigma}_{22}$ corresponding to various biaxial constant strain rate histories, under plane stress conditions.

enabled recurrence relations to be established which facilitated the calculation of the integrals at each new time. Calculations were carried out with a time increment $\Delta\tau = 0.00001$, which corresponded to a relative error of less than 0.5% in the stresses or ε_{33} at the largest solution time.

Figures 1, 2, 3 and 4 show results for plane stress conditions. Figure 1 compares stress histories $\bar{\sigma}_{11}$ and $\bar{\sigma}_{22}$ for both linear viscoelastic ($c = 0$) and non-linear viscoelastic response ($c = 1.0$) when $\varepsilon_{11} = 0.5\tau$ and $\varepsilon_{22} = 0.2\tau$. The stress histories coincide for very small times when the dilatation is so small that $a(\theta) \approx 1$ and $\xi(t) \approx t$. As the dilatation increases with time, the material time increases faster than physical time. The stresses continue to increase for linear viscoelastic reponse, while for non-linear viscoelastic response each component reaches a local maximum and then begins to decrease. Note that $\bar{\sigma}_{11}$ has a maximum at about $t_1^*/a_\mu = 0.03$ and $\bar{\sigma}_{22}$ has a maximum at about $t_2^*/a_\mu = 0.035$. Figure 2 shows the dilatation history for several values of strain rate α_1' and $\alpha_2' = 0.2$. After an initial rapid rise, the dilatation remains nearly constant at a value which depends on α_1' and α_2'. This occurs because K_∞/μ_∞ is very large and the material becomes nearly incompressible. Figures 3 and 4 show the histories $\bar{\sigma}_{11}$ and $\bar{\sigma}_{22}$ for the same combinations of α_1' and α_2' as in Figure 2. Note that the history of each stress component has a local maximum, and that its value and time of occurrence depend on the choice of α_1' and α_2'. The responses under plane strain and plane stress are compared in Figures 5, 6 and 7 for $\varepsilon_{11} = 0.5\tau$ and $\varepsilon_{22} = 0.2\tau$. Figure 5 shows that while the dilatation history in plane strain increases linearly according to the imposed relation $\theta = (\alpha_1' + \alpha_2')\tau$, the dilatation history in plane stress is bounded. Figures 6 and 7 show that this difference in the dilatation histories for plane strain and plane stress has a pronounced effect on the corresponding histories of stresses $\bar{\sigma}_{11}$ and $\bar{\sigma}_{22}$. Local maxima occur for both plane strain and plane stress, however the peak occurs earlier and is more distinct for plane strain. Moreover, the stress histories for plane strain reach minima and then approach a linear increase for $\tau > 0.03$. This latter increase occurs because, as the dilatation increases linearly with time, the reduced time becomes very large compared to the characteristic shear or bulk relaxation time. Stress relaxation then occurs very rapidly, and for $\tau > 0.03$ the material responds as if it were elastic with constant moduli K_∞ and μ_∞.

4. Stress Based Yield Criteria

As mentioned in Section 1, Shay and Caruthers (1986) considered a constitutive equation similar to that presented in Equations (2.1) and (2.2), but with a different definition of the function $a(\theta)$ in Equation (2.3). In (1987), they compared predictions of stress at yield in biaxial extension using their constitutive equation with the von Mises yield criterion. A definition of yield under biaxial deformations was not presented. In this section, we discuss some issues concerning such comparisons.

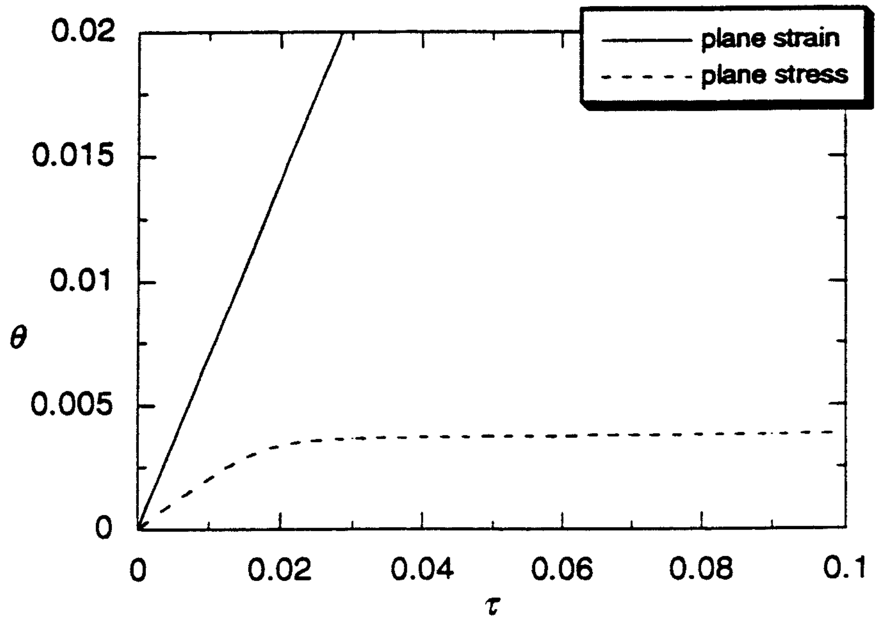

Figure 5. Dilatation histories corresponding to the biaxial strain history $\varepsilon_{11} = 0.5\tau$, $\varepsilon_{22} = 0.2\tau$. Comparison for plane strain and plane stress conditions.

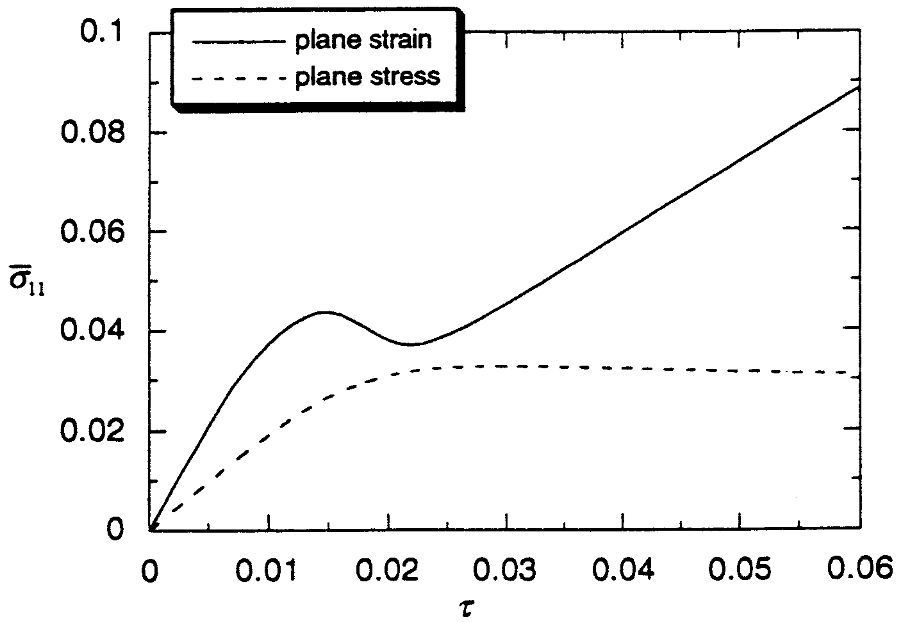

Figure 6. Histories of $\bar{\sigma}_{11}$ corresponding to the biaxial strain history $\varepsilon_{11} = 0.5\tau$, $\varepsilon_{22} = 0.2\tau$. Comparison for plane strain and plane stress conditions.

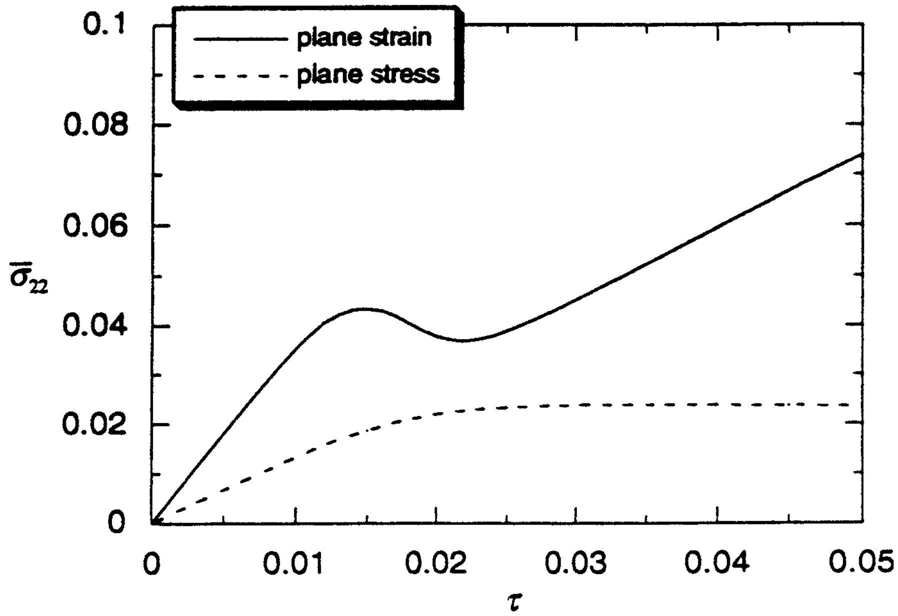

Figure 7. Histories of $\bar{\sigma}_{22}$ corresponding to the biaxial strain history $\varepsilon_{11} = 0.5\tau$, $\varepsilon_{22} = 0.2\tau$. Comparison for plane strain and plane stress conditions.

The von Mises yield criterion is defined in terms of an equivalent stress T by

$$T = [\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2]^{1/2}. \tag{4.1}$$

Consider also a generalized equivalent stress T' defined by

$$T' = [a\sigma_1^2 + b\sigma_1\sigma_2 + c\sigma_2^2]^{1/2}, \tag{4.2}$$

in which a, b, c are constants. Note that

$$T'\,\dot{T}' = a\sigma_1\dot{\sigma}_1 + b(\dot{\sigma}_1\sigma_2 + \sigma_1\dot{\sigma}_2) + c\sigma_2\dot{\sigma}_2. \tag{4.3}$$

It was shown in Section 3 that $\dot{\sigma}_1 = 0$ at some time t_1^* and $\dot{\sigma}_2 = 0$ at a different time t_2^*. Accordingly, $\dot{T} = 0$ at some time t_T which is such that $t_T \neq t_1^*$ and $t_T \neq t_2^*$. Thus, if yield is defined to occur when the von Mises equivalent stress has a local maximum, then this time will be different from the times when the individual stress components have their local maxima. Furthermore, it can be seen that a local maximum in the generalized equivalent stress T' occurs at a time which depends on the choice of constants a, b, c. Thus, if a stress based yield criterion is to be associated with the current constitutive equation, there is no unique definition of yield.

5. Yield Under Biaxial Stress Control

For uniaxial extension carried out under stress control at a constant stress rate, yield is associated with a strain-time plot which rapidly approaches a nearly vertical slope at a finite time. This corresponds to a stress-strain plot which rapidly becomes nearly horizontal at a finite stress. In this section, we consider yield under biaxial stress control, for in-plane stress histories of the form $\sigma_{11}(t) = \beta_1 t, \sigma_{22}(t) = \beta_2 t$ with $\beta_1 > 0, \beta_2 > 0$. A corresponding definition of yield would be to associate it with $\varepsilon_{11} - t$ and $\varepsilon_{22} - t$ plots which rapidly approach nearly vertical slopes at finite times.

5.1. Plane Stress

In this case, $\sigma_{33}(t) = 0$ and $\sigma_{kk}(t) = (\beta_1 + \beta_2)t$. Consider first the response in dilatation determined by Equation (2.2):

$$(\beta_1 + \beta_2)t = 3\int_0^t K[\xi(t) - \xi(s)]\dot{\theta}(s)\,\mathrm{d}s. \tag{5.1}$$

Taking the derivative of Equation (5.1) with respect to time, and rearranging terms, gives

$$K(0)\dot{\theta}(t) = (\beta_1 + \beta_2)/3 - \frac{1}{a(\theta(t))}\int_0^t \dot{K}[\xi(t) - \xi(s)]\dot{\theta}(s)\,\mathrm{d}s. \tag{5.2}$$

It is seen from Equation (5.2) that $\dot{\theta}(0) > 0$. Hence, $\dot{\theta}(t) > 0$ in some neighborhood of $t = 0$. Since $\dot{K} < 1$ because of bulk stress relaxation, the integral is negative. It follows that $\dot{\theta}(t) > \dot{\theta}(0)$ near $t = 0$. As $\theta(t)$ increases, $a(\theta(t))$ decreases and can become very small, as mentioned earlier. In this manner, $\dot{\theta}(t)$ can grow rapidly and become very large in a finite time interval.

Now, according to Equation (2.1):

$$(2\beta_1 - \beta_2)t/3 = 2\int_{0^-}^t \mu[\xi(t) - \xi(s)][\dot{\varepsilon}_{11}(s) - \dot{\theta}(s)/3]\,\mathrm{d}s, \tag{5.3}$$

$$(2\beta_2 - \beta_1)t/3 = 2\int_{0^-}^t \mu[\xi(t) - \xi(s)][\dot{\varepsilon}_{22}(s) - \dot{\theta}(s)/3]\,\mathrm{d}s. \tag{5.4}$$

Taking the derivatives of Equations (5.3) and (5.4) with respect to time, and rearranging terms, gives

$$\begin{aligned} &2\mu[0][\dot{\varepsilon}_{11}(t) - \dot{\theta}(t)/3] \\ &\quad = (2\beta_1 - \beta_2)/3 - \frac{2}{a(\theta(t))}\int_{0^-}^t \dot{\mu}[\xi(t) - \xi(s)][\dot{\varepsilon}_{11}(s) - \dot{\theta}(s)/3]\,\mathrm{d}s, \end{aligned} \tag{5.5}$$

$$2\mu[0][\dot{\varepsilon}_{22}(t) - \dot{\theta}(t)/3]$$
$$= (2\beta_2 - \beta_1)/3 - \frac{2}{a(\theta(t))} \int_{0^-}^{t} \dot{\mu}[\xi(t) - \xi(s)][\dot{\varepsilon}_{22}(s) - \dot{\theta}(s)/3]\, \mathrm{d}s. \quad (5.6)$$

Consider biaxial stress control tests for which $2\beta_1 - \beta_2 > 0$ and $2\beta_2 - \beta_1 > 0$. An argument similar to that following Equation (5.2) shows that $\dot{\varepsilon}_{11}(t) - \dot{\theta}(t)/3 > 0$ and $\dot{\varepsilon}_{22}(t) - \dot{\theta}(t)/3 > 0$ and can grow rapidly and become very large in a finite time interval. In other terms, the $\varepsilon_{11} - t$ and $\varepsilon_{22} - t$ plots can rapidly approach nearly vertical slopes at finite times. The rapid increase in slope corresponds to the rapid decrease in $a(\theta(t))$. As the factor $1/a(\theta(t))$ appears in both Equations (5.5) and (5.6), the $\varepsilon_{11} - t$ and $\varepsilon_{22} - t$ plots become nearly vertical at about the same finite time. In this case, yield is associated with a specific time. If either $2\beta_1 - \beta_2 < 0$ or $2\beta_2 - \beta_1 < 0$, a similar argument shows that $\dot{\varepsilon}_{11}(t) - \dot{\theta}(t)/3 < 0$ or $\dot{\varepsilon}_{22}(t) - \dot{\theta}(t)/3 < 0$, and the magnitude can grow rapidly and become very large in a finite time interval.

This strain response was observed by Carapellucci and Yee (1986) in experiments on yield in thin walled tubes of polycarbonate which were subjected to axial and circumferential constant stress rate histories. The stress state was equivalent to that considered here. The axial and circumferential strain versus time plots exhibited the features described above, namely, a slow increase followed by a rapid steepening so that the plots became nearly vertical at about the same time. Carapellucci and Yee observed that there is no well defined yield point on these strain-time or, equivalently, stress-strain curves. They defined yield for one of the strain-time curves by locating the intersection of the two lines drawn tangential to the initial and final slopes. This defined a time, and the stresses corresponding to this time were defined as the yield stresses.

5.2. Plane Strain

In this case, $\varepsilon_{33}(t) = 0$, $\theta(t) = \varepsilon_{11}(t) + \varepsilon_{22}(t)$ and $\sigma_{33}(t) \neq 0$. The in-plane response is given by the relations

$$\sigma_{11} = 2\int_{0}^{t} \mu[\xi(t) - \xi(s)]\dot{\varepsilon}_{11}(s)\, \mathrm{d}s + \int_{0}^{t} \left(K - \frac{2}{3}\mu\right)[\xi(t) - \xi(s)]\dot{\theta}(s)\, \mathrm{d}s \quad (5.7)$$

$$\sigma_{22} = 2\int_{0}^{t} \mu[\xi(t) - \xi(s)]\dot{\varepsilon}_{22}(s)\, \mathrm{d}s$$

$$+\int_0^t \left(K - \frac{2}{3}\mu\right)[\xi(t) - \xi(s)]\dot{\theta}(s)\,\mathrm{d}s, \tag{5.8}$$

and $\sigma_{33}(t)$ is given by

$$\sigma_{33}(t) = \int_0^t \left(K - \frac{2}{3}\mu\right)[\xi(t) - \xi(s)]\dot{\theta}(s)\,\mathrm{d}s. \tag{5.9}$$

It follows from Equations (5.7) and (5.8) that

$$\sigma_{11}(t) + \sigma_{22}(t) = (\beta_1 + \beta_2)t = 2\int_0^t \left(K + \frac{1}{3}\mu\right)[\xi(t) - \xi(s)]\dot{\theta}(s)\,\mathrm{d}s. \tag{5.10}$$

Equation (5.10) is similar in structure to Equation (5.1). By differentiating Equation (5.10) with respect to time, and recalling that $\dot{K} < 0$ and $\dot{\mu} < 0$, the same arguments used in conjunction with Equation (5.2) lead to the conclusion that $\dot{\theta}(t)$ is positive and can grow rapidly and become very large in a finite time interval.

By subtracting Equations (5.7) and (5.8), it is seen that

$$\sigma_{11}(t) - \sigma_{22}(t) = (\beta_1 - \beta_2)t = 2\int_0^t \mu[\xi(t) - \xi(s)](\dot{\varepsilon}_{11}(s) - \dot{\varepsilon}_{22}(s))\,\mathrm{d}s. \tag{5.11}$$

It can be assumed that $\beta_1 > \beta_2$ without any loss in generality. Equation (5.11) is similar in structure to Equations (5.1) and (5.10). Since $\dot{\mu} < 0$, it can be concluded that $\dot{\varepsilon}_{11} - \dot{\varepsilon}_{22}$ is positive and can grow rapidly and become very large in a finite time interval. It follows that the same conclusion applies to $\dot{\varepsilon}_{11}$. Finally, if β_2 is sufficiently close to β_1, then $\dot{\varepsilon}_{22}$ will behave in a manner similar to $\dot{\varepsilon}_{11}$. In summary, the $\varepsilon_{11} - t$ and $\varepsilon_{22} - t$ plots in plane strain have the same qualitative behavior as in plane stress and can rapidly approach vertical slopes at about the same finite time.

Numerical simulations were carried out to illustrate these qualitative conclusions using the same material properties and parameter values as in Section 3. Figure 8 shows the histories of ε_{11} and ε_{22} for both linear viscoelastic ($c = 0$) and non-linear viscoelastic response ($c = 1.0$) under plane stress with $\bar{\sigma}_{11} = 300\tau$ and $\bar{\sigma}_{22} = 200\tau$. The strain histories for linear and non-linear response are very close for short times when $\xi(t) \approx t$. When the response is non-linear, a rapid steepening in the strain-time plots begins at about $\tau = 0.8$. Calculations were terminated at $\tau = 1.0$ when the maximum strain reached a value of about 0.065. Note that these stress histories are such that $2\beta_1 - \beta_2 > 0$ and $2\beta_2 - \beta_1 > 0$ and, as shown by the analysis, the corresponding strain histories have rapidly increasing positive slopes. Figures 9, 10 and 11 show results for non-linear viscoelastic response under plane strain and plane stress conditions when $\bar{\sigma}_{11} = 300\tau$ and $\bar{\sigma}_{22} = 100\tau$. The dilatation for plane strain increases faster than for plane stress. This leads to a

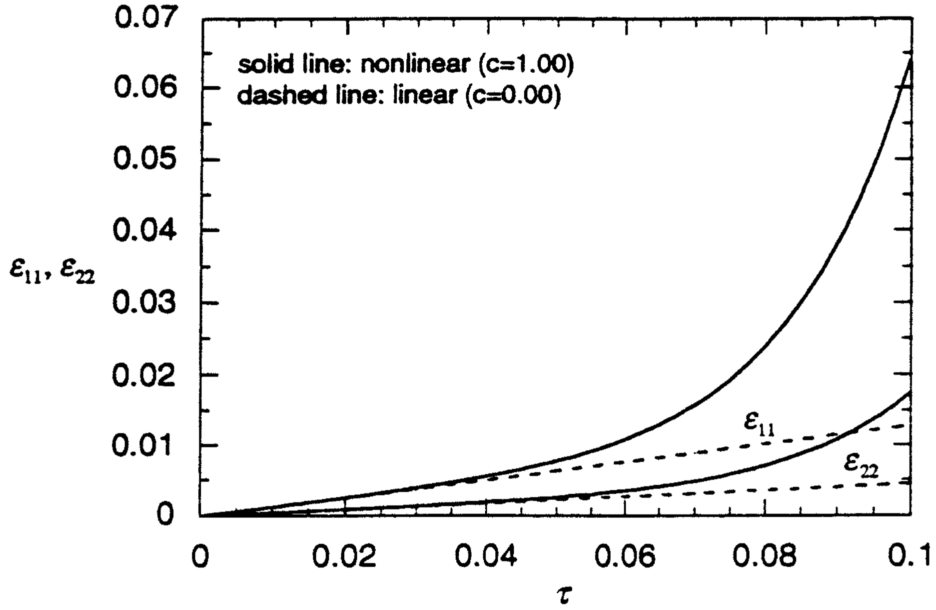

Figure 8. Histories of ε_{11} and ε_{22} corresponding to the biaxial stress history $\bar{\sigma}_{11} = 300\tau$, $\bar{\sigma}_{22} = 200\tau$ under plane stress conditions. Comparison for linear ($c = 0.0$) and non-linear ($c = 1.0$) viscoelastic response.

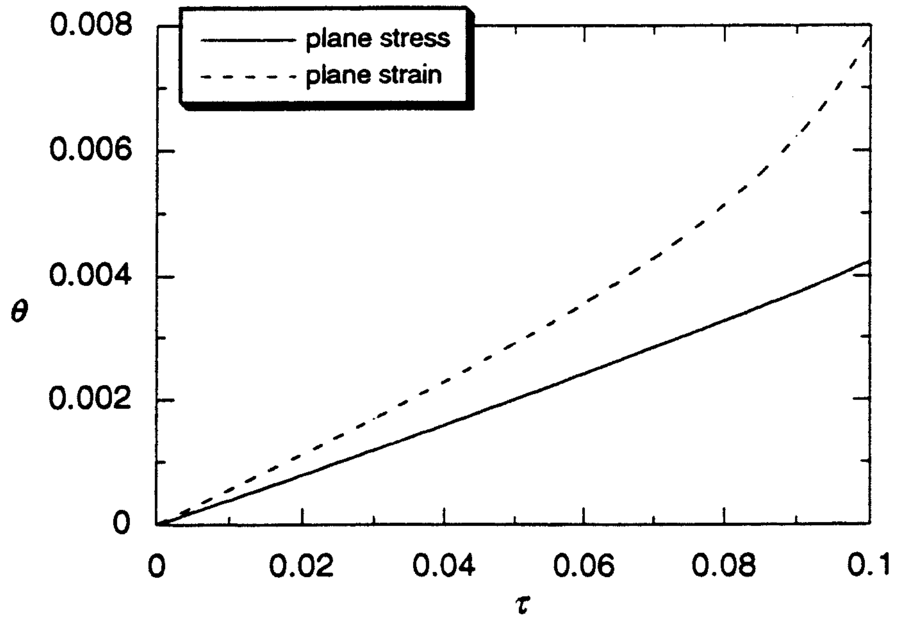

Figure 9. Dilatation histories corresponding to the biaxial stress history $\bar{\sigma}_{11} = 300\tau$, $\bar{\sigma}_{22} = 200\tau$. Comparison for plane strain and plane stress conditions.

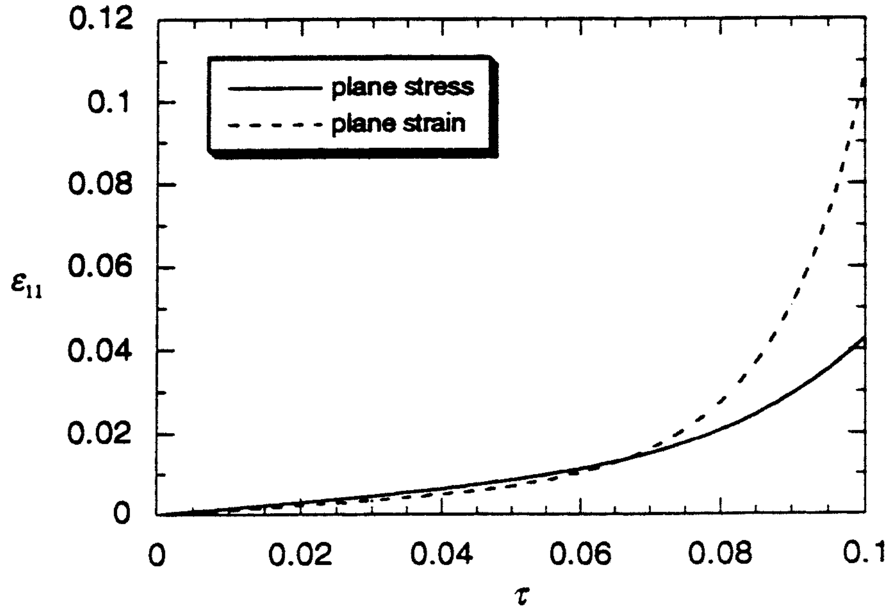

Figure 10. Histories of ε_{11} corresponding to the biaxial stress history $\bar{\sigma}_{11} = 300\tau$, $\bar{\sigma}_{22} = 100\tau$. Comparison for plane strain and plane stress conditions.

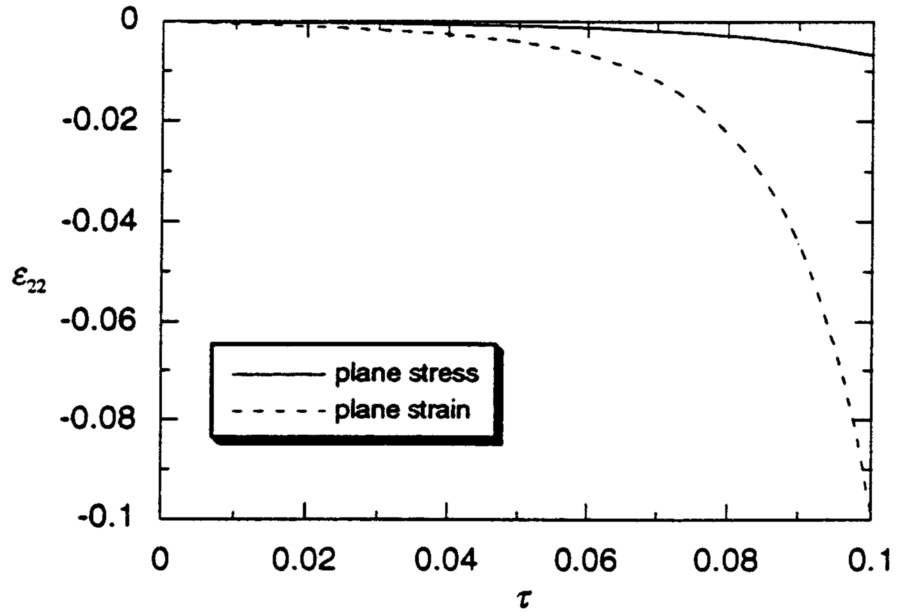

Figure 11. Histories of $\bar{\sigma}_{22}$ corresponding to the biaxial stress history $\bar{\sigma}_{11} = 300\tau$, $\bar{\sigma}_{22} = 100\tau$. Comparison for plane strain and plane stress conditions.

significant difference in the histories of ε_{11} and ε_{22} for plane strain and plane stress which is shown in Figures 10 and 11. There is a rapid steepening in the strain-time plots for plane strain at about $\tau = 0.6$ while that for plane stress appears to occur much later. Finally, note that these stress histories are such that $2\beta_1 - \beta_2 > 0$ and $2\beta_2 - \beta_1 < 0$. As indicated by the analysis, the ε_{11} *versus* time plot has a positive slope and the ε_{22} *versus* time plot has a negative slope, and the magnitudes of the slopes increase with time.

Figures 8 and 10 show that the calculated strain increases to 10% which is certainly at the limit of the linearized strain approximation. These results suggest that, in a further study of strain growth under stress control conditions, the present constitutive equation should be replaced by one which is properly frame invariant and can therefore account for finite strains.

6. Concluding Comments

The analysis and numerical results show that yield under biaxial constant strain rate conditions differs in several ways from yield under biaxial constant stress rate conditions. At yield under biaxial constant strain rate conditions, the in-plane normal stress-time plots have local maxima, but at different times. At yield under biaxial constant stress rate conditions, the in-plane extensional strain-time plots approach nearly vertical asymptotes at about the same time. In addition, these plots depend on whether plane stress or plane strain conditions apply. This dependence arises because the dilatation grows faster under plane strain conditions. In summary, yield depends on which of the four combinations of conditions is present.

The results presented here are based on specific definitions of yield. It is possible that there are other definitions of yield which should be considered. It is also possible that the above definition of yield could be satisfied for one stress or strain component, but not the other. For example, under biaxial constant strain rate conditions, a local maximum may occur in the stress-time plot for one stress component but not the other. The question then arises as to whether this should be regarded as yield. A discussion of this point is beyond the intended scope of this study.

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