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Zusammenfassung

Differenzengleichungen k-ter Ordnung ($k = 1, 2, 3, 4$), in die y und dessen Ableitungen bis zur l-ten Ordnung ($l = 1, 2, 3$) einbezogen sind, werden zur numerischen Integration der Differentialgleichung $y' = f(x, y)$, $y_0 = 0$ benützt. Die Reduktion des Abbrechfehlers erreicht man am besten, indem man eher l als k vergrössert.

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On a Mixed Boundary Value Problem for an Infinite Elastic Cone[1])

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1. Introduction

Consider an infinite right circular cone of vertex angle 2α made of an isotropic, homogeneous, elastic material. Let the vertex of the cone be at the origin of a spherical coordinate system (r, θ, φ) and let its axis lie along the polar axis $\theta = 0$. The problem is to determine the components of stress and displacement in the interior and on the boundary of the cone when the displacement is prescribed over a portion of the boundary, the remainder of the boundary being stress free. It will be assumed that the shear stress is zero on the entire boundary, there are no body forces present, and that the problem is axially symmetric. Hence the problem consists in solving the homogeneous Navier equation

$$(1 - 2\sigma)\,\nabla^2 \boldsymbol{u} + \nabla\,(\nabla \cdot \boldsymbol{u}) = 0\,, \tag{1.1}$$

for the displacement vector $\boldsymbol{u} = (u_r, u_\theta, 0)$ subject to the boundary conditions

$$\sigma_{r\theta}(r, \alpha) = 0\,, \qquad r > 0\,, \tag{1.2}$$

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$$u_\theta(r, \alpha) = g(r) , \quad 0 < r < 1 , \tag{1.3}$$

$$\sigma_{\theta\theta}(r, \alpha) = 0 , \quad r > 1 , \tag{1.4}$$

and

all stress and displacement components vanish at infinity. (1.5)

It is known [1][4]) that an axially symmetric solution of (1.1) in spherical coordinates is expressible in terms of two harmonic functions $\Phi(r, \theta)$ and $\Psi(r, \theta)$. The results are

$$u_r = \Phi_r + r \cos\theta \, \Psi_r - (2\beta - 1) \cos\theta \, \Psi , \tag{1.6}$$

$$u_\theta = \frac{1}{r} \Phi_\theta + \cos\theta \, \Psi_\theta + (2\beta - 1) \sin\theta \, \Psi , \tag{1.7}$$

where the subscripts on Φ and Ψ denote partial derivatives, and $\beta = 2(1 - \sigma)$, σ being Poisson's ratio. The non-zero stress components are then given by

$$\sigma_{rr} = 2\mu \left[\Phi_{rr} + r \cos\theta \, \Psi_{rr} - \beta \cos\theta \, \Psi_r + (2 - \beta) \frac{\sin\theta}{r} \Psi_\theta\right], \tag{1.8}$$

$$\sigma_{\theta\theta} = 2\mu \left[\frac{1}{r}\Phi_r + \frac{1}{r^2}\Phi_{\theta\theta} + (\beta - 1) \cos\theta \, \Psi_r + \beta \frac{\sin\theta}{r} \Psi_\theta + \frac{\cos\theta}{r} \Psi_{\theta\theta}\right], \tag{1.9}$$

$$\sigma_{\varphi\varphi} = 2\mu \left[\frac{1}{r}\Phi_r + \frac{\cot\theta}{r^2}\Phi_\theta + (\beta - 1) \cos\theta \, \Psi_r + \frac{1 - (\beta - 1)\sin^2\theta}{r \sin\theta} \Psi_\theta\right], \tag{1.10}$$

$$\sigma_{r\theta} = 2\mu \left[\frac{1}{r}\Phi_{r\theta} - \frac{1}{r^2}\Phi_\theta + \cos\theta \, \Psi_{r\theta} + (\beta - 1) \sin\theta \, \Psi_r - \beta \frac{\cos\theta}{r} \Psi_\theta\right], \tag{1.11}$$

where μ is the shear modulus.

2. Reduction to a Finite Wiener-Hopf Equation

We begin by defining the Mellin transforms of Φ and Ψ to be

$$\overline{\Phi}(s, \theta) = \int_0^\infty r^{s-2} \Phi(r, \theta) \, dr , \quad \overline{\Psi}(s, \theta) = \int_0^\infty r^{s-1} \Psi(r, \theta) \, dr . \tag{2.1}$$

Then by the usual process, one finds[5])

$$\overline{\Phi}'' + \cot\theta \, \overline{\Phi}' + (s - 1)(s - 2) \overline{\Phi} = 0 , \quad \overline{\Psi}'' + \cot\theta \, \overline{\Psi}' + s(s - 1) \overline{\Psi} = 0 , \tag{2.2}$$

provided the following expressions, obtained by integration by parts, vanish:

$$[r^s \Phi_r]_0^\infty , \quad [r^{s-1} \Phi]_0^\infty , \quad [r^{s+1} \Psi_r]_0^\infty , \quad \text{and} \quad [r^s \Psi]_0^\infty . \tag{2.3}$$

[4]) Numbers in brackets refer to References, page 241.

[5]) The primes denote differentiation with respect to θ.

It will be assumed that the displacement components u_r and u_θ are $0(r^{-1})$ at infinity. Then from (1.6) and (1.7) it follows that $\Phi = 0(1)$ and $\Psi = 0(r^{-1})$ as $r \to \infty$, so (2.3) all vanish at the upper limit if $\mathrm{Re}(s) < 1$. If it is also assumed that u_r and u_θ are bounded at the origin then $\Phi = 0(r)$ and $\Psi = 0(1)$ there, and (2.3) vanish at the lower limit if $\mathrm{Re}(s) > 0$.

The solutions of (2.2) which remain bounded for $\theta = 0$ are obviously

$$\bar{\Phi}(s, \theta) = A(s)\, P_{s-2}(\cos\theta)\,, \quad \bar{\Psi}(s, \theta) = B(s)\, P_{s-1}(\cos\theta)\,, \tag{2.4}$$

where $A(s)$ and $B(s)$ are to be determined from the boundary conditions (1.2)–(1.4). The requirement on the displacements in (1.5) is satisfied because of the assumed behaviour of u_r and u_θ at infinity. That the stress components also vanish at infinity follows from (1.8)–(1.11) and the order properties on Φ and Ψ established above.

To proceed from this point, it is convenient to rewrite the boundary conditions (1.3) and (1.4) as

$$u_\theta(r, \alpha) = \begin{cases} g_+(r)\,, & 0 < r < 1 \\ g_-(r)\,, & r > 1\,, \end{cases} \tag{2.5}$$

$$\sigma_{\theta\theta}(r, \alpha) = \begin{cases} 2\mu\, h_+(r)\,, & 0 < r < 1\,, \\ 0\,, & r > 1\,. \end{cases} \tag{2.6}$$

In the above equations $g_+(r)$ is, of course, the given $g(r)$, but $g_-(r)$ and $h_+(r)$ are unknown functions of r for $r > 1$ and $0 < r < 1$ respectively. The factor 2μ is inserted simply for convenience. The following order properties are assumed:

$$g_+(r) = 0(1) \quad \text{as} \quad r \to 0^+ \quad \text{and} \quad r \to 1^-\,,$$

$$g_-(r) = \begin{cases} 0(1)\,, & r \to 1^+\,, \\ 0(r^{-1})\,, & r \to \infty\,, \end{cases} \qquad h_+(r) = \begin{cases} 0(1)\,, & r \to 0^+ \\ 0(1-r)^{-1/2}\,, & r \to 1^-\,. \end{cases}$$

We now introduce the Mellin transforms

$$\left.\begin{aligned} &\bar{u}_\theta(s, \theta) = \int_0^\infty r^{s-1}\, u_\theta(r, \theta)\, dr\,, \quad \bar{\sigma}_{\theta\theta}(s, \theta) = \int_0^\infty r^s\, \sigma_{\theta\theta}(r, \theta)\, dr\,, \\ &\bar{\sigma}_{r\theta}(s, \theta) = \int_0^\infty r^s\, \sigma_{r\theta}(r, \theta)\, dr\,, \quad G_+(s) = \int_0^1 r^{s-1}\, g_+(r)\, dr\,, \\ &G_-(s) = \int_1^\infty r^{s-1}\, g_-(r)\, dr\,, \quad \text{and} \quad H_+(s) = \int_0^1 r^s\, h_+(r)\, dr\,. \end{aligned}\right\} \tag{2.7}$$

Then the boundary condition (2.5), for example, is transformed into

$$\bar{u}_\theta(s, \alpha) = G_+(s) + G_-(s) \,. \tag{2.8}$$

If one takes the transform of (1.7) according to the first of (2.7) there results

$$\bar{u}_\theta(s, \alpha) = \overline{\Phi}' + \cos\alpha\, \overline{\Psi}' + (2\,\beta - 1)\sin\alpha\, \overline{\Psi}$$

and if (2.4) are used, one finds that (2.8) becomes

$$\left.\begin{aligned} A(s)\, P^1_{s-2}(\cos\alpha) + B(s)\, [\cos\alpha\, P^1_{s-1}(\cos\alpha) + (2\,\beta - 1)\sin\alpha\, P_{s-1}(\cos\alpha)] \\ = G_+(s) + G_-(s) \,. \end{aligned}\right\} \tag{2.9}$$

In obtaining (2.9), use has been made of the fact [2, p. 63] that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\, [P_\nu(\cos\theta)] \equiv P^1_\nu(\cos\theta) \,,$$

where $P^1_\nu(\cos\theta)$ is the associated Legendre function. In a similar fashion one readily finds that the boundary conditions (1.2) and (2.6) become[6])

$$s\, A(s)\, P^1_{s-2} + B(s)\, [(s + \beta)\cos\alpha\, P^1_{s-1} + s\,(\beta - 1)\sin\alpha\, P_{s-1}] = 0 \,, \tag{2.10}$$

$$\left.\begin{aligned} A(s)\, [(s-1)^2\, P_{s-2} + \cot\alpha\, P^1_{s-2}] \\ + B(s)\, [s\,(s + \beta - 2)\cos\alpha\, P_{s-1} + (\cos\alpha\cot\alpha - \beta\sin\alpha)\, P^1_{s-1}] \\ = -H_+(s) \,. \end{aligned}\right\} \tag{2.11}$$

The desired Wiener-Hopf equation is obtained by eliminating $A(s)$ and $B(s)$ from (2.9)–(2.11). Hence from (2.10) one finds

$$A(s) = -B(s)\, \frac{(s + \beta)\cos\alpha\, P^1_{s-1} + s\,(\beta - 1)\sin\alpha\, P_{s-1}}{s\, P^1_{s-2}} \,, \tag{2.12}$$

and if one puts this relation in (2.9) there results

$$B(s) = \frac{s[G_+(s) + G_-(s)]}{\beta(\cos\alpha\, P^1_{s-1} + s\sin\alpha\, P_{s-1})} \,. \tag{2.13}$$

Finally, substitution of (2.13) and (2.12) into (2.11) gives

$$K(s)\, [G_+(s) + G_-(s)] = H_+(s) \,, \tag{2.14}$$

where

$$\left.\begin{aligned} K(s) = \left[1 + \frac{s}{\beta}\, \frac{\cot\alpha\, P^1_{s-1} - P_{s-1}}{\cot\alpha\, P^1_{s-1} + s\, P_{s-1}}\right] \left[(s-1)^2\, \frac{P_{s-2}}{P^1_{s-2}} + \cot\alpha\right] \\ - \frac{s^2\,(s + \beta - 2)\cot\alpha\, P_{s-1} + s\,(\cot^2\alpha - \beta)\, P^1_{s-1}}{\beta\,(\cot\alpha\, P^1_{s-1} + s\, P_{s-1})} \,. \end{aligned}\right\} \tag{2.15}$$

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[6]) When the argument of the Legendre functions is not indicated, it is understood to be $\cos\alpha$

Now in (2.14) $G_+(s)$ and $K(s)$ are known, whereas $G_-(s)$ and $H_+(s)$ are to be determined. If it can be shown that (2.14) holds in the strip $0 < \mathrm{Re}(s) < 1$, that $G_+(s)$ and $H_+(s)$ are regular in $\mathrm{Re}(s) > 0$, and that $G_-(s)$ is regular in $\mathrm{Re}(s) < 1$, then the determination of $G_-(s)$ and $H_+(s)$ is possible by the Wiener-Hopf technique if one can factor $K(s)$ in the form

$$K(s) = \frac{K_-(s)}{K_+(s)}, \tag{2.16}$$

in which $K_+(s)$ and $K_-(s)$ are regular and zeroless in $\mathrm{Re}(s) > 0$ and in $\mathrm{Re}(s) < 1$ respectively. From (2.15) and the asymptotic forms of the Legendre functions given in [2, p. 71] it follows that $| K(s) | = 0(| s |^p)$, where $p = 2$ unless $\alpha = \pi/2$, in which case $p = 1$. Thus it is possible [3, *p*. 42] to perform the factorization in (2.16). Hence (2.14) may be written in the form

$$K_+(s)\, H_+(s) - K_-(s)\, G_-(s) - K_-(s)\, G_+(s) = 0\,. \tag{2.17}$$

Next [3, p. 13] if $|K_-(s)\, G_+(s)| = 0(| s |^{-p})$, $p > 0$ as $| s | \to \infty$ in the strip $0 < \mathrm{Re}(s) < 1$, then the function $K_-(s)\, G_+(s) = f(s)$, say, can be decomposed into the sum $f_+(s) + f_-(s)$ in which $f_+(s)$ and $f_-(s)$ are regular in $\mathrm{Re}(s) > 0$ and $\mathrm{Re}(s) < 1$ respectively. Then (2.17) becomes

$$K_+(s)\, H_+(s) - f_+(s) = K_-(s)\, G_-(s) + f_-(s)\,. \tag{2.18}$$

Now the left side of (2.18) is regular in $\mathrm{Re}(s) > 0$, the right side in $\mathrm{Re}(s) < 1$. Since the two sides are equal in the strip $0 < \mathrm{Re}(s) < 1$, they are the analytic continuations of each other into the whole s-plane. Thus an entire function is defined, the representations of which are the two sides of (2.18) in right and left half planes. If it can be shown that each side is of algebraic growth as $| s | \to \infty$ in appropriate half planes, then by the extended form of LIOUVILLE'S theorem, the entire function must be a polynomial, say $Q(s)$. Hence one obtains from (2.18)

$$H_+(s) = \frac{Q(s) + f_+(s)}{K_+(s)} \quad \text{and} \quad G_-(s) = \frac{Q(s) - f_-(s)}{K_-(s)}\,. \tag{2.19}$$

Then from MELLIN'S inversion theorem, the functions $h_+(r)$ and $g_-(r)$ are given by

$$h_+(r) = \frac{1}{2\pi i} \int\limits_{\gamma} H_+(s)\, r^{-s-1}\, ds\,, \tag{2.20}$$

and

$$g_-(r) = \frac{1}{2\pi i} \int\limits_{\gamma} G_-(s)\, r^{-s}\, ds\,^{7)}\,. \tag{2.21}$$

[7]) In (2.20) and (2.21) and whenever it appears in the sequel, γ denotes the path $\mathrm{Re}(s) = c$ from $c - i\infty$ to $c + i\infty$, $0 < c < 1$.

To determine the complete solution one can now find $A(s)$ and $B(s)$ from (2.12) and (2.13) after which $\overline{\Phi}(s, \theta)$ and $\overline{\Psi}(s, \theta)$ are determined. Then the transforms of the stress and displacement components can be expressed in terms of $\overline{\Phi}$ and $\overline{\Psi}$ by transforming (1.6)–(1.11). Finally inversion gives the stress and displacement components themselves. Of course, the main difficulty is the determination of the factors $K_+(s)$ and $K_-(s)$, to say nothing of the integrations which arise in the inversion process.

Before considering a special case in which it is relatively easy to perform the above steps, we first show that (2.14) holds in $0 < \mathrm{Re}(s) < 1$, and verify the statements made in regard to the regions of regularity of the functions appearing in (2.14). From standard theorems [3, p. 165] on the Mellin transform and the assumed order properties on $g_+(r)$, $g_-(r)$, and $h_+(r)$, it follows that: $G_+(s)$ is regular in $\mathrm{Re}(s) > 0$, $| G_+(s) | = 0(| s |^{-1})$ as $| s | \to \infty$ in a right half plane; $G_-(s)$ is regular in $\mathrm{Re}(s) < 1$, $| G_-(s) | = 0(| s |^{-1})$ as $| s | \to \infty$ in a left half plane; and that $H_+(s)$ is regular in $\mathrm{Re}(s) > 0$, $|H_+(s)| = 0(|s|^{-1/2})$ as $|s| \to \infty$ in a right half plane. Finally it follows from (2.15) that $K(s)$ is regular in $0 < \mathrm{Re}(s) < 1$ if it can be shown that the factors $\cot\alpha\, P^1_{s-1} + s\, P_{s-1}$ and P^1_{s-2} have no zeros in this strip. We have been unable to prove this for an arbitrary α, but for the special case to be considered in the next section, $K(s)$ reduces to a function which is known to be regular and zeroless in $0 < \mathrm{Re}(s) < 1$.

3. Indentation of an Elastic Half Space by a Rigid Flat Ended Circular Punch

As an application of the above results we consider the elastic half space $0 \leqq \theta \leqq \pi/2$ on whose boundary, $\theta = \alpha = \pi/2$ the following are prescribed:

$$\left.\begin{aligned} \sigma_{r\theta}\left(r, \frac{\pi}{2}\right) = 0\,, \quad r > 0; \quad u_\theta\left(r, \frac{\pi}{2}\right) = -\varepsilon\,, \quad 0 < r < 1; \\ \sigma_{\theta\theta}\left(r, \frac{\pi}{2}\right) = 0\,, \quad r > 1\,. \end{aligned}\right\} \tag{3.1}$$

The solution of this problem is well known [4, p. 458]. It is usually formulated and solved in cylindrical coordinates, but its solution by the above method serves as a check on the method and, of course, if one desires the stress and displacement components in terms of the spherical coordinates, the present method will give them directly. To proceed then, we see from the second of (3.1) that $g_+(r) = -\varepsilon$ and then from the third of (2.7), one readily finds that

$$G_+(s) = -\varepsilon \int_0^1 r^{s-1}\, dr = -\frac{\varepsilon}{s}\,, \quad \mathrm{R}\,(s) > 0\,. \tag{3.2}$$

When $\alpha = \pi/2$ the expression (2.5) for $K(s)$ simplifies considerably. This is due to the fact that the argument of the Legendre functions is now zero and it is known that [2, p. 63]

$$P_\nu^\mu(0) = \frac{2^\mu \sqrt{\pi}}{\Gamma\left(\frac{\nu-\mu+2}{2}\right)\Gamma\left(\frac{-\nu-\mu+1}{2}\right)}. \tag{3.3}$$

Hence if one uses (3.3) with $\mu = 0, 1$ and makes use of the elementary relations for the Gamma function, after some algebraic manipulation it follows that for $\alpha = \pi/2$

$$K(s) = \frac{2}{\beta} \frac{\Gamma\left(\frac{2-s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1-s}{2}\right)}. \tag{3.4}$$

In the form (3.4) it is obvious that $K(s)$ is regular and zeroless in $0 < \mathrm{Re}(s) < 1$ and moreover the factors $K_+(s)$ and $K_-(s)$ can be determined by inspection. They are

$$K_+(s) = \beta \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1+s}{2}\right)}, \quad K_-(s) = 2 \frac{\Gamma\left(\frac{2-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}, \tag{3.5}$$

so that $K_+(s)$ and $K_-(s)$ are regular and zeroless in $\mathrm{Re}(s) > 0$ and $\mathrm{Re}(s) < 1$ respectively. Furthermore $K_+(s) = 0(s^{-1/2})$ and $K_-(s) = 0(s^{1/2})$ as $|s| \to \infty$ in appropriate half planes. Hence (2.17) becomes

$$K_+(s)\, H_+(s) - K_-(s)\, G_-(s) + f(s) = 0, \tag{3.6}$$

where

$$f(s) = \frac{2\varepsilon}{s} \frac{\Gamma\left(\frac{2-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}.$$

We must now decompose $f(s)$ into $f_+(s) + f_-(s)$ so that $f_+(s)$ and $f_-(s)$ are regular in $\mathrm{Re}(s) > 0$ and $\mathrm{Re}(s) < 1$ respectively. This is easily done by CAUCHY's theorem and the results are

$$f_+(s) = 2\varepsilon \frac{1}{2\pi i} \int_{\gamma_1} \frac{\Gamma\left(\frac{2-t}{2}\right)}{t\,\Gamma\left(\frac{1-t}{2}\right)} \frac{dt}{t-s},$$

$$f_-(s) = 2\varepsilon \frac{1}{2\pi i} \int_{\gamma_2} \frac{\Gamma\left(\frac{2-t}{2}\right)}{t\,\Gamma\left(\frac{1-t}{2}\right)} \frac{dt}{t-s},$$

where γ_1 and γ_2 are the paths $\mathrm{Re}(t) = a$ from $a + i\infty$ to $a - i\infty$ and $\mathrm{Re}(t) = b$

from $b - i\infty$ to $b + i\infty$ respectively with $0 < a < \mathrm{Re}(s) < b < 1$. We close the contours for both integrations in the left half plane since the Gamma functions are regular there. There is no contribution from the closing path since the absolute value of the integrand is $0(|t|^{-3/2})$. Hence we obtain

$$f_+(s) = \frac{2\varepsilon}{\sqrt{\pi}\, s}, \quad f_-(s) = \frac{2\varepsilon}{s}\left[\frac{\Gamma\left(\frac{2-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} - \frac{1}{\sqrt{\pi}}\right], \tag{3.7}$$

It is obvious that $f_+(s)$ is regular for $\mathrm{Re}(s) > 0$, but $f_-(s)$ appears to have a pole at $s = 0$. However, it is easy to show that $\lim\limits_{s\to 0} f_-(s) = -\varepsilon \ln 4/\sqrt{\pi}$. Thus the Wiener-Hopf equation is

$$K_+(s)\, H_+(s) + \frac{2\varepsilon}{\sqrt{\pi}\, s} = K_-(s)\, G_-(s) - \frac{2\varepsilon}{s}\left[\frac{\Gamma\left(\frac{2-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} - \frac{1}{\sqrt{\pi}}\right], \tag{3.8}$$

in which the left side is regular in $\mathrm{Re}(s) > 0$; the right side in $\mathrm{Re}(s) < 1$. The two sides are equal in the common strip so they are the analytic continuations of each other into the whole s-plane. From the order properties established in the second section, we see that the left side is $0(|s|^{-1})$ and the right side is $0(|s|^{-1/2})$ in their half planes or regularity. Hence the polynomial $Q(s)$ is identically zero and we obtain

$$H_+(s) = -\frac{2\varepsilon}{\sqrt{\pi}\, s\, K_+(s)} = -\frac{\varepsilon}{\sqrt{\pi}\,\beta}\,\frac{\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(\frac{2+s}{2}\right)}, \tag{3.9}$$

$$G_-(s) = \frac{2\varepsilon}{s\, K_-(s)}\left[\frac{\Gamma\left(\frac{2-s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} - \frac{1}{\sqrt{\pi}}\right] = \frac{\varepsilon}{s}\left[1 - \frac{\Gamma\left(\frac{1-s}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{2-s}{2}\right)}\right]. \tag{3.10}$$

Then from (3.9) and MELLIN's inversion theorem we have

$$h_+(r) = -\frac{\varepsilon}{\sqrt{\pi}\,\beta}\,\frac{1}{2\pi i}\int_\gamma \frac{\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(\frac{2+s}{2}\right)}\, r^{-s-1}\, ds\,, \quad 0 < r < 1\,. \tag{3.11}$$

To evaluate this integral we note that since $r < 1$, we must close the path in the left half plane if the contribution of the closing path is to vanish. The integrand then has simple poles at $s = -(2n+1)$, $n = 0, 1, 2, \ldots$; the residue at the n^{th} pole being

$$\frac{2\,\Gamma(n+1/2)}{\pi\,n!}\,r^{2n}\,. \tag{3.12}$$

With (3.12) and (2.6) we find

$$\sigma_{\theta\theta}\left(r,\frac{\pi}{2}\right) = 2\,\mu\,h_+(r) = -\frac{4\,\mu\,\varepsilon}{\pi\,\beta}\sum_{n=0}^{\infty}\frac{\Gamma(n+1/2)}{\sqrt{\pi}\,n!}\,r^{2n}$$

$$= -\frac{4\,\mu\,\varepsilon}{\pi\,\beta}\,(1-r^2)^{-1/2}\,,\quad 0<r<1\,.$$

Hence

$$\sigma_{\theta\theta}\left(r,\frac{\pi}{2}\right) = \begin{cases} -\dfrac{4\,\mu\,\varepsilon}{\pi\,\beta}\,(1-r^2)^{-1/2}\,, & 0<r<1 \\ 0 & ,\quad r>1\,. \end{cases} \tag{3.13}$$

Next from (3.10) and (2.5) we get

$$u_\theta\left(r,\frac{\pi}{2}\right) = g_-(r) = \varepsilon\,\frac{1}{2\,\pi\,i}\int_\gamma\left[1-\frac{\Gamma\left(\frac{1-s}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{2-s}{2}\right)}\right]r^{-s}\,\frac{ds}{s}\,,\quad r>1\,. \tag{3.14}$$

Here $r>1$ so we must close the path in the right half plane and in this region the integrand of the first integral in (3.14) is regular. Hence we have

$$u_\theta\left(r,\frac{\pi}{2}\right) = -\frac{\varepsilon}{\sqrt{\pi}}\,\frac{1}{2\,\pi\,i}\int_\gamma\frac{\Gamma\left(\frac{1-s}{2}\right)}{s\,\Gamma\left(\frac{2-s}{2}\right)}\,r^{-s}\,ds\,. \tag{3.15}$$

The integrand in (3.15) has simple poles at $s=2\,n+1$, $n=0,1,2,\ldots$, and the residue at the n^{th} pole is

$$-\frac{2}{\pi}\,\frac{\Gamma(n+1/2)}{(2\,n+1)\,n!}\,r^{-2n-1}\,.$$

Since the integration is clockwise, the integral is the negative of the sum of its residues and we have

$$u_\theta\left(r,\frac{\pi}{2}\right) = -\frac{2\,\varepsilon}{\pi}\sum_{n=0}^{\infty}\frac{\Gamma(n+1/2)}{\sqrt{\pi}\,(2\,n+1)}\,\frac{r^{-2n-1}}{n!} = -\frac{2\,\varepsilon}{\pi}\,\sin^{-1}\frac{1}{r}\,,\quad r>1\,.$$

Thus

$$u_\theta\left(r,\frac{\pi}{2}\right) = \begin{cases} -\varepsilon & ,\quad 0<r<1 \\ -\dfrac{2\,\varepsilon}{\pi}\,\sin^{-1}\dfrac{1}{r}\,, & r>1\,. \end{cases} \tag{3.16}$$

The results (3.13) and (3.16) were obtained without explicit knowledge of $A(s)$ and $B(s)$; but to determine the rest of the solution one must find these functions. From (2.12) and (2.13) it is a simple matter to obtain

$$A(s) = \frac{\varepsilon(\beta - 1)}{\beta} \frac{\sec\frac{\pi s}{2}}{s(s-1)}, \quad B(s) = -\frac{\varepsilon}{\beta} \frac{\sec\frac{\pi s}{2}}{s}.$$

Then as an example let us determine $u_r(r, 0)$. Define

$$\bar{u}_r(s, \theta) = \int_0^\infty r^{s-1} u_r(r, \theta)\, dr.$$

Then from (1.6) and (2.4) it follows that

$$\bar{u}_r(s, \theta) = -(s-1) A(s) P_{s-2}(\cos\theta) - (s + 2\beta - 1) B(s) \cos\theta\, P_{s-1}(\cos\theta).$$

Then since $P_\nu(1) = 1$ we have

$$\bar{u}_r(s, 0) = \frac{\varepsilon}{\beta} \left(\frac{\beta + s}{s}\right) \sec\frac{\pi s}{2},$$

and from the inversion theorem

$$u_r(r, 0) = \frac{\varepsilon}{\beta} \frac{1}{2\pi i} \int_\gamma \frac{\beta + s}{s} \frac{r^{-s}}{\cos\frac{\pi s}{2}}\, ds. \tag{3.17}$$

To evaluate (3.17) we complete the contour in the left or right half plane according as $r < 1$ or $r > 1$; the result is the same in both cases and we have

$$u_r(r, 0) = \varepsilon \left(1 - \frac{2}{\pi} \tan^{-1} r + \frac{2}{\beta\pi} \frac{r}{1 + r^2}\right). \tag{3.18}$$

In a similar fashion one can determine the remaining stress and displacement components on the boundary $\theta = \pi/2$ and on the axis $\theta = 0$, and the results agree with those listed in [4]. The result (3.18) is not listed there but it is probably well known. Finally, it is remarked that the solution for interior points offers no serious mathematical difficulties; however, the results are given as infinite series. Since the present special case is considered primarily as an example of the method, the results for interior points are not included.

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Zusammenfassung

Die Verfasser haben ein gemischtes Randwertproblem für einen unendlichen elastischen Kegel mit Spitzenwinkel 2α auf die Lösung einer endlichen Wiener-Hopf-Gleichung zurückgeführt. Der wesentliche Schritt besteht darin, eine Funktion $K(s)$, die in einem Streifen regulär ist, als ein Produkt von Faktoren darzustellen, die auf übereinanderliegenden Halbebenen regulär sind. In dem hier behandelten Fall für willkürliches α ist diese Methode besonders schwierig. In dem Sonderfalle des elastischen Halbraumes $\alpha = \pi/2$ können jedoch bekannte Resultate erlangt werden. Das deutet darauf hin, dass die Methode auf das Kegelproblem anwendbar ist, vorausgesetzt dass die erforderliche Faktorzerlegung durchgeführt werden kann.

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Magneto-Fluid Dynamics of Thin Bodies in Oblique Fields (II)[1]

By KEITH STEWARTSON, Durham, Great Britain[2])

1. Introduction

The motion of an electrically conducting fluid past a fixed body in the presence of a magnetic field has attracted a good deal of interest in recent years. Restricting attention to inviscid compressible fluids perhaps the earliest work [5][3]) was concerned with the motion of a conducting fluid past a perfectly conducting sphere. This work revealed an apparent non-uniqueness in the steady state solution which was removed by considering how the motion was set up from an initial state of rest. Subsequently SEARS and RESLER [4] presented a two-dimensional theory for a thin body in a perfectly conducting fluid when the magnetic field is either perpendicular (crossed field) or parallel (aligned field) to the direction of motion of the fluid. McCUNE [2] generalised the first of these to include a fluid of finite conductivity σ concentrating his attention on the flow properties when σ is large. He was able to show how the interesting and novel features found by SEARS and RESLER, for crossed fields, developed as $\sigma \to \infty$.

[1]) Part I see ZAMP *12*, 261 (1961).
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[3]) Numbers in brackets refer to References, page 255.