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The Method of Averaging for Euler's Equations of Rigid Body Motion

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Abstract. We formulate the method of averaging for perturbations of Euler's equations of rotational motion. Euler's equations are three strongly nonlinear coupled differential equations that can be viewed as a three dimensional oscillator. The method of averaging is used to determine the long-term influence of perturbation terms on the motion by averaging about the nominal rigid body motion. The treatment is applicable to a large class of motions including precession with large nutation – it is not restricted to small motions about simple spins or nearly axi-symmetric bodies. Three examples are shown that demonstrate the accuracy of the method's predictions.

Key words: Spacecraft attitude motion, method of averaging, elliptic functions.

1. Introduction

The method of averaging is a perturbation method applicable for systems whose nominal motion consists of oscillatory solutions. Typically, the method is applied to a perturbed two-dimensional oscillator or to coupled pairs of oscillators. The most widely known example is the perturbed harmonic oscillator

$$\ddot{x} + x = \varepsilon g(x, \dot{x}, t, \varepsilon), \tag{1}$$

where g represents the perturbation and $\varepsilon \ll 1$ is a scaling parameter. The method of averaging provides an approximation of the solution to the perturbed equations by averaging out short-period fluctuations while retaining long-term information. In the case of the harmonic oscillator, the method provides information on the change of the amplitude of motion and its frequency.

We formulate the method of averaging for the three strongly nonlinear equations represented by perturbations of Euler's equations of rotational motion. This is possible because the nominal (unperturbed) motion consists of three oscillatory motions. However, rather than the more familiar trig functions that naturally appear in applying averaging to Equation (1), the computations involve differentiation and integration of elliptic functions. An introduction to the method of averaging using elliptic functions is found in [3, 5, 6, 7].

The motivation for investigating Euler's equations stems from their use in modeling the motion of spinning spacecraft. References concerning spinning spacecraft motion are too numerous to list – see [11, 12, 16] for an introduction. The literature contains many references employing the method of averaging for axi-symmetric spacecraft or for asymmetric spacecraft undergoing small coning motions. In both of these cases, the equations of motion reduce to a perturbed linear oscillator of the form Equation (1) that is treated using trig functions.

Averaging of spacecraft motion using elliptic functions has been largely avoided, although a few examples exist. Chernous'ko [2] investigates the motion of an asymmetric rigid body perturbed by orientation-dependent gravity gradient torques. Hall and Rand [10] investigate the spinup dynamics of a dualspin spacecraft.

2. Formulation

Consider the equations of motion for a spinning vehicle consisting of a base rigid body $\mathcal{R}$ and movable appendages $\mathcal{W}$. The rotational equations of motion are

$$\boldsymbol{I} \cdot \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} + \boldsymbol{\omega} \times (\boldsymbol{I} \cdot \boldsymbol{\omega}) = \boldsymbol{M}(\boldsymbol{\omega}, \dot{\boldsymbol{\omega}}), \tag{2}$$

where $\boldsymbol{I}$ is the mass moment of inertia tensor of $\mathcal{R} + \mathcal{W}$ about its center of mass, $\boldsymbol{\omega}$ is the inertial angular velocity of $\mathcal{R}$, and $\boldsymbol{M}$ is composed of the external moment on $\mathcal{R} + \mathcal{W}$ and internal moments caused by the motion of $\mathcal{W}$ relative to $\mathcal{R}$. We shall restrict this investigation by not considering the orientation-dependent environmental torques (e.g., gravity or drag).

By resolving Equation (2) onto body axes (x_1, x_2, x_3) fixed in $\mathcal{R}$ and simplifying, we find:

$$\dot{\omega}_1 = \quad J_1 \omega_2 \omega_3 + \varepsilon g_1(\boldsymbol{\omega}, \dot{\boldsymbol{\omega}}), \tag{3a}$$

$$\dot{\omega}_2 = -J_2 \omega_3 \omega_1 + \varepsilon g_2(\boldsymbol{\omega}, \dot{\boldsymbol{\omega}}), \tag{3b}$$

$$\dot{\omega}_3 = \quad J_3 \omega_1 \omega_2 + \varepsilon g_3(\boldsymbol{\omega}, \dot{\boldsymbol{\omega}}), \tag{3c}$$

where

$$J_1 = \frac{I_2 - I_3}{I_1}\,, \quad J_2 = -\frac{I_3 - I_1}{I_2}\,, \quad J_3 = \frac{I_1 - I_2}{I_3}\,. \tag{4}$$

I_n denotes a diagonal element of the inertia tensor $\boldsymbol{I}$ and g_n denotes terms arising from the moments and all the off-diagonal inertia terms, scaled by a parameter ε.

2.1. Unperturbed Motion

The unperturbed system consists of Equation (3) with $\varepsilon = 0$ (the torque-free rigid body equations). The system is fully integrable, with two constants of motion: the angular momentum vector $\boldsymbol{L}$ and the rotational kinetic energy scalar $T = (\boldsymbol{\omega} \cdot \boldsymbol{L})/2$. Resolving onto the body axes, we find

$$T = \frac{1}{2}\,(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2), \tag{5a}$$

$$L^2 = I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + I_3^2 \omega_3^2. \tag{5b}$$

These equations describe two intersecting ellipsoids in the three-dimensional space $(\omega_1, \omega_2, \omega_3)$. Thus, the unperturbed solution is a motion specified by T lying on a two-dimensional ellipsoid defined by L^2 in $\boldsymbol{\omega}$-space. Tangential intersections give rise to six equilibrium points, a pair lying on each axis ω_n, that correspond to simple spin motions, i.e., motions in which $\boldsymbol{\omega}$ and $\boldsymbol{L}$ are aligned. One pair of equilibria are unstable and are connected by four heteroclinic connections (i.e., separatrices) while the other two pair are (linearly) stable. Nontangential intersections give rise to one-dimensional closed curves lying on the ellipsoid that encircle a stable equlibrium and accumulate onto a pair of separatrices. Each of these curves correspond to a complicated orientation manuever involving coning, nutation, spin, and precession. See Hughes [11] or Wittenburg [18] for an illustration. Finally, since no external torques act in the unperturbed system, $\boldsymbol{L}$ remains inertially fixed.

Knowing $\boldsymbol{\omega}$ does not completely determine the orientation of $\mathcal{R}$ with respect to inertial space; however, the orientation relative to $\boldsymbol{L}$ is determined. Define the unit vector $\boldsymbol{\ell}$ by

$$\boldsymbol{\ell} = (\ell_1, \ell_2, \ell_3) \triangleq \frac{1}{L}\,(I_1\omega_1, I_2\omega_2, I_3\omega_3), \tag{6}$$

where ℓ_n are direction cosines. Then, the direction of $\boldsymbol{\ell}$ relative to the body axes is specified by (θ, ψ) where

$$\ell_1 = \sin\psi\sin\theta, \tag{7a}$$

$$\ell_2 = \cos\psi\sin\theta, \tag{7b}$$

$$\ell_3 = \cos\theta. \tag{7c}$$

The angle θ is the coning angle and ψ is the spin angle. These angles are two rotations for Euler angles; the third rotation (i.e., the precession about $\boldsymbol{L}$) cannot be found from $\boldsymbol{\omega}$ alone.

Using the direction cosines, Equation (5) becomes

$$\frac{1}{I} = \frac{\ell_1^2}{I_1} + \frac{\ell_2^2}{I_2} + \frac{\ell_3^2}{I_3}\,, \tag{8a}$$

$$1 = \ell_1^2 + \ell_2^2 + \ell_3^2, \tag{8b}$$

where $I = L^2/(2T)$ is called the momentum parameter [11]. For each value of I,[1] the solution to Equation (8a) consists of two disjoint one-dimensional curves lying on the momentum sphere described by Equation (8b). The two curves are merely reflections of one another under the symmetry $(\ell_1, \ell_2, \ell_3) \to (-\ell_1, -\ell_2, -\ell_3)$. Thus, the unperturbed motion traverses one of the two curves: initial conditions determine which one.

2.2. Unperturbed Solution

The unperturbed solution for a general asymmetric rigid body is well known [11, 17, 18]. It is given by

$$\omega_1 = s_1\,R_1\,\mathrm{cn}(u, k), \tag{9a}$$

$$\omega_2 = s_2\,R_2\,\mathrm{sn}(u, k), \tag{9b}$$

$$\omega_3 = s_3\,R_3\,\mathrm{dn}(u, k), \tag{9c}$$

where $s_n = \pm 1$ and $s_1 s_2 s_3 = -1$. The amplitudes R_n are given by

$$R_1 = \Gamma_{13} k\Omega, \quad R_2 = \Gamma_{23} k\Omega, \quad R_3 = \Omega, \tag{10}$$

where $k \geq 0$, $\Omega \geq 0$, and $\Gamma_{nk} = \sqrt[+]{J_n/J_k}$. The argument u, linear in time, is given by

$$u = \Gamma_{13}\Gamma_{23}J_3\,\Omega t + u_0. \tag{11}$$

[1] Except for degenerate cases when $I = I_1$, $I = I_2$, or $I = I_3$.

Table 1. Convention for determining the 1 and 3-axes. I is evaluated at the initial condition.

	I_1	I_3
$I > I_2$	I_{min}	I_{max}
$I < I_2$	I_{max}	I_{min}

Ω, k, and u_0 are the three constants of integration that are related to the initial conditions. The functions $\text{cn}(u,k)$, $\text{sn}(u,k)$, and $\text{dn}(u,k)$ are called Jacobian elliptic functions. These functions are periodic in the argument u for each fixed modulus k with period[2] $4K(k)$ when $0 \leq k^2 < 1$, where $K(k)$ is the complete elliptic integral of the first kind [1].

The unperturbed solution for $\boldsymbol{\ell}$ is

$$\ell_1 = s_1\, r_1\, \text{cn}(u,k), \tag{12a}$$

$$\ell_2 = s_2\, r_2\, \text{sn}(u,k), \tag{12b}$$

$$\ell_3 = s_3\, r_3\, \text{dn}(u,k), \tag{12c}$$

where $r_n = (I_n R_n / L)$. These amplitudes are related according to

$$r_1 = \gamma_{13}\, k r_3, \quad r_2 = \gamma_{23}\, k r_3, \tag{13}$$

where

$$\gamma_{nk} = \frac{I_n}{I_k}\, \Gamma_{nk} = \sqrt[+]{\frac{j_n}{j_k}}\,, \tag{14}$$

with

$$j_1 = I_1(I_2 - I_3), \quad j_2 = I_2(I_1 - I_3), \quad j_3 = I_3(I_1 - I_2). \tag{15}$$

The amplitudes r_n depend on k but not Ω.

Solution (9–11) holds whenever I_2 denotes the middle axis of inertia, i.e., $I_{\text{max}} > I_2 > I_{\text{min}}$. We shall not consider cases for which $I_2 \approx I_{\text{min}}$ or $I_2 \approx I_{\text{max}}$ since Equation (3) then reduces to a perturbed harmonic oscillator that can be investigated using standard perturbation methods.

The solution is most easily interpreted by adopting the following convention. The momentum parameter I is known to satisfy $I_{\text{max}} \geq I \geq I_{\text{min}}$ for unperturbed motion [11]. The directions of the 1-axis and 3-axis are now chosen according to Table 1.

We shall not consider cases in which I is approximately I_2 because the motion for $\boldsymbol{\omega}$ lies near a separatrix in the phase space where averaging is not valid.

Adopting the convention listed in Table 1, all inertia parameters J_n and j_n are of the same sign. Moreover, the x_3-axis undergoes coning and nutation while precessing about $\boldsymbol{L}$. The modulus k is a measure of the coning and nutation motion (i.e., the oscillation of θ during the motion). In particular, $\theta_{\text{min}} \leq \theta \leq \theta_{\text{max}}$ where

$$\theta_{\text{min}} = \cos^{-1}\left(\sqrt{\frac{1}{1+\gamma_{13}^2 k^2}}\right), \tag{16a}$$

[2] While the smallest period for $\text{dn}(u,k)$ is $2K(k)$, it is still periodic with period $4K(k)$.

$$\theta_{\max} = \cos^{-1}\left(\sqrt{\frac{1-k^2}{1+\gamma_{13}^2 k^2}}\right). \tag{16b}$$

The constants of motion are expressed in terms of (Ω, k) according to

$$T = \frac{1}{2}\, I_3\Omega^2\,(1+\gamma_{13}\Gamma_{13}k^2), \tag{17a}$$

$$L = I_3\Omega\,\sqrt[+]{1+\gamma_{13}^2 k^2}\,. \tag{17b}$$

Thus, k depends only on I according to

$$k^2 = \frac{(I_1-I_2)}{(I_2-I_3)}\frac{(I-I_3)}{(I_1-I)}\,, \tag{18}$$

so that its value also specifies closed curves on the momentum sphere. From Equation (18), we also find that

$$0 \le k^2 < 1$$

implying that ω_1, ω_2, ω_3, and θ are all periodic functions of time (as expected). Both ω_3 and θ have nonzero mean value while ω_1 and ω_2 have zero mean value.

2.3. Method of Averaging

A brief summary of the method of averaging is described below. Consider the system of differential equations for $x \in R^2$ and $\phi \in S^1$ below:

$$\dot{x} = \varepsilon F(x,\phi), \tag{19a}$$

$$\dot{\phi} = \Lambda(x) + \varepsilon G(x,\phi), \tag{19b}$$

where $0 < \varepsilon \ll 1$ is a small parameter, $F \in R^2$ and G are both periodic in ϕ for fixed x, and $\Lambda \gg \varepsilon$. Define a near-identity transformation by

$$x = x' + \varepsilon W(x',\phi') + O(\varepsilon^2), \tag{20a}$$

$$\phi = \phi' + \varepsilon V(x',\phi') + O(\varepsilon^2). \tag{20b}$$

The transformation functions $W \in R^2$ and V are chosen to transform the differential equations for x' and ϕ' into

$$\dot{x}' = \varepsilon\,\bar{F}(x') + O(\varepsilon^2), \tag{21a}$$

$$\dot{\phi}' = \Lambda(x') + \varepsilon\,\bar{G}(x') + O(\varepsilon^2), \tag{21b}$$

where $\bar{F}$ and $\bar{G}$ denote the averages over ϕ' of $F(x',\phi')$ and $G(x',\phi')$ respectively, where

$$\underset{\phi'}{\mathrm{avg}}\langle f(x',\phi')\rangle = \frac{1}{T(x')}\int_0^{T(x')} f(x',\phi')\,\mathrm{d}\phi' \tag{22}$$

and $T(x')$ denotes the period. See [3, 8, 9, 14] for a more complete discussion.

Note that in the averaged system (21), the differential equations for x' (the slow-varying variables) are decoupled from ϕ' (the fast variable): the slow-varying variables evolve on a manifold that has been reduced by one dimension. Theorems on the averaging method guarantee that the solution for x' is an $O(\varepsilon)$ approximation for x for times t of $O(1/\varepsilon)$ [8, 9, 14]. Moreover, many qualities of the motion for x can be determined from investigating x'.

2.4. Variational Equations

Three conditions must be satisfied so that εg_n represents a small perturbation term in Equation (3). First, the moment $\|\boldsymbol{M}\|$ must be small relative to T so that the nominal motion consists of rigid body motion. Hence, define ε as $(\|\boldsymbol{M}\|/T)$ evaluated at the initial condition. Second, the body axes must be nominally principal axes, i.e.,

$$\boldsymbol{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} + \varepsilon \begin{bmatrix} 0 & I_{12}^s & I_{13}^s \\ I_{12}^s & 0 & I_{23}^s \\ I_{13}^s & I_{23}^s & 0 \end{bmatrix}, \tag{23}$$

where the off-diagonal elements remain of $O(\varepsilon)$. Obviously, the body axes may be chosen to be principal axes at all times, but such axes may be inconvenient to use if shape changes of the body occur (caused by $\mathcal{W}$ moving relative to $\mathcal{R}$). Third, the rate at which the inertia tensor $\boldsymbol{I}$ changes must be slow, i.e., $\boldsymbol{I} \triangleq \boldsymbol{I}(\tau)$ where $\tau = \varepsilon t$.

We now identify a set of slowly-varying variables and a fast varying phase angle to put Equation (3) into the form of Equation (19). Obviously, both T and L^2 are slowly-varying but we prefer to use $x = (\Omega, k)$ (the equation for τ need not be explicitly computed). Since the unperturbed system is strongly nonlinear, only certain phase angles will result in periodic variational equations [3, 4, 5]. A convenient choice is $\phi = u/(4K(k))$.

Variation of parameters is now applied by assuming a solution of the form given by (9–11) with (Ω, k, u_0) now considered as unknown functions of time. After lengthy computations (computed using an elliptic function processor written for Mathematica[3]), we find

$$\dot{\Omega} = \varepsilon F_\Omega(\Omega, k, \tau, \phi), \tag{24a}$$

$$\dot{k} = \varepsilon F_k(\Omega, k, \tau, \phi), \tag{24b}$$

$$\dot{\phi} = \Lambda(\Omega, k, \tau) + \varepsilon G(\Omega, k, \tau, \phi), \tag{24c}$$

where the functions F_Ω, F_k, G, and Λ are given in Table 2.

3. Examples

The averaged system (21) is computed from the variational equations given by (24) once the perturbation terms $g_n(\boldsymbol{\omega}, \dot{\boldsymbol{\omega}})$ are specified. Three examples are considered below.

3.1. Internal Moving Parts

Consider the case where the the external moment on the body is zero, but appendages are allowed to move and thrusters are allowed to fire [13]. The variables g_n corresponding to a

[3] Available from the author. E-mail request to coppola@caen.engin.umich.edu

Table 2. Explicit forms of the variational equations (24). Each g_n is to be evaluated on the unperturbed solution given by (9–11). $Z(u,k)$ denotes the Jacobian Zeta function (related to the incomplete elliptic integral of the second kind [1]). The arguments (u,k) have been suppressed in writing the elliptic functions.

$$
\begin{aligned}
F_\Omega(\Omega,k,\tau,\phi) &= s_3 g_3 \mathrm{dn} + \frac{s_2 g_2}{\Gamma_{23}} k\mathrm{sn} - \frac{1}{\Gamma_{23}}\frac{\mathrm{d}\Gamma_{23}}{\mathrm{d}\tau} k^2 \Omega \mathrm{sn}^2 \\
F_k(\Omega,k,\tau,\phi) &= \frac{s_1 g_1}{\Gamma_{13}\Omega}\,\mathrm{cn} + \frac{s_2 g_2}{\Gamma_{23}\Omega}(1-k^2)\mathrm{sn} - \frac{s_3 g_3}{\Omega} k\mathrm{dn} - \frac{1}{\Gamma_{13}}\frac{\mathrm{d}\Gamma_{13}}{\mathrm{d}\tau} k\mathrm{cn}^2 - \frac{1}{\Gamma_{23}}\frac{\mathrm{d}\Gamma_{23}}{\mathrm{d}\tau} k(1-k^2)\mathrm{sn}^2 \\
G(\Omega,k,\tau,\phi) &= \frac{1}{4K}\left\{ \frac{s_1 g_1(\mathrm{Zcn}-\mathrm{sndn})}{\Gamma_{13}k\Omega(1-k^2)} + \frac{s_2 g_2(\mathrm{Zsn}+\mathrm{cndn})}{\Gamma_{23}k\Omega} - \frac{s_3 g_3(\mathrm{Zdn}-k^2\mathrm{sncn})}{\Omega(1-k^2)} \right. \\
&\qquad \left. - \frac{\mathrm{cn}}{\Gamma_{13}(1-k^2)}\frac{\mathrm{d}\Gamma_{13}}{\mathrm{d}\tau}(\mathrm{Zcn}-\mathrm{sndn}) - \frac{\mathrm{sn}}{\Gamma_{23}}\frac{d\Gamma_{23}}{d\tau}(\mathrm{Zsn}+\mathrm{cndn}) \right\} \\
\Lambda(\Omega,k,\tau) &= \frac{\Gamma_{13}\,\Gamma_{23}\,J_3\,\Omega}{4K}
\end{aligned}
$$

body with internal moving parts are written

$$
\begin{aligned}
g_1 &= G_{10} + G_{11}\omega_1 + G_{12}\omega_2 + G_{13}\omega_3 \\
&\quad + \frac{I_{23}^s}{I_1}\left(\omega_3^2 - \omega_2^2\right) - \frac{I_{13}^s}{I_1}\left(\omega_1\omega_2 + \dot{\omega}_3\right) \\
&\quad + \frac{I_{12}^s}{I_1}\left(\omega_1\omega_3 - \dot{\omega}_2\right),
\end{aligned}
\tag{25}
$$

where g_2 and g_3 can be found using cyclic permutation. G_{mn} depends on $\boldsymbol{I}$ and the motion of $\mathcal{W}$ relative to $\mathcal{R}$ but not on $\boldsymbol{\omega}$. The three parameters of the form G_{nn} are given by

$$
G_{nn} = -\frac{1}{I_n}\frac{\mathrm{d}I_n}{\mathrm{d}\tau} + \frac{1}{I_n}\sum_i \frac{\mathrm{d}m_i}{\mathrm{d}\tau}\, d_{ni}^2 \tag{26}
$$

where d_{ni} measures the perpendicular distance from the x_n-axis to mass thruster m_i which is ejecting mass at the rate $\mathrm{d}m_i/\mathrm{d}\tau$. The three parameters G_{n0} are related to the external torque on $\mathcal{R}+\mathcal{W}$, torques on $\mathcal{R}$ caused by $\mathcal{W}$, and torques created by thruster firings. These parameters figure prominently in the averaged system.

3.2. Thrust Torques

Perturbations caused by small thrust torques are also described by Equation (25). However, in many applications, the amount of mass lost during the firing is negligible compared to the inertia of the vehicle. In such cases, the inertia may be approximated as constant so that the perturbation becomes $g_n = G_{n0}$.

3.3. Linear Feedback

Consider the model of a control system for a rigid spacecraft given by Equation (3) where εg_n represents a control torque and (x_1, x_2, x_3) are principal axes. The analysis for linear

Table 3. Values of the averages appearing in the averaged equations (27). The prime notation has been dropped for convenience. $K = K(k)$ and $E = E(k)$ are complete elliptic integrals.

$$\bar{F}_\Omega(\Omega, k; N_0, N_2, N_3) = \frac{\pi}{2K} N_0 + \Omega \left\{ N_2 \left(1 - \frac{E}{K}\right) + N_3 \frac{E}{K} \right\}$$

$$\bar{F}_k(\Omega, k; N_0, N_1, N_2, N_3) = -\frac{\pi k}{2K\Omega} N_0 + \frac{N_2 - N_1}{k} \left(1 - k^2 - \frac{E}{K}\right) + (N_2 - N_3)\, k \frac{E}{K}$$

$$\bar{G}(k; G_{12}, G_{21}, s_3\Gamma_{12}) = -\frac{\pi}{8k^2K^2} \left\{ s_3\Gamma_{12}G_{21} \left(1 - \frac{E}{K}\right) + \frac{G_{12}}{s_3\Gamma_{12}} \left(1 - \frac{1}{1-k^2}\frac{E}{K}\right) \right\}$$

where

$$N_0 = s_3 G_{30}, \quad N_1 = G_{11} - \frac{1}{\Gamma_{13}}\frac{\mathrm{d}\Gamma_{13}}{\mathrm{d}\tau}, \quad N_2 = G_{22} - \frac{1}{\Gamma_{23}}\frac{\mathrm{d}\Gamma_{23}}{\mathrm{d}\tau}, \quad N_3 = G_{33}$$

feedback control torques can be recovered from Equation (25) by taking $G_{n0} = 0$, $I^s_{nm} = 0$, and each G_{nm} constant. Note that the linear terms here are considered as perturbations on the nonlinear unperturbed system, in contrast to traditional perturbation approaches that treat nonlinear terms as perturbations on a linear system.

4. Averaging Results

The averaged system (21) corresponding to Equation (25) is found to be

$$\dot{\Omega} = \varepsilon \bar{F}_\Omega(\Omega, k; N_0, N_2, N_3), \tag{27a}$$

$$\dot{k} = \varepsilon \bar{F}_k(\Omega, k; N_0, N_1, N_2, N_3), \tag{27b}$$

$$\dot{\phi} = \Lambda(\Omega, k, \tau) + \varepsilon \bar{G}(k; G_{12}, G_{21}, s_3\Gamma_{12}), \tag{27c}$$

where the values of the averages $\bar{F}$ and $\bar{G}$ are given in Table 3. The prime notation has been dropped for convenience. The averaged system depends on only seven parameters (out of a total of 18) with the slow-flow equations depending only on the four parameters denoted N_n. These four parameters determine the qualitative behavior of the motion – the other parameters do not significantly affect the long-term motion. Note in particular that the off-diagonal elements I^s_{ij} do not appear in the averaged system.

Additionally, one sees that if $N_0 = 0$, the slow-flow equations themselves decouple with Equation (27b) being a first order nonlinear differential equation in k and possibly τ. The motion for Ω is then found from quadrature of Equation (27a).

4.1. Internal Motions

When the perturbation term consists only of internal motions (not external torques or thruster firings), then the total angular momentum of $\mathcal{R} + \mathcal{W}$ is conserved. If, in addition, the internal motions do not generate any internal angular momentum, then $G_{n0} = 0$ and $\boldsymbol{L}$ is conserved both in Equation (3) and in the averaged system.

Equation (27b) becomes

$$\dot{k} = \varepsilon \frac{1}{k} \left\{ \alpha\, k^2 \frac{E}{K} + \beta \left(1 - k^2 - \frac{E}{K} \right) \right\}, \tag{28}$$

where

$$\alpha = \frac{1}{\gamma_{32}} \frac{\mathrm{d}\gamma_{32}}{\mathrm{d}\tau}\,, \quad \beta = \frac{1}{\gamma_{12}} \frac{\mathrm{d}\gamma_{12}}{\mathrm{d}\tau}\,. \tag{29}$$

To find an approximate solution, we first compute the equation for r_1, i.e.,

$$\frac{\dot{r}_1}{r_1} = \varepsilon \frac{\beta}{k^2} \left(1 - \frac{E}{K} \right), \tag{30}$$

and approximate the right-hand side by performing a Taylor series in k about $k = 0$. To lowest order, the right-hand side becomes $\varepsilon\beta/2$ making Equation (30) integrable. The solution to the approximate equation is

$$r_1 = r_0 \sqrt{\gamma_{12}} = r_0 \sqrt[4]{\frac{j_1}{j_2}}\,, \tag{31}$$

where r_0 is an arbitrary constant. Approximate solutions for the remaining slowly-varying quantities are given below:

$$r_2 = \frac{r_0}{\sqrt{\gamma_{12}}}\,, \tag{32a}$$

$$r_3 = \sqrt{1 - r_0^2 \gamma_{12}}\,, \tag{32b}$$

$$k^2 = \frac{r_0^2 \gamma_{31} \gamma_{32}}{1 - r_0^2 \gamma_{12}}\,, \tag{32c}$$

$$\Omega = \frac{L}{I_3} \sqrt{1 - r_0^2 \gamma_{12}}\,, \tag{32d}$$

$$T = \frac{L^2}{2I_3} \left(1 - r_0^2 (\gamma_{12} - \gamma_{32} \Gamma_{13}) \right), \tag{32e}$$

and $R_n = (L r_n / I_n)$. Thus, the averaging method provides an *explicit algebraic* approximation to the motion of a vehicle with internal motions. This result was first developed for sufficiently small coning motions in [15] using another technique.

Figure 1 compares the actual motion for Equation (3) with the averaged solution for the case of a mass deploying along the end of a boom along the x_3-axis. The moments of inertia are given by

$$\begin{aligned} I_1 &= 10 + ms^2, \\ I_2 &= 11 + ms^2, \\ I_3 &= 20, \end{aligned}$$

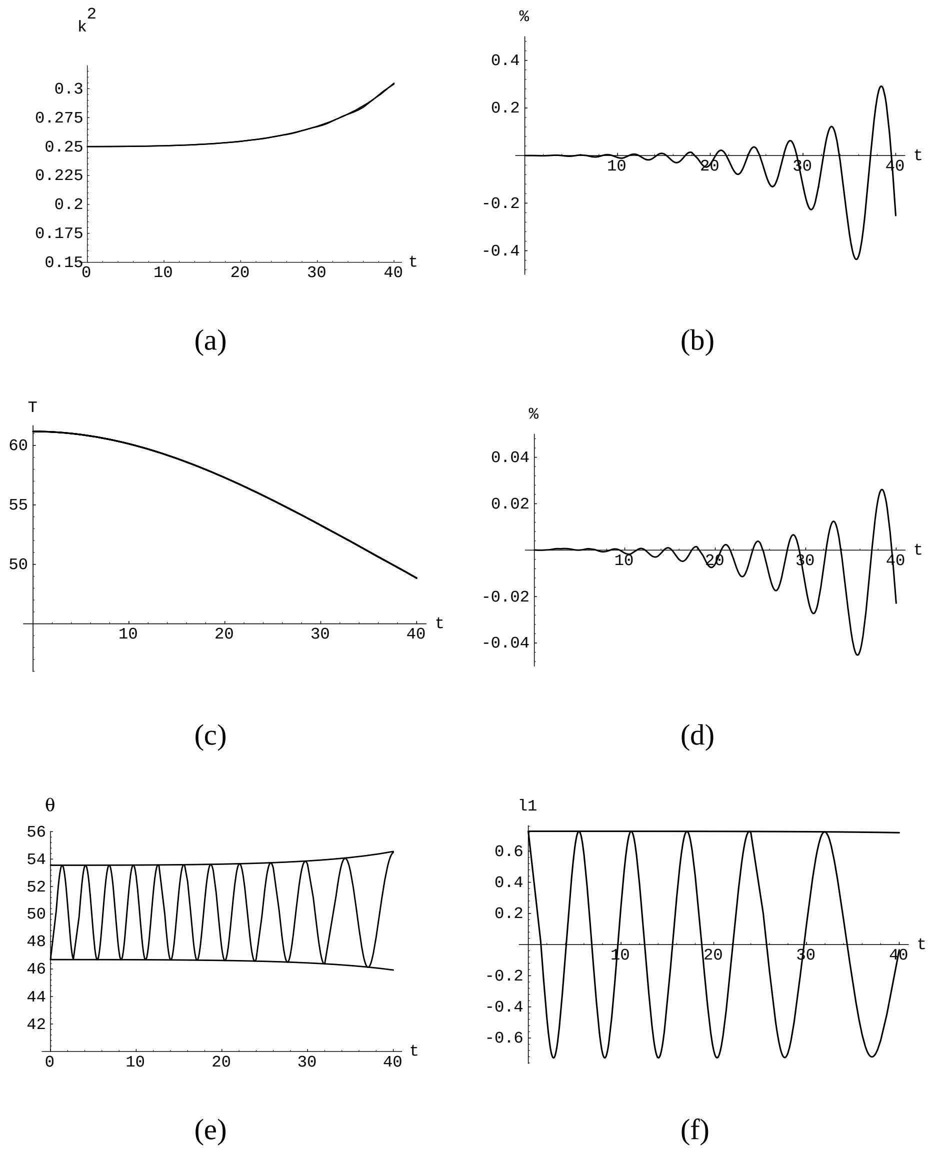
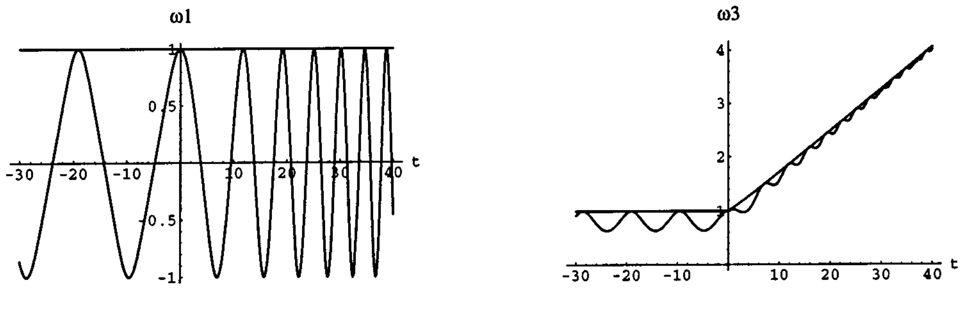


Figure 1. Boom Deployment. $\varepsilon = 0.1$. (a) Plot of k^2 for both actual and averaged solution. (b) Relative %-error between actual and averaged solution for k^2. (c) Plot of T for both actual and averaged solution. (d) Relative %-error between actual and averaged solution for T. (e) Plot of actual θ (in deg) and the averaged solutions for $\theta_{\min}$ and $\theta_{\max}$. (f) Plot of actual ℓ_1 and the averaged solution for r_1.

where $m = 0.04$, $\dot{s} = 0.25$, $s(0) = 0$, and $\varepsilon = 0.1$. The angular momentum is $L = 40$ and the initial value of k^2 is 0.25. Values for k^2, Ω, θ, and r_1 were computed for the actual motion of Equation (3) by computing T and L from Equation (5) and using the formulas given in Section 2.1. It is seen that the averaging solution is an excellent approximation to the actual motion.



(a) (b)

Figure 2. Thruster firing. $\varepsilon = 0.1$. Before $t = 0$ the thruster is off and the body is performing purely rigid body motion. At $t = 0$, the thruster is turned on resulting in an increase of Ω and decrease of k. The oscillations are the result of numerical integration of (3) while the lines are the predicted amplitudes based on the averaging approximation.

4.2. Thruster Firing

For the thruster firing example, the averaged system becomes

$$\dot{\Omega} = \varepsilon \frac{\pi}{2K} N_0, \tag{33a}$$

$$\dot{k} = -\varepsilon \frac{\pi k}{2K\Omega} N_0. \tag{33b}$$

Only the torque about the x_3-axis leads to any significant long-term motion. Equation (33) has the integral of motion $k\Omega$ that remains constant during the motion. Recalling Equation (10), we see that torque does not change the amplitudes of ω_1 or ω_2. Using the constant of motion, Equation (33b) becomes

$$\dot{k} = -\varepsilon \frac{\pi N_0}{2k_0\Omega_0} \frac{k^2}{K}, \tag{34}$$

where (Ω_0, k_0) denote initial conditions. When $N_0 > 0$, $k = 0$ is an attractive equilibrium: the torque serves to reduce k until a simple spin about the x_3-axis is achieved; however, the spin rate Ω increases. When $N_0 < 0$, $k = 0$ is an unstable equilibrium: the torque serves to increase the nutation amplitude while reducing Ω. The coning increases until the motion nears the separatrix (at $k = 1$) at which point the averaging solution is no longer valid.

Figure 2 compares the motion of Equation (3) with the averaged solution based on Equation (34) for the case where $I_1 = 3$, $I_2 = 4$, $I_3 = 5$, $\varepsilon = 0.1$, $G_{30} \neq 0$, and all other G_{mn} are zero. For $t < 0$, $G_{30} = 0$ and the motion is purely rigid body motion described by $k = 0.6$ and $\Omega = 1.0$. At $t = 0$ the thruster is turned on with $G_{30} = 0.8$. As predicted, the amplitude of w_1 remains fixed while the amplitude of w_3 grows. Note also that the period of oscillation decreases.

4.3. Limit Cycles Produced by Feedback

In the linear feedback example, all inertias are constant and $G_{30} = 0$, so that the slow-flow equation (27b) is given by Equation (28) where $\alpha \triangleq N_2 - N_3$ and $\beta \triangleq N_2 - N_1$ are both constant parameters. A nontrivial equilibrium $\hat{k}$ of Equation (28) exists if

$$\beta = \frac{\hat{k}^2 \hat{E}}{\hat{E} - (1 - \hat{k}^2)\hat{K}}\, \alpha, \tag{35}$$

where $\hat{E} = E(\hat{k})$ and $\hat{K} = K(\hat{k})$. This condition is bounded by the lines $\beta = \alpha$ and $\beta = 2\alpha$ in the (α, β) parameter plane. The equilibrium is stable if $\alpha > 0$ and unstable if $\alpha < 0$.

Considering now the implications for the momentum sphere, we conclude that the nontrivial equilibrium $k = \hat{k}$ corresponds to a *limit cycle* on the momentum sphere with the same stability properties. The limit cycle exists independent of Ω: in fact, Ω may increase or decrease with no change in the limit cycle shape.

The limit cycle bifurcates from a particular unperturbed trajectory (depending on $\hat{k}$) as ε passes through zero. The bifurcation is comparable to the well-known birth of a limit cycle in the van der Pol oscillator wherein a particular unperturbed trajectory becomes a limit cycle as $\varepsilon \uparrow 0$. (The bifurcation is sometimes called an Andronov bifurcation.) The limit cycle trajectory, then, is very nearly a particular coning motion of the x_3-axis of $\mathcal{R}$ of the unperturbed motion. Thus, Ω only affects the rate at which the limit cycle is traversed (i.e., the spin rate of the motion).

The solution for Ω can be found from quadrature, once the solution to Equation (28) is determined. Note, however, that even on the limit cycle described by $k = \hat{k}$, Ω need not remain constant. In fact, $d\Omega/dt = 0$ only when

$$N_2 = \left(\frac{\hat{E} - (1 - \hat{k}^2)\hat{K}}{\hat{E} - \hat{K}} \right) N_1, \tag{36a}$$

$$N_3 = \left(\frac{\hat{E} - (1 - \hat{k}^2)\hat{K}}{\hat{E}} \right) N_1. \tag{36b}$$

Even when Equation (36) is satisfied, a constant value of Ω is not obtained until the motion for $k(t)$ reaches $k = \hat{k}$. The final value of Ω cannot be found without performing the quadrature (which in turn depends on the initial value of k). Hence, Equation (36) does not imply the existence of a limit cycle in $\boldsymbol{\omega}$-space. Moreover, since the analysis is really for the primed (averaged) variables, Equation (36) really implies that $\mathrm{d}\Omega/\mathrm{d}t = O(\varepsilon^2)$, not 0, resulting in a slowly-varying (rather than constant) value of Ω.

Figure 3 compares the motion of Equation (3) with that predicted by averaged solution at $k = \hat{k}$ for the case where $I_1 = 3$, $I_2 = 4$, $I_3 = 5$, $G_{11} = -1.35614$, $G_{22} = 1.26209$, $G_{33} = -0.187738$, $\omega_1(0) = 0.5$, $\omega_2(0) = 0.5$, and $\omega_3(0) = 1.0$. These values give $\varepsilon = 0.1$, $\hat{k} = 0.25$, $\alpha = 1.44982$, and $\beta = 2.61822$ so that Equation (36) is satisfied. Figures 3a and 3b show the limit cycle that the averaging method predicts.

Values for k^2, Ω, θ, and r_1 were computed for the actual motion of Equation (3) by computing T and L from Equation (5) and using the formulas given in Section 2.1. It is seen that as time increases the actual motion consists of small oscillations about the constant solution determined from $k = \hat{k}$. These oscillations can be determined by computing the transformation functions W and V in Equation (20) but was not done here.

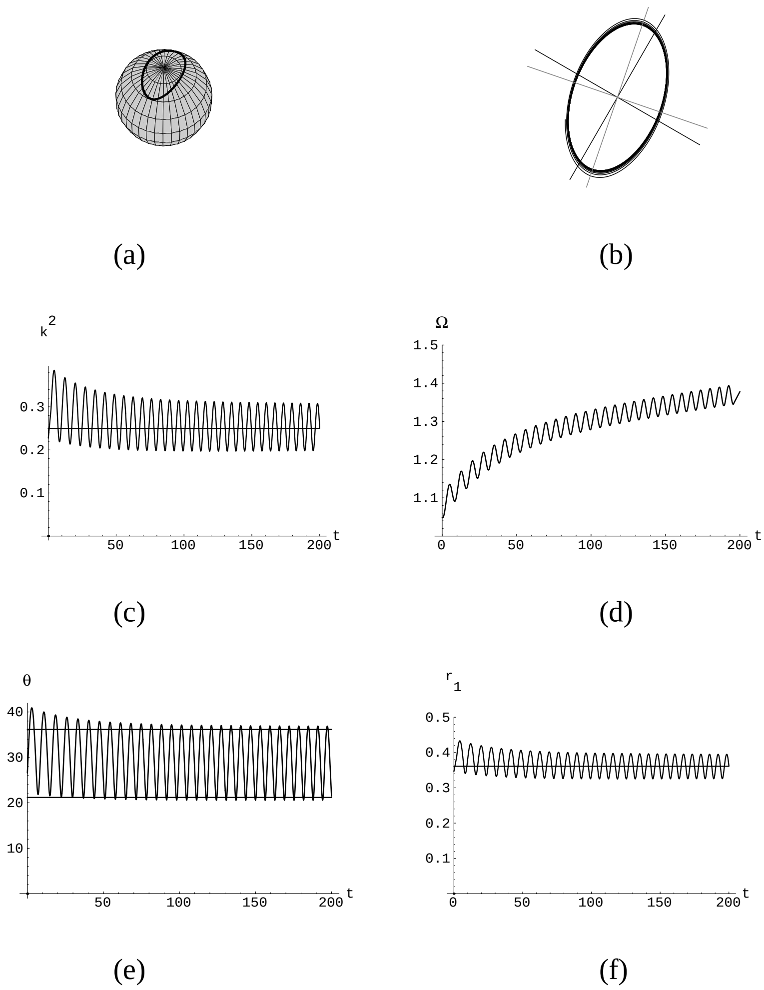


Figure 3. Limit Cycle. $\varepsilon = 0.1$. (a) Limit cycle on the momentum sphere. (b) Projection onto $\ell_1\ell_2$ plane (axes shown black); gray axes align with symmetry lines of limit cycle that are rotated w.r.t axes. (c) Plot of actual k^2 and the line $k^2 = \hat{k}^2$. (d) Plot showing actual Ω increasing slowly. (e) Plot of actual θ (in deg) and the lines $\theta = \theta_{\min}(\hat{k})$, $\theta = \theta_{\max}(\hat{k})$. (f) Plot of actual r_1 and the line $r_1 = r_1(\hat{k})$.

5. Conclusions

The method of averaging has been applied to the strongly nonlinear system of Euler's equations for rotational motion. The formulation uses the elliptic function solution that appears from the torque-free asymmetric rigid body equations. The averaging procedure averages the influence of perturbation terms over nominal rigid body motion – motion that includes nutation and possibly large coning motions. This generalizes the weakly nonlinear cases described by a nearly axi-symmetric spacecraft or by a small coning motion assumption.

An averaged system has been computed for a perturbation involving linear terms in the angular velocity components. The averaged system shows that the long-term motion is determined by only four parameters out of a total of 18. The reduction of the number of parameters greatly simplifies a qualitative investigation.

Three examples concerning internal motion, thrust torques, and feedback control have been investigated. In each case, the predictions made using the averaged system compare very well with the actual motion, both quantitatively and qualitatively.

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