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Cracked orthotropic strip with clamped boundaries

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1. Introduction

One of the basic aims of the *linear elastic fracture mechanics* (*LEFM*) in recent years is the determination of stress intensity factors in complicated situations regarding geometric features and/or mechanical behavior [1]. This reflects the general claim for fracture mechanics solutions to represent engineering reality instead of idealized situations. Of course, mathematical complexities set some bounds up to this policy and then one has to resort to numerical methods. However, there are still many problems in the realm of *LEFM* which admit an analytical treatment. One of these is the problem solved here.

This concerns a long strip made by *orthotropic* material containing a long crack. The case of *clamped* strip boundaries will be discussed here. The alternative way of loading, i.e. the *shear free* strip boundaries, has been considered in a recent paper of ours [2]. The present work follows the analysis therein closely.

Knauss [3] and Rice [4] have solved the respective isotropic problem, whereas Nilsson [5] and Popelar et al. [6] considered dynamic crack motion in an elastic and viscoelastic strip. Relative to the present work are the ones in [7–11] where, however, different geometric features were encountered.

The solution here was accomplished by Fourier transforms and the *Wiener-Hopf technique.* The procedure will be described briefly.

2. Governing equations

Consider a linear elastic orthotropic body in the form depicted in Fig. 1. Then, with respect to the principal material-axes, the elastic constitutive expression relating the in-plane stresses and displacements is [12]

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix} = \begin{bmatrix}c_{11} & c_{12} & 0\\ c_{12} & c_{22} & 0\\ 0 & 0 & c_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix} \tag{1}$$

where the components of the stiffness matrix are given in terms of the engineering material constants as follows

$$c_{11} = E_1/(1-\nu_1\nu_2),\quad c_{12} = \nu_2 E_1/(1-\nu_1\nu_2),\quad c_{22} = E_2/(1-\nu_1\nu_2),\quad c_{66} = G.$$

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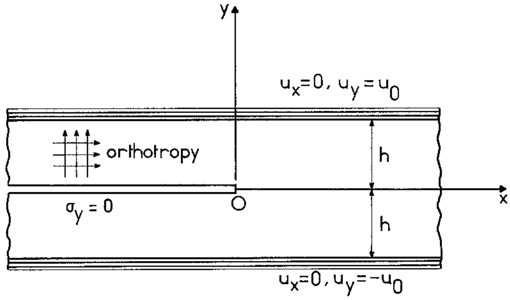

Figure 1
A long orthotropic strip with clamped boundaries containing a crack opened by vertical displacements.

In the above relations, the subscripts 1 and 2 refer to the principal directions of material symmetry which coincide here with the x and y reference axes. Only four elastic constants are independent, the fifth given by

$$\nu_2 E_1 = \nu_1 E_2 .$$

Following Ref. [2], we introduce the ϕ- and ψ-displacement potential by the relations

$$u_x = \frac{\partial}{\partial x}(\phi + \psi), \quad u_y = \frac{\partial}{\partial y}(\alpha\phi + b\psi), \quad u_z = 0 \tag{2}$$

where

$$\alpha = \frac{c_{11}\beta_1 - c_{66}}{c_{12} + c_{66}} = \frac{(c_{12} + c_{66})\beta_1}{c_{22} - c_{66}\beta_1}, \quad b = \frac{c_{11}\beta_2 - c_{66}}{c_{12} + c_{66}} = \frac{(c_{12} + c_{66})\beta_2}{c_{22} - c_{66}\beta_2} \tag{3}$$

and β_1, β_2 the roots of the characteristic equation

$$c_{11}c_{66}\beta^2 + (c_{12}^2 + 2c_{12}c_{66} - c_{11}c_{22})\beta + c_{22}c_{66} = 0. \tag{4}$$

It can be shown further that the displacement potentials satisfy the following Laplace type differential equations

$$\frac{\partial^2\phi}{\partial x^2} + \beta_1\frac{\partial^2\phi}{\partial y^2} = 0, \quad \frac{\partial^2\psi}{\partial x^2} + \beta_2\frac{\partial^2\psi}{\partial y^2} = 0 \tag{5}$$

and the stresses are given in terms of ϕ and ψ as

$$\sigma_x = \frac{(1+\alpha)c_{66}}{\beta_1}\frac{\partial^2\phi}{\partial y^2} + \frac{(1+b)c_{66}}{\beta_2}\frac{\partial^2\psi}{\partial y^2} \tag{6.1}$$

$$\sigma_y = -(1+\alpha)c_{66}\frac{\partial^2\phi}{\partial x^2} - (1+b)c_{66}\frac{\partial^2\psi}{\partial x^2} \tag{6.2}$$

$$\tau_{xy} = c_{66}\left[(1+\alpha)\frac{\partial^2\phi}{\partial x\,\partial y} + (1+b)\frac{\partial^2\psi}{\partial x\,\partial y}\right]. \tag{6.3}$$

3. Analysis

The semi-infinite crack is opened by constant vertical displacements u_0 applied to the strip boundaries, whereas these boundaries are clamped in the x-direction. According to Popelar and Atkinson [6] this type of loading is more convenient for fracture studies than the shear-free boundaries. To apply the Wiener-Hopf technique the following auxiliary problem is considered

$$u_x(x, h) = 0 \quad \text{for} \quad -\infty < x < \infty \tag{7.1}$$

$$u_y(x, h) = 0 \quad \text{for} \quad -\infty < x < \infty \tag{7.2}$$

$$\tau_{xy}(x, 0) = 0 \quad \text{for} \quad -\infty < x < \infty \tag{7.3}$$

$$\sigma_y(x, 0) = \sigma_0 \quad \text{for} \quad -\infty < x < 0 \tag{7.4}$$

$$u_y(x, 0) = 0 \quad \text{for} \quad 0 < x < \infty \tag{7.5}$$

Then, by a trivial superposition we may arrive again at the original problem. The proper value of σ_0 in the above boundary conditions and for plane-stress may be derived by Eq. (1) as

$$\sigma_0 = -(c_{12}\nu_2 + c_{22})\frac{u_0}{h}. \tag{8}$$

Obviously, such a superposition does not affect the value of the stress intensity factor obtained by solving problem (7).

Now we introduce two as yet unknown functions the determination of which completes the solution of problem (7):

$$\sigma_y(x, 0) = m(x) \quad \text{for} \quad 0 < x < \infty \tag{9.1}$$

$$u_y(x, 0) = n(x) \quad \text{for} \quad -\infty < x < 0. \tag{9.2}$$

Applying the Fourier transform Eqs. (5) become [13]

$$-\omega^2\phi^*(\omega, y) + \beta_1\frac{\partial^2}{\partial y^2}\phi^*(\omega, y) = 0 \tag{10.1}$$

$$-\omega^2\psi^*(\omega, y) + \beta_2\frac{\partial^2}{\partial y^2}\psi^*(\omega, y) = 0 \tag{10.2}$$

which have general solutions of the form ($\gamma_j = \beta_j^{-1/2}$ $(j = 1, 2)$)

$$\phi^*(\omega, y) = A(\omega)e^{\gamma_1\omega y} + B(\omega)e^{-\gamma_1\omega y} \tag{11.1}$$

$$\psi^*(\omega, y) = C(\omega)e^{\gamma_2\omega y} + D(\omega)e^{-\gamma_2\omega y}. \tag{11.2}$$

The transforms of the functions of interest are defined as

$$m_+^*(\omega) = (2\pi)^{-1/2}\int_0^\infty m(x)e^{i\omega x}\,dx \tag{12.1}$$

$$n_-^*(\omega) = (2\pi)^{-1/2}\int_{-\infty}^0 n(x)e^{i\omega x}\,dx. \tag{12.2}$$

Then, application of the Fourier transform to the boundary conditions in conjunction with the transformed Eqs. (6) results in an algebraic system of five equations with the six unknown functions $A(\omega)$, $B(\omega)$, $C(\omega)$, $D(\omega)$, $m_+(\omega)$ and $n_-(\omega)$. Some tedious algebra eliminates $A(\omega), \ldots, D(\omega)$ and reduces the system to the following Wiener-Hopf equation

$$m_+^*(\omega) = -\frac{c_{66}}{\gamma_1\gamma_2(\alpha - b)}K(\omega)n_-^*(\omega) - \frac{\sigma_0}{i(2\pi)^{1/2}\omega} \tag{13}$$

where the *Kernel* $K(\omega) = \omega F(\omega)[G(\omega)]^{-1}$ is given by

$$\begin{aligned} F(\omega) = {} & (1+\alpha)\{2\gamma_1\gamma_2(1+\alpha)b - (1+b)\gamma_2[(b\gamma_2+\alpha\gamma_1)\cosh(\gamma_1 h\omega - \gamma_2 h\omega) \\ & - (b\gamma_2 - \alpha\gamma_1)\cosh(\gamma_1 h\omega + \gamma_2 h\omega)]\} + (1+b)\{2\gamma_1\gamma_2(1+b)\alpha \\ & - (1+\alpha)\gamma_1[(b\gamma_2+\alpha\gamma_1)\cosh(\gamma_1 h\omega - \gamma_2 h\omega) + (b\gamma_2 - \alpha\gamma_1) \\ & \cdot \cosh(\gamma_1 h\omega + \gamma_2 h\omega)]\}, \\ G(\omega) = {} & (b\gamma_2 - \alpha\gamma_1)\sinh(\gamma_1 h\omega + \gamma_2 h\omega) \\ & - (b\gamma_2 + \alpha\gamma_1)\sinh(\gamma_1 h\omega - \gamma_2 h\omega). \end{aligned} \tag{14}$$

The Kernel takes the following asymptotic forms for small and large ω's

$$\lim_{\omega \to 0} K(\omega) = \frac{\gamma_1\gamma_2 \cdot (\alpha - b)^2}{(\alpha\gamma_1^2 - b\gamma_2^2)h} \tag{15}$$

$$\lim_{\omega \to \infty} \frac{K(\omega)}{\omega} = (1+\alpha)(1+b)(\gamma_2 - \gamma_1). \tag{16}$$

Following now the classical Wiener-Hopf method [14] and the procedure in Refs. [2, 5, 15] we may find

$$m_+^*(\omega) = -\frac{\sigma_0 K_+(\omega)}{i(2\pi)^{1/2}\omega}\left[\frac{1}{K_+(\omega)} - \frac{1}{K_+(0)}\right] \tag{17}$$

and further

$$\lim_{\omega \to \infty} m_+^*(\omega) = -\frac{\sigma_0}{i(2\pi)^{1/2}\omega} + \frac{\sigma_0}{i(2\pi)^{1/2}\omega^{1/2}K_+(0)} \lim_{\omega \to \infty} \frac{K_+(\omega)}{\omega^{1/2}} \tag{18}$$

or [13]

$$\lim_{x \to 0^+} m(x) = -\frac{\sigma_0}{\pi^{1/2}}\left[\frac{(\gamma_2 - \gamma_1)(\alpha\gamma_1^2 - b\gamma_2^2)(1+\alpha)(1+b)h}{\gamma_1\gamma_2(a-b)^2}\right]^{1/2} x^{-1/2}. \tag{19}$$

The above expression is the asymptotic form of the *cleavage* $\sigma_y(x, 0)$ – stress near the crack tip. Since the *stress intensity factor* is given by

$$K_I = \lim_{x \to 0^+} [(2\pi x)^{1/2} \cdot \sigma_y(x, 0)] \tag{20}$$

we may easily find that in our case

$$K_I = \left[\frac{2(\gamma_2 - \gamma_1)(\alpha\gamma_1^2 - b\gamma_2^2)(1+\alpha)(1+b)}{\gamma_1\gamma_2(\alpha - b)^2}\right]^{1/2} \frac{(c_{12}v_2 + c_{22})u_0}{h^{1/2}}. \tag{21}$$

4. Results and discussion

Obtaining the isotropic result of Rice [4] as a limiting case of our analysis seems to be cumbersome. This is due to the fact that both numerator and denominator in Eq. (21) tend to zero for $\alpha = b$, $\gamma_1 = \gamma_2$, viz. for an isotropic material. Of course, application of L'Hospital rule may lead to an analytic result after some tedious algebra. However, in order to check our final result (21), we chose to work numerically. We consider the stress intensity factor for a nearly isotropic material with the following mechanical constants:

$E_1/E_2 = 1.3/1.0$, $\nu_1/\nu_2 = 0.325/0.250 = 1.3$, $c_{66} = E_2/2(1+\nu_2) = 0.4$ (all values have been normalized in respect to E_2-modulus). In order to get the isotropic SIF, the values $E = E_2 = 1.0$ and $\nu = \nu_2 = 0.250$ were used.

In addition, we compare the results of (21) with those obtained by our formula for the shear-free type of boundary conditions in the same strip [2]. It is noticed that the latter case is reduced immediately to the isotropic one (see [5] for an analogy in elastodynamics).

The following stress intensity factor values were found in the three cases:

$$K_I^{\text{iso}} = 1.032\, u_0 h^{-1/2},\ K_I^{\text{orth1}} = 1.145\, u_0 h^{-1/2} \text{ (clamped boundaries)}$$

and

$$K_I^{\text{orth2}} = 1.105\, u_0 h^{-1/2} \text{ (shear-free boundaries)}.$$

The above results clearly show the validity of Eq. (21).

On the other hand, it is interesting to explore the effect of the degree of orthotropy on the stress intensity factor. In particular, in what follows we consider the effect of high orthotropy. This was accomplished by considering large differences in the orthotropic constants in the directions parallel and perpendicular to the crack line. The material behavior of plywood was utilized.

In the first case the strong direction coincides with the crack axis: $E_1 = 24.600$, $E_2 = 1.000$, $\nu_1 = 0.298$, $\nu_2 = 0.012$ and $G = 0.750$. Equation (21) then gives $K_I^{(\alpha)} = 1.346\ u_0 h^{-1/2}$. In the second case the weak direction coincides with the crack axis: $E_1 = 1.000$, $E_2 = 24.600$, $\nu_1 = 0.012$, $\nu_2 = 0.298$ and $G = 0.750$. In that case Eq. (21) gives $K_I^{(b)} = 30.549\ u_0 h^{-1/2}$. This great difference in SIF values for case (α) and (b) is mainly due to the term c_{22} in the numerator of Eq. (21).

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Abstract

The stress intensity factor at the tip of a semi-infinite crack in an orthotropic infinite strip was determined. Clamped strip boundaries were considered.

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