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FINE # First-order Modal Theories*

Abstract. This paper is part of a general programme of developing and investigating particular first-order modal theories. In the paper, a modal theory of propositions is constructed under the assumption that there are genuinely singular propositions, ie. ones that contain individuals as constituents. Various results on decidability, axiomatizability and definability are established.

In some recent work ([7], [8], [9], [10]), I have attempted to carry out a dual programme of developing a general model-theoretic account of first-order modal theories, on the one hand, and of studying particular theories of this sort, on the other. The two parts of the programme are meant to interact, with the second providing both motivation and application for the first. The present paper belongs to the second part of the programme and deals with the question of giving a correct essentialist account of propositions.

My approach is distinctive in two main ways, one linguistic and the other metaphysical. On the linguistic side, I have let the variables for propositions be both nominal and objectual. That is to say, the variables occupy the same position as names and are interpreted in terms of a range of objects, which, in the present case, turn out to be propositions. This approach stands in contrast to the earlier work of Prior [17], Bull [1], Fine [4], Kaplan [14] and Gabbay [12], [13], in which the variables are sentential (they occupy the same position as sentences) and are interpreted either substitutionally or in terms of a range of *intensional* values.

Grammatically, the distinction between the two approaches is quite sharp. On the nominal account, the propositional variables attach to predicates to form formulas and are not themselves formulas; while on the sentential account, the variables attach directly to connectives to form formulas and are, by themselves, formulas. For example, the expression $\Box (p \supset q)$ is a formula when 'p' and 'q' are sentential variables, but is not even well-formed when they are nominal. To produce a corresponding formula with nominal variables, one should introduce a truth-predicate and then use $\Box (Tp \supset Tq)$.

Whether there is a deeper, more philosophic, distinction between the two approaches is another matter. My own view is that nominal-cum-objectual quantification is fundamental and that all other forms of quantification are ultimately to be explained in terms of it. This is not to say that the other forms are illegitimate, but merely that they stand in need of analysis. Thus the use of these other forms is not ruled out in the formulation of a theory, when there is no requirement that the formulation be basic.

It might be thought that the sentential account will also suffer from certain logical drawbacks; for on it, there is no natural way of expressing the identity or existence of propositions. Now it is in fact true of most of the earlier work that questions of identity and existence were not considered. The one exception is Prior, who introduced a sta-bility operator, which corresponds to the existence-predicate, and who was prepared to countenance a connective for identity (see [16] and [18], pp. 53-56). But as the example of Prior makes clear, there is nothing in the sentential notation as such to block expressive parity with the nominal account. Indeed, thinking in more general terms, we might always suppose that the values of the nominal variables for propositions should be intensional values for the sentential variables and that, for each predicate of propositions, there should be corresponding connective on sentences.

The reasons, then, for preferring the nominal account are ultimately philosophical, not logical, and those of another philosophical persuasion will probably be able to adjust my symbolism and its interpretation to suit their own preferences.

On the metaphysical side, my assumptions are far more drastic in their consequences. I have adopted what one might call a platonic and objectual conception of propositions. By platonism here I mean, roughly, that the domain of propositions is not constrained by the limited means of expression of a finitary language. In particular, the propositions will be closed under arbitrarily long conjunctions, whether finite or not, and may be about arbitrarily many individuals. It should be possible to work out the theories for various non-platonic stances, as in [4], but this is not something that I have done.

Objectualism is a form of structuralism. This is the view that pro-positions have a quite definite structure. The proposition is actually regar-ded as being built up in a certain way out of certain constituents. Thus the structure, i.e. the manner of composition and the constituents, is internal to the proposition and may be said, in a quite natural sense, to explain its identity. What the structure of a proposition is will depend upon the form of structuralism. But on most accounts, the structure will abstract from, and correspond more or less closely to, the syntactic structure of a sentence, if any, which expresses the proposition. For

example, a (genuine) subject- predicate sentence will express a proposition that is composed, in a predicative manner, of a subject- and predicate constituent which correspond, respectively, to the subject- and predicate-expressions in the sentence.

Objectualism is that species of structuralism which allows a proposition to have individual, as opposed to intensional, constituents. There are various ways, in principle, in which an individual may enter into a proposition, but the most characteristic way is as a subject-constituent. If, for example, the proposition to the effect that Socrates is a philosopher is genuinely of subject-predicate form, then the subject-constituent will not be something like an intension or individual concept, but will be Socrates himself.

Let us say that a proposition is *singular* if it contains an individual constituent and is *purely general* otherwise. Then an objectualist admits singular propositions, whereas his opponent does not. However, it is not just that the one accepts a proper subclass of the propositions accepted by the other. For given the difference on objectualism, the construction of even purely general propositions may well be different.

Speaking with rough historical accuracy, we may say that Russell and Frege were both structuralists, but that Russell was an objectualist while Frege was not.

The issue of objectualism is one with many ramifications in both the philosophy of language and metaphysics. However, its interest for us is rather special. Our language for the modal theory of propositions is extremely limited in expressive power — in addition to the usual logical notions, it only contains a predicate for truth; and our concern is merely with how objectualism effects the truths of such a language.

There are, in fact, two main consequences for the language, one on identity and the other on existence, those very topics that are usually ignored on the sentential account. The one consequence is that necessarily equivalent propositions may be distinct. This may happen for general structural reasons or for more distinctively objectual ones. For example, the propositions expressed by $\forall x (x = x)$ and $\forall x \exists y (x = y)$ may be distinguished in terms of their quantificational structure (or what corresponds to it), whereas the propositions expressed by .Socrates = Soctrates' and 'Plato = Plato' may be distinguished in terms of their respective individual constituents.

The other, and more important, consequence is that propositions may contingently exist. Given that a proposition is constructed from certain individuals, it is plausible to suppose that the proposition is existent (or actual) only if the individuals are. But then if the individuals contingently exist, so does the proposition. For example, the proposition to the effect that Socrates exists will itself exist only if Socrates does.

In working out the details of there consequences, some special con-

siderations are called for. In regard to identity, we may merely state, e.g., that there are so many distinct but necessarily equivalent propositions, since there is no direct way of talking in the language about the structure of a proposition. In regard to existence, though, it is not sufficient to state that there are contingently existing propositions, or even so many of them, for it is also necessary to make various other determinations of the existence of propositions.

To explain this matter more fully, let us introduce some terminology. Given a proposition, let its *truth-set* be the set of possible worlds in which it is true and its *existence-set* the set of possible worlds in which it exists. For example, the truth-set of the proposition to the effect that Socrates does not exist is the set of possible worlds in which Socrates does not exist, while its existence-set is the complement of its truth-set. Let the *modal value* of a proposition be the pair consisting of its truth-set and its existence-set. Then, in this terminology, our problem is to determine for a pair of sets of worlds, when a proposition has that pair as its modal value.

This problem can be solved by means of the following criterion:

(1)   there is a proposition with modal value $\langle U, V \rangle$ iff there is a subset J of individuals in some possible world such that J determines the identity of U and V is the set of possible worlds in which all of the individuals of J exist.

The notion of determines here may be explained, as in [6], in terms of automorphisms. It may then be proved, for a suitable choice of an ideal (i.e. infinitary) language, that:

(2)   the set J of individuals determines the identity of U iff there is a sentence of the ideal language which is true in exactly the worlds of U and whose (rigid) names refer to exactly the individuals in J.

Given (2), (1) is equivalent to the following linguistic criterion for the representation of modal values.

(3)   there is a proposition with modal value $\langle U, V \rangle$ iff there is a sentence φ of the ideal language such that U is the set of possible worlds in which φ is true and V is the set of possible worlds in which all of the referents of names in φ exist.

The statement (3) may itself be justified in other terms. Let the *objectual content* of a *proposition* be the set of its individual constituents and the *objectual content* of a *sentence* the set of referents of its (rigid) names. Moreover, let the *truth-set* of a *sentence* be the set of possible worlds in which it is true. Then (3) may be based on the following two assumptions:

(4)   a proposition exists (in a possible world) iff all of its individual constituents do:

(5)   there is a proposition with given truth-set and objectual content iff there is a sentence of the ideal language with the same truth-set and objectual content.

To justify the right-to-left direction of (5), let the proposition, for a given sentence of the ideal language, be the one expressed by the sentence. Clearly, the sentence and the proposition have the same truth-set. Moreover, given:

(6)   the individual constituents of a proposition expressed by a sentence are the referents of the names in the sentence;

the sentence and proposition will also have the same objectual content. To justify the left-to-right direction of (5), we must appeal to the expressive power of the ideal language. Let $\varrho = \varrho(i_1, i_2, \ldots)$ be a proposition with individual constituents $i_1, i_2, \ldots$. Then there is a corresponding relation $R = \lambda x_1, x_2, \ldots \varrho(x_1, x_2, \ldots)$, which is purely qualitative. Now the primitive relation-symbols of the ideal language are so chosen that for any purely qualitative relation there is a name-free formula with the same possible worlds intension. Therefore there is such a formula $\varphi(x_1, x_2, \ldots)$ for R. But then the sentence $\varphi(n_1, n_2, \ldots)$, for $n_1, n_2, \ldots$ the respective names of $i_1, i_2, \ldots$, will have the same truth-set and objectual content as ϱ.

(6) may itself be justified on the basis of three further assumptions. The first is that (6) itself holds for atomic sentences. The second is that the logico-syntactic operations on sentences correspond to operations on propositions, that, for example, the conjunction of two sentences expresses the conjunction of the propositions expressed by the conjuncts. The third is to the effect that objectual content accumulates in the natural way under the application of the logical operations to propositions, that, for example, the objectual content of a conjunctive proposition is the union of the objectual contents of its conjuncts. Such a principle of accumulation is stated in [11] and used in the semantical analysis of Parry's system of analytic implication.

The other main assumption, (4), follows naturally from the constructive aspect of objectualism. Given that a proposition is built up from its constituents, it is difficult to see how the proposition could exist unless its constituents did; and since neither the manner of composition nor the other constituents are a source of contingency, the existence of the individual constituents will be both a necessary and a sufficient condition for the existence of the proposition. In respect to the existence criterion, then, propositions are just like sets or, indeed, like any other complexes. I now wish to repudiate the suggestion in [5] (pp. 127-8) and [6] (p. 136) that the existence criterion for intensional entities, such as propositions, is different from that for extensional entities, such as sets. In both cases, it is in terms of constituents. Indeed, unless propo-

sitions were complexes or structured entities, it is difficult to see, on the present platonic view, how their necessary existence might plausibly be denied. The reason for the apparent difference in criteria is that, in [6], the propositions are not given in terms of their structure and hence some detective work, of a rather special nature, needs to be done to recover the underlying objectual content.

Note that the statement and justification of the criterion in (1) above does not presuppose the details of any particular structuralist account. The nearest the discussion gets to such details is in the justification of (6), where it is assumed that there are logical operations on propositions corresponding to the syntactic operations on sentences. But only a very limited use is made of this assumption, viz., that embedded in the accumulation principle. Such questions as whether conjunction is a constituent of a conjunctive proposition or whether the order of the conjuncts makes a difference to its identity need not be considered.

It has been usual in the literature on modal logic, my own work included, to identify propositions with their truth-sets. This practice is harmless enough if it is actually thought that necessarily equivalent propositions are identical. But without this presupposition, the practice can run into certain dangers. Some properties of propositions only depend upon their truth-sets, the most notable examples being the modal properties of necessary and possible truth. Such properties, then, can be replaced by the corresponding properties of their truth-sets. But other properties depend upon more than the truth-set, and I am not thinking here merely of intentional properties, like those for belief and knowledge, but also of more logical ones, like existence or identity to a given proposition. If propositions are to be identified with truth-sets, then such properties will either be ignored or not properly considered.

As an example of the first danger, I might cite the neglect of the identity relation in the recent work on propositions in modal logic. It might be thought that the introduction of this relation is a trivial matter. But, in fact, if one adopts a platonist and anti-structuralist position, a decidable system is turned into an undecidable upon the more addition of identity and into an unaxiomatizable one when the intended interpretation is maintained (theorems 7 and 8 below).

As an example of the second danger, I might cite my own work on the existence of intensional entities in [6]. The significance of the definitions there is to some extent vitiated by a lack of correspondence with the intuitive properties of the intensional entities themselves. For example, the definition of "existence" for sets of possible worlds V is such that V "exists" in a world w iff *some* proposition with V as its truth-set exists in w. But then there may be no single proposition which exists in exactly those worlds in which V "exists", and so even the above correspondence may not be preserved for higher types.

The problems over existence may be avoided by using modal values in place of truth-sets. But there will still be problems over identity, which will be compounded when we move to entities of higher type or to some other extensions of the language. In order to avoid these dangers, it may often be wise to talk directly both about the propositions and their truth-sets. In this way, the connection between the two can be explicitly worked out without prejudging any questions about the propositions themselves. This is the approach adopted in the present paper, and it is one that might have been used to resolve the aforementioned difficulties in [6].

The reluctance to talk about propositions has, I think, more often arisen from despair than conviction. It has been felt that in the absence of a fully worked out structuralist criterion, nothing useful can be said about the propositions themselves. But this is a mistake. One may develop a theory without a full grasp of the entities with which it deals. In the case at hand, the previous discussion makes it clear how one can determine which pairs of sets of worlds are the truth-and existence sets of a proposition without presupposing the details of any particular objectualist theory. Indeed, the main interest in the techniques of [6] is that they show how determinate is the shadow that the structure of propositions casts on their possible world representation.

The plan of the present paper is as follows. The first section sets out the underlying formal language and its general interpretation. For further discussion of these matters, the reader might consult the first three sections of [7]. The second section deals with various anti-objectualist theories of propositions. Their characteristic feature is that propositions are assumed to exist necessarily. The next two sections set out and develop those conditions on a modal structure which are justified by an objectualist and platonic conception of propositions. The resulting theory is not axiomatizable, but the fifth section presents a partial axiomatization of it and the sixth supplies some standard metatheoretical results. The seventh section considers various extensions of the basic theories, two of which are obtained by introducing propositional abstraction and quantification over sets of propositions. The final section deals with the actualist demand of defining all predicates in terms of those that are only true of the actuals of each world. It is shown that the normal truth-predicate is not definable in terms of actualist truth alone, but is definable in terms of actualist truth and a predicate for actualist strict implication.

1. Language and models

Our language $\mathscr{L}$ will be a modal first-order one in the sense of [7] or [15]. The logical, or rather fixed, vocabulary includes a two-place predicate $=$ for identity and a one-place predicate E for existence. In

addition, there is a single non-logical predicate T for truth. In keeping with our nominalistic approach to the quantifiers, T will aply to a variable ϱ of the language. The result '$T\varrho$' may then read as 'ϱ is true' or as 'the proposition ϱ is true'.

It is customary to use Roman letters, such as 'x', 'y' and 'z', for the individual variables of a first-order language. However, in order to emphasize the fact that our variables are to range over propositions, I shall use the Greek letters 'ϱ', 'σ' and 'τ', with or without subscripts, in their place.

A (modal) structure for the language may also be raken in the sense of [7] or [15]. However, since there is only one non-logical predicate, the definition may be simplified somewhat and a structure $\mathfrak{A}$ may be regarded as a triple $(W, \overline{A}, t)$ in which W is a non-empty set, $\overline{A}_w$ is, for each $w \in W$, a set, at least one of which is non-empty, and t is a set of pairs (w, e) in which $w \in W$ and $e \in \overline{A}_v$ for at least one $v \in W$. The outer domain A of the structure $\mathfrak{A}$ may then be defined as $\{e\colon$ for some $w \in W,\ e \in \overline{A}_w\}$.

Intuitively, W is the set of all possible worlds, $\overline{A}_w$ is, for each w, the set of propositions which exist (or are actual) in W, t is the set of pairs (w, e) for which e is a proposition true in the world w, and A is the set of all possibly existing propositions. There is some difficulty in talking intuitively about propositions within a modal framework, for one may mean actual propositions or possible (i.e. possibly existing) propositions. Usually, the former is meant at the level of the object-language and the latter at the level of semantics. However, the appropriate qualifications will be made explicit when they are important. In the intuitive talk of propositions I shall use the variables $\varrho, \sigma, \dots$, as in the object language. This dual use of the symbols should cause no confusion, though.

A truth-definition may be given by adding each possible object as a name of itself to the language. The clauses for the necessity operator, the existence- and truth-predicates, and the existential quantifiers are then as follows:

(i) $w \vDash \Box\varphi$ iff $v \vDash \varphi$ for all $v \in W$;

(ii) $w \vDash Ee$ iff $e \in \overline{A}_w$;

(iii) $w \vDash Te$ iff $(w, e) \in t$;

(iv) $w \vDash \exists\varrho\varphi(\varrho)$ iff $w \vDash \varphi(e)$ for some e in $\overline{A}_w$.

The clause for T does not require $e \in \overline{A}_w$. Later, in section 8, we shall consider some questions which arise from adding this condition. Clauses (i) and (iv) give rise to a certain modal predicate logic, viz. **S5** with actualist quantifiers. This reflects my belief in an **S5**-logic for metaphysical necessity and in the primacy of the actualist quantifiers. However, it would be possible, either on philosophical or technical grounds, to consider other clauses for the quantifiers and for necessity.

We shall often follow standard conventions and terminology in logic. We shall also adopt a special convention under which a formula φ will be depicted as $\varphi(\varrho_1, \ldots, \varrho_n)$ when $\varrho_1, \ldots, \varrho_n$ are *exactly* the free variables of φ.

In stating conditions on the structures $\mathfrak{A}$ for $\mathscr{L}$, we shall find it helpful to introduce some terminology for truth-sets, existence-sets and modal values. Let $\mathfrak{A} = (W, \overline{A}, t)$ be a structure for $\mathscr{L}$, w a member of W, and e of A. Then we put:

$$
\begin{aligned}
ts(e) &= \{w \in W : \langle w, e \rangle \in t\}; \\
es(e) &= \{w \in W : e \in \overline{A}_w\}; \\
mv(e) &= \langle ts(e), es(e) \rangle; \\
TS_w &= \{ts(e) : e \in \overline{A}_w\}; \\
MV_w &= \{mv(e) : e \in \overline{A}_w\}; \\
TS &= \bigcup_{w \in W} TS_w = \{ts(e) : e \in A\}; \\
MV &= \bigcup_{w \in W} MV_w = \{mv(e) : e \in A\}.
\end{aligned}
$$

Intuitively, $ts(e)$, $es(e)$ and $mv(e)$ are the truth-set, existence-set and modal value, respectively, of e; TS_w and MV_w are the collections of truth-sets and modal values, respectively, of propositions in w; and TS and MV are the collections of all truth-sets and modal values respectively. In the above notation, mention of the underlying structure $\mathfrak{A}$ has been suppressed; but it may, if desired, be explicitly indicated by means of a superscript.

2. Anti-objectualist theories

Our main concern is with theories based upon an abjectualist conception of propositions. However, it will be of interest to begin with a study of anti-objectualist theories, both because of their intrinsic interest and for purposes of comparison.

All of the anti-objectualist theories considered will presuppose the necessary existence of propositions and, as always, a Platonic stance on propositions. They will, for the most part, only differ in their assumptions on the identity of proposition. The first system is based on the anti-structuralist assumption that propositions with the same modal value are identical. Its semantics is determined by the following conditions:

Constant Domain (CD). For any $e \in A$, $ts(e) = W$;

Modal Criterion (MC). $mv(e) = mv(f)$ implies that $e = f$ for all $e, f \in A$;

Platonism (P). For each $V \subseteq W$, $V \in TS$.

The first condition expresses the necessary existence of propositions; the second the identity of propositions with the same modal value; and

the third the Platonic assumption that each set of worlds is the truth-set of some proposition. Given a set of conditions, as above, let us say that a sentence is *valid relative to the conditions* if it is true in all models whose structure satisfies the conditions.

In order to formulate the corresponding theory, we make the following definitions:

$$\varrho \approx_t \sigma \quad \text{for} \quad \Box\,(T\varrho \equiv T\sigma);$$
$$\varrho \approx_e \sigma \quad \text{for} \quad \Box\,(E\varrho \equiv E\sigma);$$
$$\varrho \approx_{te} \sigma \quad \text{for} \quad (\varrho \approx_t \sigma) \wedge (\varrho \approx_e \sigma).$$
$$\text{We} \qquad \text{for} \quad T\varrho \wedge \forall\sigma\big(T\sigma \supset \Box(T\varrho \supset T\sigma)\big).$$

The first three defined expressions say, respectively, that ϱ and σ have the same truth-conditions, the same existence-conditions, and the same truth- and existence-conditions or modal value. The last says that ϱ is a true world-proposition, i.e. that ϱ is true and necessarily implies all truths.

The axioms of our theory are then as follows:

Necessary Existence. $\Box\forall\varrho\;\;\Box E\varrho$;

Modal Criterion Axiom. $\Box\forall\varrho\;\;\Box\forall\sigma[\varrho \approx_{te} \sigma \supset \varrho = \sigma]$;

Simple Comprehension. $\Box\forall\varrho_1, \ldots, \varrho_n\;\exists\sigma\Box(T\sigma = \varphi)$, where φ is a formula whose free variables include $\varrho_1, \ldots, \varrho_n$ but not σ;

World-Proposition. $\Box\exists\varrho W\varrho$.

Necessary Existence and Modal Criterion directly express the conditions of a constant domain and modal criterion for the identity of propositions. Simple Comprehension says that, necessarily, for any condition φ of the language and propositions $\varrho_1, \ldots, \varrho_n$, there is a proposition σ which is true exactly when $\varrho_1, \ldots, \varrho_n$ satisfy the condition. World Propositions says that necesarily there is a true world-proposition.

Note that Comprehension, in a system with nominal quantifiers, is not a purely logical axiom. On the other hand, in a system with sentential quantifiers it is. For from $\Box(A \equiv A)$, $\exists p\Box(p \equiv A)$ follows by Specification.

Given Necessary Existence, the Modal Criterion axiom may be replaced by:

T-Criterion. $(\Box)\,[\varrho \approx_t \sigma \supset \varrho = \sigma]$;
and the prefix $\forall\varrho_1 \ldots \forall\varrho_n\exists\sigma\Box$ in Simple Comprehension by $\forall\varrho_1 \ldots \forall\varrho_n\exists\sigma$, without loss of deductive power. Similarly, in the pressence of Constant Domain, the modal criterion condition may be replaced by:

$$ts(e) = ts(f) \quad \text{implies} \quad e = f \quad \text{for any} \quad e, f \in A.$$

However, for later purposes, we shall find it useful to use the stronger formulations here.

Let us call the theory resulting from these axioms MC. (M for modal criterion and C for constant domain. The Platonism is taken for granted.) Then the theory and the semantics outlined above are equivalent to the system $S5\pi^+$ and its semantics as given in [4]. To be exact, for any formula φ of $\mathscr{L}$, let φ^* be the result of replacing each individual variable ϱ_i by the sentential variable p_i, and each of the atomic subformulas $E\varrho_i$, $T\varrho_i$ and $\varrho_i = \varrho_j$ by $(\forall p_1)(p_1 \supset p_1)$, p_i, and $\square(p_i = p_j)$ respectively. Conversely, given a formula A of the language of $S5\pi^+$, let A' be the result of replacing each quantifier $\forall p_i$ by $\forall \varrho_i$ and each sentential variable p_i, not attached to a quantifier, by $T\varrho_i$. Then it is an easy matter to show that the translations preserve validity and theoremhood in the respective systems. That is:

THEOREM 1 (i).   *The sentence φ is valid relative to the conditions* CD, MC *and* P *above iff φ' is a valid sentence of $S5\pi^+$.*

(ii).   *The sentence φ is a theorem of* MC *iff φ' is a theorem of $S5\pi^+$.*

(iii).   *The formulas $\varphi = \varphi^{*\prime}$ are both valid for the three conditions and provable in* MC.

Given this theorem and the soundness, completeness and decidability results for $S5\pi^+$, we may show:

COROLLARY 1.   *The theory* MC *is sound and complete for its semantics and is decidable.*

Of course, the methods used in establishing the various results for $S5\pi^+$ might be applied directly to MC.

Let TS_n be the sentence:

$$\square \exists \varrho_1 \ldots \exists \varrho_n [\, \bigwedge_{1 \leqslant i < j \leqslant n} \sim (\varrho_i \approx_t \varrho_j).$$

This sentence says, for any given n, that there are at least n (actually existing) propositions which differ in their truth-conditions. Then by a corresponding result for $S5\pi^+$ and Theorem 1, it may be shown that:

COROLLARY 2.   *Each sentence of* MC *is provably equivalent to a truth-functional compound of the sentences $TS_1, TS_2, \ldots$.*

Let $MCInf$ be the result of adding all of the sentences $TS_1, TS_2, \ldots$ as axioms to MC. Then given the previous corollary, it follows that:

COROLLARY 3.   *The theory* $MCInf$ *is negation-complete*, i.e., *for each sentence φ of $\mathscr{L}$, either φ or $\sim\varphi$ is a theorem of* MC.

The interest of $MCInf$ is that its theorems are *exactly* the truths of the langauge $\mathscr{L}$ that should be accepted by one who is Platonist, anti-objectualist and modalist in his attitude towards propositions. Given the corollary, it suffices to show that each of the sentences TS_i is true (under the intended interpretation of the language). But this might be

shown (without presupposing the truth of objectualism or platonism) in something like the following way. For each $i = 1, 2, \ldots$ let ϱ_i be the proposition that there exist at least i cats. (Choose some other example if cats do not please.) Then these propositions differ in their truth-sets, since for each $i = 1, 2, \ldots$, it is possible that there are exactly i cats. Moreover, each of these propositions is purely general and hence exists necessarily. Therefore the sentences $TS_1, TS_2, \ldots$ are all true.

Although they shall not be given here, results for the analogues of other extensions of $S5\pi$ — might be established in a similar way.

Let us now drop the anti-structuralism of the preceding approach, but retain the anti-objectualism and the Platonism. It would then appear reasonable to adopt the following condition and axioms in favour of the Modal Criterion ones:

Diversity (D). Given any $e \in A$, there are infinitely many $f \in A$ for which $mv(f) = mv(e)$;

Diversity Axioms. $\Box \forall \varrho \Diamond \exists \sigma_1 \ldots \Diamond \exists \sigma_n (\varrho \approx_{te} \sigma_1 \wedge \ldots \wedge \varrho \approx_{te} \sigma_n \wedge \bigwedge_{1 \leqslant i < \leqslant nj} \sigma \neq \sigma_j)$, for $n = 1, 2, \ldots$.

The condition says that there are infinitely many propositions with the same modal value as a given proposition. In the presence of Constant Domain, the condition merely says that there are infinitely many propositions with the same *truth-set* as a given proposition. The axioms, taken conjointly, exactly express the condition. Note that these axioms, unlike Inf which is also formulated with identity, are not expressible in the language of $S5\pi^+$.

The condition (or axioms) may be given the following intuitive justification. Let σ be any proposition. Granted that:

(1)   there are infinitely many propositions $\sigma_1, \sigma_2, \ldots$ which necessarily exist and are necessarily true;
it follows that we may form the conjunctions $\tau_i = \varrho' \cdot \sigma_i$ of the proposition ϱ with each of the propositions σ_i. Clearly, each proposition τ_i has the same truth-set as ϱ; and by the cumulation principle, they also have the same existence-set as ϱ. But granted that:

(2) $\varrho \cdot \sigma_i \neq \varrho \cdot \sigma_j$ for $i \neq j$, it follows that the propositions $\tau_1, \tau_2, \ldots$ are distinct.

The most vulnerable premises in the above argument have been labelled (1) and (2). Both premisses seem reasonable on a structuralist conception of propositions; and of the two premisses, the first may be justified in terms of an example say the series of propositions that $\psi = \psi$, $\{\psi\} = \{\psi\}, \ldots$. For each of these propositions is clearly necessarily true, and, since they each have null objectual content, they also necessarily exist. However, I know of no justification of these premises that is independent of structuralist considerations.

Diversity does not say exactly how many propositions have the same modal value as a given proposition, and the above argument only tells us that there are as many such propositions as there are necessarily true and existent propositions. It would be good to know more about the cardinalities of sets of propositions with the same modal value. But, fortunately for us, such information is not required, since nothing that can be said in the language $\mathscr{L}$ will turn upon it. Indeed, the diversity axioms will be the only ones that will require structural considerations that are independent of the cumulation principle.

Let the theory which results from replacing Modal Criterion with Diversity be called DC. Then it will follow from a later result (Lemma 14) that:

LEMMA 4. *Each sentence of $\mathscr{L}$ is provably equivalent in DC to an identity-free sentence of $\mathscr{L}$.*

Given this result and Theorem 3, it may then be shown that:

COROLLARY 5. *The theory DC is sound and complete for its semantics and is decidable*; and

COROLLARY 6. *The theory DCInf, obtained by adding Inf to DC, is negation-complete.*

By considerations similar to those adduced for *MCInf*, if follows that *DCInf* is the theory that should be adopted by one who is Platonist, anti-objectualist and yet structuralist in his attitude towards propositions.

Let us now take a neutral stand on the identity of propositions. Thinking semantically gives us the theory C^* of all sentences valid in structures which satisfy Constant Domain and Platonism. In regard to this theory, it may be shown that:

THEOREM 7. *C^* is not axiomatizable.*

SKETCH OF PROOF. The second-order theory of a symmetric relation is not axiomatizable. Indeed, it is equivalent in undecidability to full second-order logic. Now the second-order theory of a symmetric relation can be embedded in the modal theory. For given a sentence A of the second-order theory, let A^* be the result of replacing each atomic formula Rxy in A by $\exists \tau \, \exists \tau' [\tau \neq \tau' \land \Box (T\tau \equiv (T\varrho \lor T\sigma)) \land \Box (T\tau' \equiv (T\varrho \lor T\sigma))]$, each identity formula $x = y$ by $\varrho \approx_t \sigma$, each membership formula $x \in X$ by $\Box(T\varrho \supset T\varrho')$, each individual quantifier $\exists x$ by $\exists \varrho (\Diamond W\varrho \land \ldots)$, and each set quantifier $\exists x$ by $\exists \varrho'$. Then it may readily be shown that the sentence A is a theorem of the classical theory iff A^* is a theorem of the modal theory.

Retaining the neutral stand on identity but thinking syntactically gives the theory C obtained by dropping Modal Criterion from MC. In regard to this theory, it may be shown that:

THEOREM 8. *The theory C is not decidable.*

PROOF: The first-order theory of a symmetric relation is undecidable (see [2] and [3]). But then the previous translation A^* (without set variables) may be used to embed the classical theory in the modal theory.

The above two results are remarkable. The theory MC (or, equivalently, $S5\pi^+$) is decidable. If the axioms are retained, but for a neutral stand on the identity of propositions, then the result C is undecidable. If the semantics is retained, but for the neutral stand on identity, then the result C^* is not even axiomatizable.

There are some extensions of C, which fall short of either MC or DC, but which may be of independent interest. We might, for example, consider axioms of the sort:

$$\exists_n \sigma (\sigma \approx_t \varrho) \supset \exists_n \sigma \square (T\sigma \equiv -T\varrho)$$

(for $\exists_n$ the quantifier "there are at least n"). Such an axiom reflects the fact that the negations of distinct propositions are distinct, a fact that cannot be directly expressed within the language. These intermediate systems are perhaps worthy of further study; and it may be that some of them are decidable or complete for a Platonic semantics.

3. Objectualist conditions on a structure

We now wish to consider what conditions should be imposed upon a structure, given an objectualist and Platonic stand on propositions. There are three conditions in all. Of these, the first two are relatively straight-forward, but the third, Automorphism, is not. We shall first state it, then present some partial reformulations and, finally, outline its justification.

The first condition is Diversity. Since it has already been considered in section 2 and since its justification there did not presuppose anti-objectualism, we shall not consider it any further here.

The second condition is *World Actualism* (WA), as explained on p. 148 of [6]. Given a structure $\mathfrak{A} = (W, \overline{A}, t)$ for $\mathscr{L}$ and $w \in W$, let $\mathfrak{A}_w = (A, \overline{A}_w, t_w)$, where $t_w = \{e \in A : \langle w, e \rangle \in t\}$, and let $\overline{\mathfrak{A}}_w = (\overline{A}_w, \overline{t}_w)$, where $\overline{t}_w = \{e \in \overline{A}_w : \langle w, e \rangle \in t\}$.
Then World Actualism states:

$$\overline{\mathfrak{A}}_w = \overline{\mathfrak{A}}_v \quad \text{implies} \quad \mathfrak{A}_w = \mathfrak{A}_v \quad \text{for all} \quad w, v \in W.$$

A structure $\mathfrak{A}$ is said to be *differentiated* if $\mathfrak{A}_w = \mathfrak{A}_v$ implies $w = v$ for all $w, v \in W$. Since the addition or deletion of copies of worlds make

no difference to the evaluation of modal formulas in a structure, we shall henceforth assume that all structures are differentiated. Under this assumption, World Actualism may be replaced with the condition:

$$\overline{\mathfrak{A}}_w = \overline{\mathfrak{A}}_v \quad \text{implies} \quad w = v \quad \text{for all} \quad w, v \in W.$$

If the nature of a structure $\mathfrak{A}$ for $\mathscr{L}$ is spelt out, then this condition becomes:

if $w \neq v$, then either there is an $e \in A$ in $\overline{A}_w - \overline{A}_v$ or $\overline{A}_v - \overline{A}_w$ or there is an $e \in A$ such that exactly one of w and v is in $ts(e) \cap es(e)$.

World Actualism is a reasonable condition for propositional structures. For given two distinct possible worlds, there will be something the case in the one but not the other. But then the proposition that states that this is the case will be true and existent in the one world but not the other.

To state the final condition, let us introduce some terminology. Given a structure $\mathfrak{A} = (W, \overline{A}, t)$, the pair $\alpha = (\alpha_1, \alpha_2)$ is said to be an *automorphism on* $\mathfrak{A}$ if

(i) α_1 and α_2 are permutations on W and A respectively,

(ii) $e \in \overline{A}_w$ iff $\alpha_2(e) \in \overline{A}_{\alpha_1(w)}$,

(iii) $\langle w, e \rangle \in t$ iff $\langle \alpha_1(w), \alpha_2(e) \rangle \in t$.

In other words, an automorphism is a permutation on the worlds and propositions which respects the truth- and existence- conditions of the propositions. The above notion is merely a special case of the general notion explained on p. 149 of [6].

Let B be a subset of A and V of W. Say that the automorpism $\alpha = \langle \alpha_1, \alpha_2 \rangle$ if *fixed* on B if $\alpha_2(e) = e$ for all $e \in B$; and that B *determines* V if $\alpha_1[V] = V$ whenever the automorphism $\alpha = \langle \alpha_1, \alpha_2 \rangle$ is fixed on B. Intuitively, a set of propositions determines a set of worlds if the identity of the latter set can be determined on the basis of the propositions alone. For a *collection* of propositions $B \subseteq A$, let the existence-set $es(B)$ be $\{w : w \in es(e) \text{ for all } e \in B\}$. Thus the existence-set for a collection is the set of worlds in which all of the propositions in the collection exist.

The final condition can now be stated:

Automorphism. If a subset B of some $\overline{A}_u$ determines $V \subseteq W$ then $\langle V, es(B) \rangle \in QP$.

In other words, if the identity of the set of worlds V can be determined on the basis of the propositions B of some world, then some proposition has V as its truth-set and exists in exactly those worlds in which all of the propositions of B exist.

The exact import of the automorphism condition is hard to appreciate. However, its meaning for truth-sets may be considerably simplified. Taking the lead from section IV of [6], say, for a given structure $\mathfrak{A}$ and $w, v, u \in W$, that u and v are *indiscernible relative to* w — in symbols,

$u \equiv_w v$ — if for all $V \in TS_w$, $u \in V$ just in case $v \in V$. Now say that $u \approx_w v$ if there is an automorphism $a = \langle a_1, a_2 \rangle$ fixed on $\bar{A}_w$ for which $a_1(u) = v$.

Then it may be shown that:

LEMMA 9. *For $\mathfrak{A}$ a structure and $w, v, u \in W$,*

(i) $u \approx_w v$ *implies* $u \equiv_w v$, *and*

(ii) $u \equiv_w v$ *implies* $u \approx_w v$, *when*

$\mathfrak{A}$ *satisfies Automorphism.*

PROOF: (i) Suppose $u \approx_w v$. So for some automorphism $a = \langle a_1, a_2 \rangle$, a is fixed on $\bar{A}_w$ and $a_1(u) = v$. Choose an arbitrary member V of TS_w and suppose $u \in V$. Then for some $e \in \bar{A}_w$, $ts(e) = V$. Since $u \in V$, $(u, e) \in t$; since a is an automorphism, $\langle a_1(u), a_2(e) \rangle \in t$; and since a is fixed on $\bar{A}_w$, $\langle v, e \rangle \in t$, i.e. $v \in V$. In the same way it may be shown that $v \in V$ implies $u \in V$; and so $u \equiv_w v$.

(ii). Suppose that $\mathfrak{A}$ satisfies Automorphism and that not $u \approx_w v$. Let $V = \{t \in W : u \approx_w t\}$. Then it is readily shown that $\bar{A}_w$ determines V. So $V \in TS_w$ by the Automorphism condition. But $v \notin V$, and therefore not $u \equiv_w v$.

From the above lemma it follows that:

THEOREM 10. *For $\mathfrak{A}$ a structure satisfying Automorphism, the following three conditions are equivalent:*

(i) $V \in TS_w$;

(ii) V *is closed under* $\equiv_w$;

(iii) $\bar{A}_w$ *determines V.*

PROOF: (i) $\Rightarrow$ (ii). By definition of $\equiv_w$.

(ii) $\Rightarrow$ (iii). $\bar{A}_w$ determines V if V is closed under $\approx_w$. But then the implication follows by lemma 9 (i)

(iii) $\Rightarrow$ (i). By Automorphism.

It would be good if a similarly simple criterion for the representation of modal values in a world could be found, but I see no way of finding one.

The Automorphism condition may appear complicated and unnatural, but it may be given an intuitive justification. Suppose that B is a set of propositions from some possible world and that w is a possible world. Then it may be shown that:

(*) there is a proposition τ_w whose existence-set is $es(B)$ and whose truth-set is $T_w = \{v \in W :$ there is an automorphism a which is fixed on B and for which $a_1(w) = v\}$.

The justification of (*) goes as follows. For each world w, let the complete description ϱ_w be the result of saying:

(i) of the true, false, existent and non-existent propositions, respectively, that they are true, false, existent and non-existent;

(ii) of distinct propositions that they are distinct;

(iii) of all propositions that they are all the propositions.

Let the complete description σ of all worlds be the result of saying:

(iv) of each proposition ϱ_w that it is possible;

(v) of all propositions ϱ_w that their disjunction is necessary.

Given ϱ_w and σ, let τ_w be the result of existentially generalising on all of the propositions in the conjunction of ϱ_w and τ which are not in B (whith the quantifiers possibilist). Then it may be shown, on the basis of the construction of τ_w, that the truth-set of τ_w is T_w as required. Moreover, since τ_w is constructed by purely logical means from exactly the propositions in B, its existence-conditions are exactly $es(B)$ by the cumulation principle for objectual content.

Now suppose that the subset B of some $\overline{A}_w$ determines V, as in the condition. Let τ be the disjunction of all of the propositions τ_w in (∗) for which $w \in V$. Then its truth-set is the union of all the sets T_w for which $w \in V$. But since B determines V, this union is simply V. Moreover, by the cumulation principle again, the existence-set for τ is still $es(B)$.

The above argument is rather informal and sketchy. A more formal and detailed version of the argument, though at the level of sentences not propositions, may be found in section V of [6].

In view of the above justification, it is natural to wonder to what extent Automorphism is adequate in its postulation of modal values. This question may be answered by appeal to underlying individual structures. Let $\mathfrak{I} = (W, \overline{I}, v)$ be a modal structure for an *arbitrary* modal language L of relation-symbols. To distinguish the structures for L and $\mathscr{L}$, we shall call the former *individual* and the latter *propositional*. In thinking of $\mathfrak{I}$, we should suppose that:

(i) W is the set of all possible worlds;

(ii) I is the set of all possible individual constituents of propositions;

(iii) each relation-symbol in L is purely qualitative; and

(iv) the possible worlds intension of each purely qualitative relation on I is, in principle, expressible in terms of the relation-symbols of L.

Let $MV(\mathfrak{I}) = \{\langle U, V\rangle : \text{for some subset } J \text{ of an } \overline{I}_w, J \text{ determines } U$ and $V = \{w \in W: J \subseteq \overline{I}_w\}\}$. Then by the discussion in the introduction, it follows that:

(I) $MV(\mathfrak{I})$ is the set of modal values of (genuine) propositions.

Given the assumptions (i) − (iv), it also follows that:

(II) Any automorphism $a = \langle a_1, a_2\rangle$ on $\mathfrak{I}$ induces an automorphism β on the collection of all genuine propositions (with respect to truth and existence at a world).

For a given a and a proposition $\varrho = \varrho(i_1, i_2, \ldots)$ with individual constituents $i_1, i_2, \ldots$, let $a_3(\varrho) = \varrho(a_2(i_1), a_2(i_2), \ldots)$. With the help of (iii) and (iv), it may then be seen that $\langle a_1, a_3\rangle$ is the required automorphism β.

From (I) alone it follows that any automorphism $a = \langle a_1, a_2 \rangle$ on $\mathfrak{I}$ can be extended to an automorphism on the modal values of $\mathfrak{I}$. Thus what (II) adds is that the cardinalities of propositions with the same modal value should be in accord.

Let us say that an individual structure $\mathfrak{I} = (W, \bar{I}, v)$ *underlies* the propositional structure $\mathfrak{A} = (W, \bar{A}, t)$ if:

    (i) $MV(\mathfrak{I}) = MV$; and

    (ii) for any automorphism $a = \langle a_1, a_2 \rangle$ on $\mathfrak{I}$, there is an automorphism $\beta = \langle a_1, a_3 \rangle$ on $\mathfrak{A}$.

(Recall the very different definitions of $MV(\mathfrak{I})$ and MV). Then from (I) and (II) it follows that each (genuine) propositional structure possesses an underlying individual structure. It is therefore of great interest to show that:

THEOREM 11. *A propositional structure $\mathfrak{A}$ satisfies the Automorphism condition iff some individual structure $\mathfrak{I}$ underlies $\mathfrak{A}$.*

PROOF: $\Leftarrow$. Suppose that $\mathfrak{I} = (W, \bar{I}, v)$ underlies $\mathfrak{A} = (W, \bar{A}, t)$. Assume that a subset B of some $\bar{A}_w$ determines $V \subseteq W$. Then the satisfaction of Automorphism requires that $\langle V, es(B) \rangle \in MV^{\mathfrak{A}}$. Let $J = \bigcap \{\bar{I}_w : w \in es(B)\}$. Then the following may be shown:

    (1) J is a subset of some $\bar{I}_w$.

Pf. Since B is a subset of some $\bar{A}_w$.

    (2) If the automorphism $a = \langle a_1, a_2 \rangle$ of $\mathfrak{I}$ is fixed on J then there is an automorphism $\beta = \langle a_1, a_3 \rangle$ of $\mathfrak{A}$ that is fixed on B.

Pf. Suppose a is an automorphism on $\mathfrak{I}$ that is fixed on J. Since $\mathfrak{I}$ underlies $\mathfrak{A}$, there is an automorphism $\beta = \langle a_1, a_3 \rangle$ of $\mathfrak{A}$. Let e be any member of B, with $mv(e) = \langle U, V \rangle$. Then J determines both U and V. For since $\mathfrak{I}$ underlies $\mathfrak{A}$, there is a subset K of some $\bar{I}_w$ such that K determines U and $V = \{w \in W : K \subseteq \bar{I}_w\}$. Now, by definition, $K \subseteq J$. But, as should be clear, K determines V, and so J also determines both U and V. Since a_1 is fixed on J, $a_1(U) = U$ and $a_1(V) = V$. But then $mv(e) = mv(a_3(e))$ for all $e \in B$. So by rearranging members of A with the same modal value, it is possible to find an a_3 that is fixed on B.

    (3) J determines V (w.r.t. the automorphism of $\mathfrak{I}$).

Pf. Suppose the automorphism $a = \langle a_1, a_2 \rangle$ of $\mathfrak{I}$ is fixed on J. By (2) above, there is an automorphism $\beta = \langle a_1, a_3 \rangle$ of $\mathfrak{A}$ that is fixed on B. But since B determines V, $a_1[V] = V$, as required.

    (4) $es(B) = \{w \in W : J \subseteq \bar{I}_w\}$

Pf. If $w \in es(B)$, then $J \subseteq \bar{I}_w$ by the definition of J. For each subset K of I, let $W(K) = \{w \in W : K \subseteq \bar{I}_w\}$. Since $\mathfrak{I}$ underlies $\mathfrak{A}$, there is, for each $e \in B$, a subset K_e of I such that $W(K_e) = es(e)$. Clearly, $J \subseteq K_e$ for each $e \in B$. So $W(J) \subseteq W(K)$ for each such e. But then $W(J) \subseteq \bigcup_{e \in B} W(K)$ $= es(B)$.

From (1), (3) and (4) and the fact that $\mathfrak{I}$ underlies $\mathfrak{A}$, it follows that $\langle V, es(B) \rangle \in MV^{\mathfrak{A}}$, as required.

$\Rightarrow$. Suppose $\mathfrak{A}$ satisfies Automorphism. Intuitively, it would appear to be hard to find an underlying individual structure $\mathfrak{I}$, since many real individual structures could result in the same propositional one. But mathematically the solution is simple, since we may let $\mathfrak{I}$ be $\mathfrak{A}$ itself. Condition (ii) in the definition of 'underlies' is automatically satisfied upon letting $\beta = a$. As for condition (i), suppose that $\langle U, V \rangle \in MV^{\mathfrak{A}}$; so that for some $e \in A$, $mv(e) = \langle U, V \rangle$. Let $J \subseteq I = A$ be $\{e\}$. Then it is readily shown that J is a subset of some $\bar{I}_w$, that I determines e and that $V = \{w \in W: J \subseteq \bar{I}_w\}$. Therefore, $\langle U, V \rangle \in MV(\mathfrak{A})$. Now suppose that $\langle U, V \rangle \in MV(\mathfrak{A})$, so that for some subset B of an $\bar{I}_w$, B determines U and $V = \{w \in W: B \subseteq \bar{A}_w\} = es(B)$. Then by a direct application of the Automorphism condition, $\langle U, V \rangle \in MV^{\mathfrak{A}}$.

The adequacy of Automorphism condition can be discerned from the above result. For given that $\mathfrak{A}$ possesses an underlying individual structure $\mathfrak{I}$, any further determination of $\mathfrak{A}$ must depend upon the specific identity of $\mathfrak{I}$ and the cardinalities of propositions with the same modal value. In the absence of any such information, Automorphism gives the most that can be said of the genuine propositional structure.

Any condition on the underlying structure will have its effect on the propositional structure. We already have an example of this in World Actualism, which transfers from the individual to the propositional structure. Later, in regard to the Extendibility conditions of [6], we shall come across other examples.

4. Consequences of the conditions

In this section, I work out several elementary consequences of the preceding conditions on a structure. Some of these results will be used for later proofs; and some are merely stated for their intrinsic interest. By each result I have stated the conditions upon which its proof depends.

First, we shall give a general result on modal value. Say that two structures $\mathfrak{A} = (W, \bar{A}, t)$ and $\mathfrak{B} = (V, \bar{B}, s)$ are *MV-equivalent* if $W = V$ and $MV^{\mathfrak{A}} = MV^{\mathfrak{B}}$, and that the sequences $e_1, \ldots, e_n \in A$ and $f_1, \ldots, f_n \in B$ are *MV-equivalent* if $mv^{\mathfrak{A}}(e_i) = mv^{\mathfrak{B}}(f_i)$ for $i = 1, \ldots, n$. Then it may be shown, without any conditions on the structure, that·

LEMMA 12. *Suppose that the structures $\mathfrak{A}$ and $\mathfrak{B}$ are MV-equivalent and that $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ are MV-equivalent sequences in $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Then:*

$(\mathfrak{A}, w) \vDash \varphi(e_1, \ldots, e_n)$ iff $(\mathfrak{B}, w) \vDash \varphi(f_1, \ldots, f_n)$ for any identity-free formula $\varphi(\varrho_1, \ldots, \varrho_n)$ of $\mathscr{L}$ and any $w \in W$.

PROOF: By a straightforward induction of φ.

This result helps to explain the significance of modal values. For according to it, there is no harm in talking about modal values instead of propositions in the language $\mathscr{L}$, as long as no use is made of identity. If identity is used, then there may be a shift in truth-value, since $e_1 = e_2$ may be true in $\mathfrak{A}$ even though $f_1 = f_2$ is not true in $\mathfrak{B}$. However, by a careful examination of the role of identity, the above result may be extended.

Say that a formula φ is *loose* if no bound variable flanks an identity--sign. Say that a variable ϱ in φ is *loose* if no free occurrences of ϱ in flank an identity-sign. Then it may be shown that:

LEMMA 13. *Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are MV-equivalent structures, $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ elements in A and B respectively, and $\varphi = \varphi(\varrho_1, \ldots \ldots, \varrho_n)$ a loose formula such that* (i) $mv^{\mathfrak{A}}(e_i) = mv^{\mathfrak{B}}(f_i)$ *whenever ϱ_i is loose in φ, and* (ii) $e_i = e_j$ *iff $f_i = f_j$ whenever both ϱ_i and ϱ_j are not loose in φ. Then:*

$$(\mathfrak{A}, w) \vDash \varphi(e_1, \ldots, e_n) \quad iff \quad (\mathfrak{B}, w) \vDash \varphi(f_1, \ldots, f_n)$$

for any $w \in W$.

PROOF: Again, by induction on φ.

The above result may be strengthened with the help of Diversity, First, it may be shown that each sentence φ is equivalent to another φ^* that does not contain identity. To explain the translation $*$, suppose that $\chi = \forall \varrho \psi = \forall \varrho \psi(\varrho, \sigma_1, \ldots, \sigma_n)$ is an arbitrary universal formula. For each $i = 1, \ldots, n$, let ψ^i be the result of replacing each occurrence of $\varrho = \sigma_i$ or $\sigma_i = \varrho$, with ϱ and σ_i both free in ψ, by $\top = \forall \tau (T\tau \supset T\tau)$; and let ψ^f be the resulting of replacing each occurrence of $\varrho = \sigma_i$ or $\sigma_i = \varrho$, with ϱ and σ_i both free in ψ, by $\bot = -\top$. Let $\chi^+ = \forall \varrho \psi^f \wedge \bigwedge_{i=1}^{n} (E\sigma_i \supset \psi^i(\sigma_i, \sigma_1, \ldots, \sigma_n))$; and for any formula φ, let φ^* be the result of replacing each universal subformula χ of φ by χ^+, working successively outwards. Note that φ^* is always loose and hence is identity-free when φ is a sentence. It may be shown that:

LEMMA 14 (D). *For any elements $e_1, \ldots, e_n$ of the structure $\mathfrak{A}$ and formula $\varphi(\varrho_1, \ldots, \varrho_n)$:*

$$(\mathfrak{A}, w) \vDash (e_1, \ldots, e_n) \quad iff \quad (\mathfrak{A}, w) \vDash \varphi^*(e_1, \ldots, e_n).$$

PROOF: By induction on φ. The key point is to show that if $(\mathfrak{A}, w)$ $\vDash \forall \varrho \psi^*(\varrho, e_1, \ldots, e_n)$ then $(\mathfrak{A}, w) \vDash \psi^{*f}(e_1, \ldots, e_n)$. But this may be shown with the help of the condition D and Lemma 13.

Given two MV-equivalent structures $\mathfrak{A}$ and $\mathfrak{B}$, say that the sequences $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ from A and B respectively are MVI-equivalent if they are MV-equivalent and, whenever $1 \leqslant i < j \leqslant n$, $e_i = e_j$ iff $f_i = f_j$. Then for structures satisfying D, it may be shown that:

LEMMA 15 (D). *Suppose that* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are MV-equivalent, and that* $e_1, \ldots, e_n$ *and* $f_1, \ldots, f_n$ *are MVI-equivalent. Then*:

$$(\mathfrak{A}, w) \vDash \varphi(e_1, \ldots, e_n) \quad \textit{iff} \quad (\mathfrak{B}, w) \vDash \varphi(f_1, \ldots, f_n).$$

PROOF: The following statements are equivalent: $(\mathfrak{A}, w) \vDash \varphi(e_1, \ldots, e_n)$ $\ldots, e_n)$; $(\mathfrak{A}, w) \vDash \varphi^*(e_1, \ldots, e_u)$ (by Lemma 14); $(\mathfrak{B}, w) \vDash \varphi^*(f_1, \ldots, f_n)$ (by Lemma 13 and the looseness of φ); $(\mathfrak{B}, w) \vDash \varphi(f_1, \ldots, f_n)$ (by Lemma 14 again). The result may also be proved by a direct induction.

Say that two models $\mathfrak{M} = (\mathfrak{A}, w)$ and $\mathfrak{N} = (\mathfrak{B}, v)$ are elementarily equivalent if $\mathfrak{M} \vDash \varphi$ iff $\mathfrak{N} \vDash \varphi$ for each sentence φ of $\mathscr{L}$. Then an immediate consequence of the previous lemma is:

LEMMA 16 (D). *Suppose that* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are MV-equivalent and that* $w \in W = V$. *Then the models* $(\mathfrak{A}, w)$ *and* $(\mathfrak{B}, w)$ *are elementarily equivalent.*

What this result showns is that, given D, the only relevance of a structure for the truth-values of the sentences of $\mathscr{L}$ lies in its set of modal values $MV^{\mathfrak{A}}$. The condition D is essential to the truth of this result. Without it the sentence $(\forall \varrho)(\forall \sigma)((\Box T \varrho \wedge \Box T \sigma) \supset \varrho = \sigma)$, for example, might be true in the one model but not the other.

We shall next establish some of the consequences of Automorphism. The first of these results states that the class of propositions is closed under the operations of forming existential propositions, negations and conjunctions or, rather, the class of modal values is closed under the corresponding operations.

LEMMA 17 (A).   (i)   *If* $\langle U, V \rangle \in MV^{\mathfrak{A}}$,   *then* $\langle V, V \rangle \in MV^{\mathfrak{A}}$;

                    (ii)   *If* $\langle U, V \rangle \in MV^{\mathfrak{A}}$ *then* $\langle W - U, V \rangle \in MV^{\mathfrak{A}}$

                    (iii)   *If* $\langle U_i, V_i \rangle \in MV^{\mathfrak{A}}$ *for all i in a non-empty* *set I, then* $\langle \bigcap_{i \in I} U_i, \bigcap_{i \in I} V_i \rangle \in MV^{\mathfrak{A}}$ *as long as* $\bigcap_{i \in I} V_i$ *is non-empty.*

PROOF: (i) & (ii). Suppose that $\langle U, V \rangle \in MV^{\mathfrak{A}}$. Then for some $e \in A$, $mv(e) = \langle U, V \rangle$. Let $B = \{e\}$. Then it is readily shown that B determines both V and $W - U$.

(iii) For each $i \in I$, let $e_i \in A$ be such that $mv(e_i) = \langle U_i, V_i \rangle$. Let $B = \{e_i : i \in I\}$. Then it may be shown that B determines $\bigcap_{i \in I} U_i$ and that $es(B) = \bigcap_{i \in I} V_i$.

Secondly, it may be shown, with the help of World Actualism, that world-propositions necessarily exist:

LEMMA 18. (WA, A). *For any* w, $\{w\} \in TS^{\mathfrak{A}}$.

PROOF: Suppose $w \in W$. Then $\overline{A}_w$ determines $\{w\}$. For suppose that α is an automorphism fixed on $\overline{A}_w$ for which $\alpha_1(w) = v$. Then $\overline{\mathfrak{A}}_w = \overline{\mathfrak{A}}_v$. So by WA, $w = v$. Since $\overline{A}_w$ determines $\{w\}$, $\langle \{w, es(\overline{A}_w)\} \rangle \in MV^{\mathfrak{A}}$, by A.

The above result allows two worlds to be distinguished in terms of propositional truth, and not merely in terms of propositional existence.

The final result says that extraneous objectual content can be added to a proposition without upsetting its truth-conditions:

LEMMA 19 (A).   *If $\langle U, V \rangle$, $\langle U', V' \rangle \in MV^{\mathfrak{A}}$ and $V \cap V'$ is non-empty, then $\langle U, V \cap V' \rangle \in MV^{\mathfrak{A}}$.*

PROOF: Suppose the antecedent. Now $\langle W - U', V' \rangle \in MV$ by Lemma 17(ii), $\langle \psi, V' \rangle \in MV$ by Lemma 19(iii), and $\langle W, V' \rangle \in MV$ by Lemma 17(ii) again. So $\langle U, V \cap V' \rangle \in MV$ by Lemma 17(iii).

5.   A partial axiomatization

It will later turn out that the theory determined by the conditions D, WA and A is not axiomatizable. In this section, we will present a partial axiomatization of the theory, one that is fairly natural in itself and will serve for various working purposes. Each of the axioms is presented in turn, along with some of its consequences.

In determining these consequences, I will usually assume the strong completeness of S5 with respect to the possible worlds semantics. That is, in order to establish $\Delta \vdash \varphi$ I shall show, instead, that $\Delta \vDash \varphi$. But, of course, the derivations themselves could equally well have been presented.

Comprehension.   This takes the form:

$\Box \forall \varrho_1 \ldots \forall \varrho_n \exists \sigma [\Box (T\sigma \equiv \varphi) \wedge \Box (E\sigma \wedge \ldots \wedge E\varrho_n))]$, where $\varrho_1, \ldots, \varrho_n$ ar are *exactly* all the free variables of the formula φ and ϱ itself does not occur free in φ.

This axiom-scheme says, in regard to a condition φ, that necessarily for all propositions $\varrho_1, \ldots, \varrho_n$ there is a proposition σ which is true exactly when $\varrho_1, \ldots, \varrho_n$ satisfy the condition and existent exactly when all of $\varrho_1, \ldots, \varrho_n$ exist. If the condition φ expresses the relation R, then σ may be taken to be the singular proposition to the effect that $\varrho_1, \ldots, \varrho_n$ satisfy R. Alternatively, σ may be taken to be the proposition expressed by the result of substituting rigid names of $\varrho_1, \ldots, \varrho_n$ for the variables of φ.

In the formulation of the axiom, it is essential that the quantifiers $\forall \varrho_1 \ldots \forall \varrho_n \exists \sigma$ not be layered with modal operators. For example, the sentence $\Box \forall \varrho_1 \Box \forall \varrho_2 \Diamond \exists \sigma [\Box (T\sigma \equiv T\varrho_1 \vee T\varrho_2) \wedge \Box (E\sigma \equiv E\varrho_1 \wedge E\varrho_n)]$ is not valid in the intended semantics. To see this intuitively, let ϱ_1 and ϱ_2 be two propositions that cannot co-exist (because their individual constituents cannot co-exist). Then there will be no possibly existing proposition with the same existence-conditions as the disjunction of ϱ_1 and ϱ_2, let alone the same truth-conditions. This intuitive proof may then easily be turned into a formal demonstration.

Note that the axiom-scheme includes both a statement of the truth- and existence-conditions for σ. Thus it is, in the terminology of [9], both

a principle of internal and external existence. In the formulation of modal set theory, these two types of principle could be separated since there was a direct means, in the language, of talking about the constituents of sets. However, in the absence of any comparable rescources in $\mathscr{L}$, it is necessary to combine the two types of principle.

The existence-conditions for σ in the axiom may be justified by direct appear to the cumulation principle. For we may suppose that σ is constructed from $\sigma_1, \ldots \varrho_n$ and hence has, as its objectual content, the union of the objectual contents of $\varrho_1, \ldots, \varrho_n$. However, this application of the principle depends upon the language containing only purely general primitives, such as truth. In an extension of the language which was not of this sort, the existence-condition would need to be appropriately modified.

Let us set up the following abbreviations:

$\sigma \; Ex \; \varrho$ for $\Box (T\sigma \equiv E\varrho) \wedge \Box (E\sigma \equiv E\varrho)$;

$\sigma \; Neg \; \varrho$ for $\Box (T\sigma \equiv -T\varrho) \wedge \Box (E\sigma \equiv E\varrho)$;

$\tau \; Conj \; \varrho, \sigma$ for $\Box \big(T\tau \equiv (T\varrho \wedge T\sigma)\big) \wedge \Box \big(E\tau \equiv (E\varrho \wedge E\sigma)\big)$;

$\tau \; Disj \; \varrho, \sigma$ for $\Box \big(T\tau \equiv (T\varrho \vee T\sigma)\big) \wedge \Box \big(E\tau \equiv (E\varrho \wedge E\sigma)\big)$.

Note that $\sigma \; Neg \; \varrho$ does not express that σ is the negation of ϱ, but merely that σ has the same modal value as the negation of ϱ, and similarly for the other notions. It may be proved, by direct application of Comprehension, that:

LEMMA 20.   *From the Comprehension scheme may be derived*:

$\Box \, \forall \varrho \; \exists \, \sigma (\sigma \; Ex \; \varrho)$;

$\Box \, \forall \varrho \; \exists \, \sigma (\sigma \; Neg \; \varrho)$;

$\Box \, \forall \varrho, \sigma \; \exists \, \tau (\tau \; Conj \; \varrho, \varrho)$;

$\Box \, \forall \varrho, \sigma \; \exists \, \tau (\tau \; Disj \; \varrho, \sigma)$.

Repeated applications of Comprehension are often useful in proof. As an example, let us show that the (compatible) existence-conditions of one proposition can be combined with the existence-conditions of another proposition, without thereby upsetting its truth-conditions.

LEMMA 21. $(\Box) \big[\Diamond (E\varrho \wedge E\sigma) \supset \Diamond \exists \, \tau \big(\Box (T\tau \equiv T\sigma) \wedge \Box \big(E\tau \equiv (E\varrho \wedge E\sigma)\big)\big) \big]$ *is derivable from Comprehension.*

PROOF: Proceeding semantically, suppose that $\mathfrak{A}$ verifies Comprehension and that $e, f \in \bar{A}_w$. By repeated applications of lemma 20, it may be shown that there are g_1, g_2, g in $\bar{A}_w$ such that $g_1 \; Neg \; e$, $g_2 \; Disj \; g_1, e$ and $g \; Conj \; f, g_2$ are all true in $\mathfrak{A}$. But then it is easily shown that $\mathfrak{A} \vDash \Box (Tg \equiv Tf) \wedge \Box \big(Eg \equiv (Ee \wedge Ef)\big)$.

The above proof establishes for the object-language what is established meta-theoretically in Lemma 19.

Covering. This is the following axiom:
$$\Box\forall\varrho\ \Box\big[T\varrho \supset \exists\,\sigma\big(T\sigma \wedge \Box(T\sigma \supset T\varrho)\big)\big].$$

It says that for any true (but possibly non-existent) proposition there is a true and existent proposition which necessarily implies it. As such, it is a kind of actualist demand on propositions. As an example, consider the proposition ϱ that Socrates does not exist. This is true (but non-existent) in certain worlds. By Covering, there is, in each such world, a true but existing proposition σ which implies ϱ. Indeed, if $a_1, a_2, \ldots$ are all of the existing individuals in the world, we may let σ be the proposition that $a_1, a_2, \ldots$ are all of the existing individuals.

Covering implies some other sentences of the same form. Let Covering $(F, E$ and $NE)$ be the sentences $\Box\forall\varrho\ \Box\big(\varphi \supset \exists\,\sigma(T\sigma \supset \sigma)\big)$, for φ the formulas $-T\varrho$, $E\varrho$ and $-E\varrho$, respectively. These sentences say that the falsehood, the existence and the non-existence of propositions are "covered" by the existing propositions in any world. Covering itself may be dubbed "Covering (T)". Then:

LEMMA 22. *Given Compr., Covering (T) provably implies Covering $(F, E$ and $NE)$.*

PROOF: From Compr. can be derived $\Box\forall\sigma\exists\,\varrho\ \Box(T\varrho \equiv \sigma)$, for $\varphi = -T\sigma$, $E\sigma$ or $-E\sigma$. The implications then follow.

From the definition of a world-proposition, it follows that a true world-proposition necessarily implies all true propositions. Given the covering theorems, this result can be extended to all conditions:

THEOREM 23. *The sentences Covering $(T, F, E$ and $NE)$ provably imply $(\Box)[(W\varrho \wedge \varphi) \supset \Box(T\varrho \supset \varphi)]$, for any formula φ not containing the variable ϱ free.*

PROOF: It suffices to establish the theorem for the cases in which $\varphi = T\sigma$, $-T\sigma$, $E\sigma$ or $-E\sigma$; for then it follows that $(\mathfrak{A}, w) \vDash We$ and $(\mathfrak{A}, v) \vDash Te$ imply $\mathfrak{A}_w = \mathfrak{A}_v$, from which the general result follows.

For $\varphi(\sigma)$ in one of these cases, then, suppose that $(\mathfrak{A}, w) \vDash We \wedge \varphi(f)$. By Covering (T) and Lemma 22, there is a $g \in \overline{A}_w$ such that $(\mathfrak{A}, w) \vDash \Box(Tg \supset \varphi(f))$. But by the definition of W, $(\mathfrak{A}, w) = \Box(Te \supset Tg)$. Therefore $(\mathfrak{A}, w) \vDash \Box(Te \supset \varphi(f))$, as required.

Note that although General World-Proposition is a scheme, it follows from a finite number of axioms.

World-Proposition. Recall that this is the axiom $\Box\exists\varrho W\varrho$. It is important to appreciate that this axiom is independent of Covering and has, indeed, a different philosophical basis. World-proposition draws on a form of Platonism according to which there exists a logical product of all the true existing propositions. Covering, on the other hand, draws on a form of Actualism according to which no merely possible proposition can better describe the world than an actual one.

From Theorem 23 and the logical truth of $(\Box)[W\varrho \supset T\varrho]$, it follows that $(\Box)[(W\varrho \wedge \varphi) \supset \Box(W\varrho \supset \varphi)]$ is derivable from the Covering sentences. Therefore the conditions (i) and (ii) of Lemma 4 in [9] are satisfied for the theory with Comprehension, World-Proposition and Covering as axioms. Hence the possible worlds semantics may be represented within that theory, as explained in Lemma 4, Corollary 5, Lemma 6, Corollary 7 and Corollary 8 of that paper.

We shall not need this representation here, but shall merely use the following definitions:

$\Pi\varrho\varphi$ for $\exists\sigma\big(W\sigma \wedge \Box\forall\varrho \ \Box(W\sigma \supset \sigma)\big)$,   where σ is not free in φ;

$Q\varrho$ for $\Diamond W\varrho$;

$T(\varphi, \varrho)$ for $\Box(T\varrho \supset \varphi)$;   and

$E(\sigma, \varrho)$ for $\Box(T\varrho \supset \exists\sigma)$.

In order to make the role of variables qualified by Q clear, we shall usually use $a, b, c, \ldots$ for them, instead of $\varrho, \sigma, \tau, \ldots$. A more detailed account of the reduction of possible worlds to propositions may be found in [5], though the discussion there does not have the benefit of the general results in [9].

Conjunctive Closure. We may define when a proposition has the same truth- and existence-conditions as the conjunction of all (actual) propositions ϱ which satisfy a certain condition ψ, by putting:

$\sigma \ Conj_\varrho \psi$ for $\forall a\Big[Wa \supset \Big(\Box\big(T\sigma \equiv \Pi\varrho\big((T(\psi, a) \wedge E(\varrho, a)) \supset T\varrho\big)\big) \wedge$

$\wedge \ \Box\big(E\sigma \equiv \Pi\varrho\big((T(\psi, a) \wedge E(\varrho, a)) \supset E\varrho\big)\big)\Big)\Big]$.

Note the use of Wa to secure back-reference to the actual world. The second conjunct then says that σ is true exactly when all of the propositions which exist and satisfy the condition ψ in the actual world are true, and the third conjunct that σ exists when all of the above-mentioned propositions exist. One might equally well have used $\exists a(Wa \wedge \ldots)$ in place of $\forall a[Wa \supset \ldots]$ in the above axiom, but it would have then not been independent of World-Proposition.

The axiom-scheme of *Conjunctive Closure* is:

$(\Box) \ [\exists\sigma(\sigma \ Conj_\varrho\psi)]$, where σ is a variable distinct from ϱ and not free in ψ.

A special case of the above axiom-scheme may be obtained from Comprehension. This case is obtained by altering the prefix $(\Box)$ to $\Box\forall\varrho_1 \ldots$ $\ldots \forall\varrho_n$ and by dropping the third conjunct in the definition of $Conj_\varrho$. The derivation then proceeds by letting φ in Comprehension be the formula $\Pi\varrho\big((T(\psi, a) \wedge E(\varrho, a)) \supset T\varrho\big)$ and by supposing (intuitively) that a is a true world-proposition. However, as we shall see, there is no way

of deriving the full scheme, with arbitrary parameters, from the other axioms.

It is important to distinguish between the proposition which (in a given world) is the conjunction of all propositions which satisfy a condition $\psi(\varrho)$ and the proposition expresses by $\forall\varrho(\psi(\varrho) \supset T\varrho)$. Since the domain of propositions can vary, these propositions may differ in both their truth- and existence-conditions. For example, if $\psi(\varrho)$ is the condition T, then the conjunction (in a given world) is a true world-proposition, perhaps existing contingently, whereas the other propositions is a necessary existent and truth.

Diversity. The final axiom-scheme is Diversity, as in section 2.

Putting all of the axioms together (Comprehension, Covering, World--Proposition, Conjunctive Closure and Diversity) gives the theory DV (D for diversity, V for varying domain). Various results within the full theory are of interest. Define arbitrarily disjunction, in analogy to arbitrary conjunction, by:

$$\sigma \; Disj_\varrho\psi \;\; \text{for} \;\; \forall a\Big[Wa \supset \Big(\square\big(T\sigma \equiv \Sigma\varrho\big(T(\psi, a)\wedge E(\varrho, a)\wedge$$

$$\wedge\, T\varrho\big)\wedge \square\big(E\sigma \equiv \Pi\varrho\big((T(\psi, a)\wedge E(\varrho, a))\supset E\varrho\big)\big)\Big)\Big].$$

Then the closure of the actual propositions under arbitrary disjunctions may be established:

LEMMA 24. ($\square$) $[\exists\sigma(\sigma \; Disj_\varrho \; \psi)]$, *for* σ *a variable distinct from* ϱ *and not free in* ψ, *is a theorem of DV.*

PROOF: Use Conjunctive Closure and the closure under negation from Comprehension.

Given the closure principles for conjunction and disjunction, it can be shown that for any condition there is an actual proposition which approximates to the satisfaction of the condition from either above or below:

LEMMA 25. *For* φ *a formula in which the distinct variables* σ *and* τ *are not free, the following two sentences are theorems of DV:*

   (i)   ($\square$) $\Big[\exists\sigma\Big(\square(T\sigma \supset \psi)\wedge\forall\tau\big(\square(T\tau \supset \psi) \supset \square(T\tau \supset T\sigma)\big)\Big)\Big]$

   (ii) ($\square$) $\Big[\exists\sigma\Big(\square(\psi \supset T\sigma)\wedge\forall\tau\big(\square(\psi \supset T\tau) \supset \square(T\sigma \supset T\tau)\big)\Big)\Big].$

PROOF: For (i), apply Conjunctive Closure to the formula $\square(T\varrho \supset \psi)$; and for (ii), apply Lemma 23 to the formula $\square(\psi \supset T\varrho)$.

From Conjunctive Closure, it may also be shown that for any non--empty set of worlds (as given by a condition), there is a smallest existence-set to contain the given set:

LEMMA 26. *For σ and τ distinct variables not free in φ, the sentence:*

$$(\Box) \left[\Diamond\varphi \supset \Sigma\sigma\Big(\Box(\varphi \supset E\sigma) \land \Pi\tau\big(\Box(\varphi \supset E\tau) \supset \Box(E\sigma \supset E\tau)\big)\Big)\right]$$

is a theorem of DV.

PROOF: Choose a world at which φ is true, and then, within that world, apply Conjuncive Closure to $\Box(\varphi \supset E\varrho)$.
It then follows that in any world there is a proposition with maximum objectual content:

LEMMA 27. $\Box \exists\sigma \, \forall\tau \Box (E\sigma \supset E\tau)$ *is a theorem of DV.*

PROOF: Given the true world-proposition ϱ, apply Lemma 26 to the case in which φ is $T\varrho$.

The world-relative notion of indiscernibility may be defined within the object-language by putting:

$$b \equiv_a c \text{ for } (\Pi\tau)\big(E(\tau, a) \supset T(\tau, b) \equiv T(\tau, c)\big).$$

The indiscernibility criterion (Theorem 10) for the existence of propositions may then be derived within the object-language.

LEMMA 28. *The sentence:*

$$(\Box) \Big[Qa \supset \Sigma\varrho'\big(\varrho' \approx_t \varrho \land E(\varrho', a) \equiv \Pi b \Pi c\big((Qb \land Qc$$
$$\land T(\varrho, b) \land b \equiv_a c\big) \supset T(\varrho, c)\big)\big)\Big]$$

is a theorem of DV.

PROOF: The left-to-right direction is straighforawrd. To establish the other direction, show first that $(\Box) \Big[\Pi a, b(Qa \land Qb \supset \Sigma\varrho_b\big(E(\varrho_b, a)$ $\land \Pi c\big(Qc \supset \big(T(\varrho_b, c) \equiv (b \equiv_a c)\big)\big)\big)\Big]$ is a theorem by letting ϱ_b be the conjunction of all propositions σ in a that satisfy the condition $T(\sigma, b)$. Then let ϱ' in the lemma be the disjunction of all the propositions ϱ_b for which $T(\varrho, b)$ holds.

6. Some Meta-Theorems

In this section we shall establish some standard meta-logical results for the theory DV and its intended semantics. It will be shown that DV is sound, that its axioms are independent, that the theory DV itself is not decidable, and that the theory for the intended semantics is not even axiomatizable.

Soundness. To establish the validity of World-Proposition and Conjunctive Closure, we need to establish that various defined notions have their intended meaning.

LEMMA 29. *Suppose that* $\mathfrak{A}$ *satisfies the condition of* lemma 18, *viz. that* $\{w\} \in TS_w^{\mathfrak{A}}$ *for each* $w \in W$. *Then:*

(i) $(\mathfrak{A}, w) \vDash We$ *iff* $ts(e) = \{w\}$

(ii) $(\mathfrak{A}, w) \vDash \Pi\varrho\ \varphi(e)$ *iff* $(\mathfrak{A}, w) \vDash \varphi(e)$ *for all* e *in* A;

(iii) *If* $(\mathfrak{A}, v) \vDash We$, *then* $(\mathfrak{A}, w) \vDash T(f, e)$ *iff* $v \in ts(f)$;

(iv) *If* $\mathfrak{A}, v) \vDash We$, *then* $(\mathfrak{A}, w) \vDash E(f, e)$ *iff* $v \in es(f)$;

(v) $(\mathfrak{A}, (w) \vDash f\ \mathrm{Conj}_\varrho\ \psi(e, e_1, \ldots, e_n)$ *iff* $ts(f) = \bigcap\{ts(e) : e \in \overline{A}_w$ *and* $(\mathfrak{A}, w) \vDash \psi(e, e_1, \ldots, e_n)\}$ *and* $es(f) = \bigcap\{es(e) : e \in \overline{A}_w$ *and* $(\mathfrak{A}, w) \vDash \psi(e, e_1, \ldots, e_n)\}$

PROOF: Straightforward.

To establish the validity of Comprehension, we also need to show that the language $\mathscr{L}$ respects automorphisms:

LEMMA 30. *Suppose that* $a = \langle a_1, a_2 \rangle$ *is an automorphism on* $\mathfrak{A}$ *and that* $e_1, \ldots, e_n$ *are elements of* $\mathfrak{A}$. *Then:*

$$(\mathfrak{A}, w) \vDash \varphi(e_1, \ldots, e_n) \quad iff \quad (\mathfrak{A}, a_1(w)) \vDash \varphi(a_2(e_1), \ldots, a_2(e_n)).$$

PROOF: By a straightforward induction of φ.

THEOREM 31. (Soundness of *DV*). *The axioms Diversity, World-Proposition, Covering, Conjunctive Closure and Comprehension are true in any structure which satisfies the conditions of Diversity, Automorphism and World Actualism.*

PROOF: The truth of Diversity in structures which satisfy the corresponding condition is clear. In structures satisfying A and WA, the truth of World-Proposition follows from lemmas 18 and 29(i), the truth of Covering from Lemma 18 alone, and the truth of Conjunctive Closure from lemmas 18, 17(iii) and 29(v). As for Comprehension, choose any formula $\varphi = \varphi(\varrho_1, \ldots, \varrho_n)$, structure $\mathfrak{A}$, world w in W, and elements $e_1, \ldots, e_n$ of $\overline{A}_w$. Let $V = \{v \in W : v \vDash \varphi(e_1, \ldots, e_n)\}$ and $B = \{e_1, \ldots, e_n\}$. Then B determines V. For let $a = \langle a_1, a_2 \rangle$ be an automorphism that is fixed on B and suppose that $v \in V$, i.e. that $v \vDash \varphi(e_1, \ldots, e_n)$. By lemma 30, $a_1(v) \vDash \varphi(a_2(e_1), \ldots, a_2(e_n))$; and so, by a fixed on B, $a_1(v) \vDash \varphi(e_1, \ldots, e_n)$. But then $a_1(v) \in V$ and V is determined by B. By the automorphism condition, $\langle V, es(B) \rangle \in MV^{\mathfrak{A}}$. Therefore there is an $f \in A$ for which $mv(f) = \langle V, es(B) \rangle$ and, giving σ the value f, shows that Comprehension is satisfied.

Although condition WA was used in establishing the truth of World--Proposition and Conjunctive Closure, its use, in fact, is not essential. On the other hand, a rather extensive use is made of condition A in establishing the truth of all but the Diversity axioms.

Independence. It will be useful, in establishing Independence, to be able to ignore the Diversity axioms. This is achieved by means of the following result:

LEMMA 32. *Let $\mathfrak{A}$ be a structure and $\mathfrak{B}$ an MV-equivalent structure which satisfies the Diversity axioms. Then $\mathfrak{A}$ verifies World-Proposition or Covering or an instance of either Comprehension or Conjunctive Closure just in case $\mathfrak{B}$ does.*

PROOF: Since World-Proposition and Covering are identity-free, the result for these axioms follows from Lemma 12. For Comprehension, let $\varphi(\varrho_1, \ldots, \varrho_n)$ be an arbitrary formula, w a member of W, and $e_1, \ldots, e_n$ elements of B. By Lemma 14, there is an identity-free formula $\varphi^*(e_1, \ldots, e_n)$ such that:

(1) $\mathfrak{B} \vDash \Box \big(\varphi(e_1, \ldots, e_n) \equiv \varphi^*(e_1, \ldots, e_n)\big).$

Therefore the statement:

(2) $(\mathfrak{B}, w) \vDash \exists \sigma \big[\Box \big(T\sigma \equiv \varphi(e_1, \ldots, e_n)\big) \wedge \Box \big(E\sigma \equiv (Ee_1 \wedge \ldots \wedge Ee_n)\big)\big]$

is equivalent to the statement:

(3) $(\mathfrak{B}, w) \vDash \exists \sigma \big[\Box \big(T\sigma \equiv \varphi^*(e_1, \ldots, e_n)\big) \wedge \Box (E\sigma \equiv (Ee_1 \wedge \ldots \wedge Ee_n)\big)\big].$

But then $\mathfrak{B}$ will verify Comprehension iff $\mathfrak{B}$ verifies all identity-free instances of Comprehension. By Lemma 12, $\mathfrak{B}$ will verify an identity-free instance of Comprehension iff $\mathfrak{A}$ does, and so the result for Comprehension is established.

The proof for Conjunctive Closure is similar.

THEOREM 33. *The five axioms of DV are independent.*

PROOF: Let us establish the independence of each axiom in turn. By Lemma 32, Diversity need not be considered in establishing the independence of the other axioms.

Diversity. Let $\mathfrak{A} = (W, \overline{A}, t)$, where $W = \{w\}$, $\overline{A}_w = \{e, f\}$ and $t = \{\langle e, w \rangle\}$. Then it is readily shown that $\mathfrak{A}$ verifies all of the axioms, but Diversity.

World-Proposition. Let $\mathfrak{A} = (W, A, t)$, where A is an atomless set-algebra on W, each $\overline{A}_w$ is A, and $t = \{\langle V, w \rangle : V \subseteq W$ and $w \in V\}$. By Proposition 3 of [11], $\mathfrak{A}$ verifies Comprehension and, given its constant domain, $\mathfrak{A}$ also verifies Covering. Since A is atomless, $\mathfrak{A}$ trivially verifies Conjunctive Closure and fails to verify World-Proposition.

Covering. Let $\mathfrak{A} = (W, \overline{A}, t)$ be a structure verifying the T-Criterion ($\Box$) $[\varrho \approx_t \sigma \supset \varrho = \sigma)$ for which $W = \{1, 2, 3\}$, $TS_1 = TS_2 = \{\psi, \{1, 2\}, \{3\}, \{1, 2, 3\}\}$ and $TS_3 = \mathscr{P}(W)$. Then it may be shown that $\mathfrak{A}$ establishes the independence of Covering. The only difficult case is Comprehension, for which it must be shown that, for $e_1, \ldots, e_n \in \overline{A}_1 = \overline{A}_2$, $(\mathfrak{A}, 1) \vDash \varphi(e_1, \ldots, e_n)$ iff $(\mathfrak{A}, 2) \vDash \varphi(e_1, \ldots, e_n)$.

Comprehension. Let $\mathfrak{A}$ be the structure $(W, \overline{A}, t)$, where $W = \{w\}$, $\overline{A}_w = \{e\}$, and $t = \{\langle e, w \rangle\}$.

Conjunctive Closure. This is the most difficult case. Let $\mathfrak{A} = (W, \overline{A}, t)$ be a structure verifying the T-Criterion for which $W = \{0, 1, 2, \ldots\}$, $\overline{A}_0 = TS_0 = \{V \subseteq W : (1 \in V \ \& \ V \text{ is co-finite}) \text{ or } (1 \notin V \ \& \ V \text{ is finite})\}$, and $\overline{A}_k = TS_k = \mathscr{P}(W)$ for all $k > 0$. It is readily shown that $\mathfrak{A}$ verifies World-Proposition and Closure.

To take care of Comprehension, some preliminary results are required. Let a_1 be a permutation on W. Then a_1 induces, in the obvious way, a permutation a_2 on $\mathscr{P}(W)$. The first result is:

(1) if a_1 is a permutation for which $a_1(0) = 0$ and $a_1(1) = 1$ and if a_2 is the permutation on $\mathscr{P}(W)$ induced by a_1, then $\langle a_1, a_2 \rangle$ is an automorphism on the structure $\mathfrak{A}$.

PROOF: Straightforward.
Say that w, v are *indiscernible w.r.t.* a subset X of $\mathscr{P}(W)$ if, for all $V \in X$, $w \in V$ iff $v \in V$. Then:

(2) for $V_1, \ldots, V_n \subseteq W, w, v > 1$, and w, v indiscernible w.r.t. $\{V_1, \ldots \ldots, V_n\}$, $(\mathfrak{A}, w) \vDash \varphi(V_1, \ldots, V_n)$ iff $(\mathfrak{A}, v) \vDash \varphi(V_1, \ldots, V_n)$

PROOF: By induction on φ. For the case in which $\varphi(V_1, \ldots, V_n)$ is of the form $\exists \varrho \psi(\varrho, V_1, \ldots, V_n)$, (1) is required. For suppose $w \vDash \exists \varrho \psi(\varrho, V_1, \ldots, V_n)$. Then $w \vDash \psi(V, V_1, \ldots, V_n)$ for some $V \subseteq W$. Let $a = \langle a_1, a_2 \rangle$ be the permutation which interchanges w and v. By (1) and Lemma 30, $a_1(w) = v \vDash \psi(a_2(V), a_2(V_1), \ldots, a_2(V_n))$. But since w, v are indiscernible w.r.t. $\{V_1, \ldots, V_n\}$, $a_2(V_i) = V_i$ for $i = 1, \ldots, n$, and so $v \vDash \exists \varrho \psi(\varrho, V_1, \ldots, V_n)$.
From (2) it follows that:

(3) if $V_1, \ldots, V_n \in \overline{A}_0$, then the set $\{w \in W : w \vDash \varphi(V_1, \ldots, V_n)\}$ is either finite or co-finite.
Given (3), it may readily be shown that $\mathfrak{A}$ verifies Comprehension.

As for Conjunctive Closure, let ψ be the formula $\Box(T\tau \supset T\varrho)$. Then if the axiom is verified, $(\mathfrak{A}, 0) \vDash \exists \sigma(\sigma \operatorname{Conj}_\varrho \Box(T\{1\} \supset T\varrho))$. But, by Lemma 29(v), this requires that $\{1\} \in TS_0$, which is not so.

Decidability. There are two results, one for the theory DV itself, and the other for its intended semantics.

THEOREM 34. *The theory DV is undecidable.*

PROOF: It is readily shown that the first-order theory T of a reflexive anti-symmetric and transitive relation R is undecidable — indeed, that it is equivalent in undecidability to full first-order logic. The theory T can then be embedded in DV. To show this, let:

$$R\varrho \quad \text{abbreviate} \quad \Diamond(W\varrho \wedge \forall \sigma \Box(E\varrho \supset E\sigma)).$$

Thus R is true of those propositions that possibly have maximal truth and objectual content. Given a sentence φ of T, let φ^* be the result of

replacing each atomic formula Rxy in φ by $E(\varrho, \sigma)$, i.e. by $\square(E\sigma \supset E\varrho)$, each identity $x = y$ by $\varrho =_{te} \sigma$, and each quantifier $\exists x$ by the relativized quantifier $\Sigma\varrho(R\varrho \wedge \ldots)$. It then follows that:

(∗) the classical sentence φ is a theorem of T iff φ^* is a theorem of DV. The proof of the left-to-right direction is straigtforward. To establish the other direction, suppose that φ is not a theorem of T. Then φ is false for some reflexive and transitive ordering $(W, \leqslant)$. Let $\mathfrak{A} = (W, \bar{A}, t)$ be a modal structure satisfying Diversity for which $MV_w = \{\langle U, V\rangle: U, V \subseteq W$ and V contains w and is $\leqslant$-closed, i.e. $u \in V$ & $u \leqslant v$ implies $v \in V\}$. Then it may be shown that the axioms of DV are verified by $\mathfrak{A}$, that φ^* itself is not, and that, consequently, φ^* is not a theorem of MC,

THEOREM 35. *The theory* P *of all structures satisfying conditions* D. WA, *and* A *is not axiomatizable.*

PROOF: Let T^+ be the second-order theory of a reflexive, anti-symmetric and transitive relation R with greatest element. (The quantification is over arbitrary sets and all valid sentences are to be theorems.) Extend the translation ∗ in the proof of Theorem 34 to sentences φ of T^+ by replacing each membership formula $x \in X$ by $\square(T\varrho \supset T\varrho')$ and each set quantifier $\exists X$ by $\Sigma\varrho'$. Then it may be shown that:

(†) the sentence φ is a theorem of T^+ iff $\Diamond \Pi\varrho\, E\varrho \supset \varphi^*$ is a theorem of P.

The proof of the left-to-right direction is relatively straightforward. (But note that the condition $\Diamond \Pi\varrho E\varrho$ is required in order that the quantifiers $\Sigma\varrho'$ should, in effect, range over all sets of worlds.) The other direction may be established by the same construction as before.

Finite Axiomatizability. If the axiom $(\square)E\varrho$ for Necessary Existence is added to DV, then the resulting system is finitely axiomatizable. (This follows with the help of Proposition 3 in [4].) It is then natural to wonder whether the original theory DV is itself finitely axiomatizable. Some instances of Comprehension and Closure have great deductive power. For example, from $\square\exists\varrho\square(T\varrho \equiv \forall\sigma(\sigma = \sigma))$ and $\square\forall\sigma\square\exists\varrho\square(T\varrho \equiv \exists\tau\Diamond(-E\tau \wedge T\sigma))$ can be derived $\square\exists\varrho\square(T\varrho \equiv \exists\varrho_1-(-E\sigma_1\Diamond\exists\sigma_2(-E\sigma_2\wedge \ldots \ldots \Diamond -E\sigma_n)\ldots))$ for all $n = 1, 2, \ldots$. However, I suspect, although I have no proof, that the theory DV is not finitely axiomatizable and that, indeed, neither of the schemes for Comprehension or Conjunctive Closure can be replaced by finitely many instances in the presence of the other. It may also be true, though this is a stronger claim, that neither of the schemes, in the presence of the other, can have its parameters restricted to a fixed finite number. This would be in contrast to classical (though not perhaps modal) set theory, in which the use of n-tuples enables one to manage with one parameter.

7. Extensions

There are four types of extension to the theory DV that we shall consider. The first two result from adding propositional abstracts and quantifiers over sets of propositions to the language. The remaining two result from generalising on the comprehension scheme or from adding other axioms to the theory.

Propositional Abstracts. We shall use § as a symbol for propositional abstraction. If φ is a sentence, $\S\varphi$ will denote the proposition expressed by φ. For example, '§ Grass is green' will denote the proposition that grass is green. Given the admission of genuinely singular propositions, the application of § to a *formula* φ is not problematic. For given that φ expresses a relation, $\S\varphi$ will denote, for a given assignment of values, the proposition to the effect that those values satisfy the relation. For example, '§ x is mortal' will express the singular proposition to the effect that Socrates is mortal, when Socrates is the value of the variable x.

Propositional, like class, abstracts may not denote or, at least, not denote an object in the range of the variables, be they actualist or possibilist. Suppose, for example, that ϱ and σ are propositions that cannot co-exist. Then the abstract $\S(E\varrho \wedge E\sigma)$ will not denote a possibly existing proposition. Such abstracts may be assigned a denotation, but it must then be a virtual proposition, i.e. one outside of the range of the quantifiers.

Virtual classes may be introduced in the same way. However, the reasons for positing virtual objects are less compelling in the case of propositions than of sets. For the supposition that all class abstracts denote a real set leads to contradiction, whereas the corresponding supposition for propositional abstract leads only to the conclusion that all propositions necessarily exist.

The possibility that propositional abstracts may not denote or, at least, not denote a real proposition is of some philosophical interest. For it has been thought that, with the help of §, any sentential connectives C might be eliminated in favour of a predicate c of propositions, with $c\S\varphi_1 \ldots \S\varphi_n$ being used in place of $C\varphi_1 \ldots \varphi_n$. But if the predicates only apply to real propositions, such an elimination will not, in general, work; for all abstracts which denote virtual propositions (or not at all) will have to be treated in some uniform manner. There will be no way, for example, of distinguishing between the truth of $\Diamond(E\varrho \vee E\sigma)$ and the falsehood of $\Diamond(E\varrho \wedge E\sigma)$ in case ϱ and σ cannot co-exist.

In another respect, class and propositional abstracts differ; for whereas the former will not, in general, denote rigidly, the latter will. The reason for this difference is that the denotation of class abstract $\{x : \varphi\}$ is determined on the basis of the extension of φ, which may vary from world to world, whereas the denotation of a propositional abstract $\S\varphi$

is determined by the intension of φ, which remains constant in the different worlds.

Let us add the symbol § to our previous modal language $\mathscr{L}$. The formation rules are extended in the obvious way, with §φ being a term when φ is a formula. In particular, iterations of § are allowed, so that §T§$T\varrho$, for example, is a legitimate term.

A theory T with Comprehension and Modal-Criterion as theorems may be extended to the new language by adding the following axioms:

(i) ($\square$) [T §$\varphi \equiv \varphi$];

(ii) ($\square$) [E §$\varphi \equiv (E\varrho_1 \wedge \ldots \wedge E\varrho_n)$]$, where $\varrho_1, \ldots, \varrho_n$ are exactly the free variable s to occur in φ;

(iii) ($\square$) [$s = t \equiv s \approx_{te} t$], where either s or t is an abstract.

(In the underlying logic, it should be supposed that each term is a rigid designator, but not necessarily of a possible. Thus Specification $(\forall \varrho \varphi(\varrho) \wedge \wedge Et) \supset \varphi(t)$ is valid whenever t is a term free for ϱ in $\varphi(\varrho)$, but $\diamond Et$ is not valid for all terms. In the sequel I shall think semantically, although a strongly complete logic could easily be provided.)

By using the translation suggested by the above axioms, it may be shown that each formula is provably equivalent to one without abstracts and that the new theory is a conservative extension of the original one (i.e. no new theorems without abstracts are provable).

In the manner of Fine [9] or, originally, of Scott [19], any structure $\mathfrak{A} = (W, \overline{A}, t)$ for the initial theory T may be extended in a natural way to a structure $\mathfrak{B} = (W, B, \overline{A}, s)$ for the new theory (where B serves as the domain of all objects, virtual and real). With each pair $p = \langle U, V \rangle$ for which $U, V \subseteq W$, associate an object $o(p)$ in such a way that:

(a) $p \neq q$ implies $o(p) \neq o(q)$;

(b) $o(p)$ is the entity e in A if $p = mv^{\mathfrak{A}}(e)$, and $o(p) \notin A$ otherwise. Let $B = \{o(p)\colon p = \langle U, V \rangle$, and either $o(p) \in A$ or $(o(p) \notin A$ and $V = \psi)\}$, and let $s = \{\langle w, e \rangle\colon w \in W, e \in B$ and w belongs to the first component of $o^{-1}(e)\}$. In setting up the truth-definition for $\mathfrak{B}$, let the denotation of $\varphi(e_1, \ldots, e_n)$, for $e_1, \ldots, e_n \in A$, be $o(\langle \{w \in W\colon (\mathfrak{A}, w) \vDash \varphi(e_1, \ldots, e_n)\}$, $\{w \in W\colon e_1, \ldots, e_n \in \overline{A}_w\}\rangle)$. It may now be shown that the structure $\mathfrak{B}$ verifies the equivalences (i)-(iii) listed above (thereby providing a new proof of the conservative extension result).

If Modal Criterion is not a theorem of the initial theory, then the adoption of (ii) and (iii) becomes problematic. The problem with (ii) is not its truth, but the requirement, imposed by the Specification scheme $(\forall \varrho \ \varphi(\varrho) \wedge Et) \supset \varphi(t)$, that all existents fall within the range of the quantifier. All that Comprehension asserts is the existence of a proposition with the same modal value as §φ, but not the existence of §φ itself. However, if the variables are supposed to range over all propositions or if §φ is merely supposed to denote one of the propositions with a given modal value, then this problem will not arise.

The problem with (iii) is more serious; for in the absence of Modal Criterion, it is not clear what to put in its place. Indeed, an adequate solution requires a detailed account of the identity of propositions. On certain structuralist views, it may be possible to give a reduction for identities between abstracts. For example, if propositional identity is tied to sentential structure, we might adopt the following reductive theses:

(iv) $(\Box)$ $[(\S\varphi = \S\psi) = \bot$, if φ and ψ are formulas which do not only differ in their free variables;

(v) $(\Box)$ $(\S\varphi = \S\psi) \equiv (\varrho_1 = \sigma_1 \wedge \ldots \vee \varrho_n = \sigma_n)$, where φ and ψ only differ in their free variables and where $\varrho_1, \ldots, \varrho_n$ and $\sigma_1, \ldots, \sigma_n$ are the free variables of φ and ψ respectively in their order of occurrence (countaing each distinct occurrence separately).

However, even in these more favourable cases, it does not seem possible to give a reduction for identities of the form $\varrho = \S\varphi$. Such reductions would seem to require the introduction of a vocabulary for describing the structure of an arbitrary proposition.

   Although elimination of $\S$ cannot be effected in many of these systems, it is still possible to prove conservative extension results by semantical methods. Let us illustrate in the case of a theory with Diversity and Comprehension as axioms. Let $\mathfrak{A} = (W, \overline{A}, t)$ be a countable structure for such a theory, and let $\mathscr{L}_A$ be the result of enriching the original language $\mathscr{L}$ with the objects of A as constants. With each sentence φ of $\mathscr{L}_A$, associate an object $o(\varphi)$ in such a way that:

   (a) $\varphi \neq \psi$ implies $o(\varphi) \neq o(\psi)$;

   (b) $o(\varphi)$ is an entity e in A such that $ts(e) = \{w \in W: (\mathfrak{A}, w) \vDash \varphi\}$ and $es(e) = \{w \in W:$ all objects of A in φ belong to $w\}$ if there is such an entity and $o(\varphi) \notin A$ otherwise;

   (c)   the range of o contains A.

Let $\mathfrak{B}$ be the structure $(W, B, \overline{A}, s)$ in which $B = \{o(\varphi): \varphi$ a sentence of $\mathscr{L}_A\}$ and $s = \{\langle w, e\rangle: w \in W, \ e \in B$ and $(\mathfrak{A}, w) \vDash o^{-1}(e)\}$. In setting up the truth-definition for $\mathfrak{B}$, let the denotation of $\S\varphi$, for φ a sentence of $\mathscr{L}_A$, be $o(\varphi)$. Then it may be shown that $\mathfrak{B}$ verifies the original theory and, in addition, the axioms (i), (ii) and (iv) listed above.

   Sets of Propositions. The presence of Comprehension and World-Proposition somewhat mar the simplicity and elegance of our system. However, following the suggestion of Fine [5], p. 121, the use of these axioms may be avoided by introducing quantification over sets of propositions. In order for the theory to reamin one-sorted, a new style of variable $x, y, z \ldots$, ranging indifferently over propositions and sets of propositions is required. In addition, there are two new predicates: a monadic predicate S for being a set of propositions; and a dyadic predicate for membership. Let us use:

   Px for $-Sx$.

Then given the intended range of the variables, P is the predicate for being a proposition. The new language will be called $\mathscr{L}^s$.

We shall use $\varrho, \sigma, \tau, \ldots$ for those variables which are relativized (in the usual way) to propositions, and $X, Y, Z, \ldots$ for those variables which are relativized to sets. We shall also suppose that the language contains both possibilist and actualist quantifiers as primitive. The axioms of the theory (to be dubbed DV^s) are then Covering along with:

Comprehension^s. $\Box \forall x_1, \ldots, \forall x_n \exists \sigma \lceil \Box (T\sigma = \varphi) \wedge \Box \lceil E\sigma = (Ex_1, \wedge \ldots \wedge$ $\wedge Ex_n) \rceil \rceil$, where φ is a formula of $\mathscr{L}^s$, $x_1, \ldots, x_n$ are exactly the free variables of φ, and σ itself does not occur free in φ;

Type $\quad\quad\quad (\Box)\ [x \in y \supset (Px \wedge Sy)]$

Rigidity $\quad\quad (\Box)\ [Sx \supset \Box Sx]$

$\quad\quad\quad\quad\quad (\Box)\ [x \in y \supset \Box x \in y]$

Existence $\quad\quad (\Box)\ [EX = \Pi\varrho(\varrho \in X \supset E\varrho)]$

Abstraction $\quad (\Box)\ \exists X \forall \varrho(\varrho \in X \equiv \varphi)$, for φ any formula of $\mathscr{L}^s$ in which X is not free;

Extensionality. $\quad \Box \forall X \forall Y\ \big(X = Y \equiv \forall \varrho(\varrho \in X \equiv \varrho \in Y)\big)$.

If possibilist quantifiers are not used, then an equivalent system can be obtained by replacing the Existence axiom with the two axioms $(\Box)$ $[EX \supset (\varrho \in X \supset E\varrho)]$ and $(\Box)\ \forall X \lceil \forall \sigma \in X \Box (T\varrho \supset E\sigma) \supset \Box (T\varrho \supset EX)) \rceil$. I shall not give an exact statement or proof of the result, but shall merely refer the reader to the proof of an analogous result in Theorem 3(ii) of [9].

The above axiom system is very natural and, as we shall later show (Corollary 38), both World-Proposition and Conjunctive Closure are derivable from it. From this point of view, then, the presence of these axioms in DV arise from certain inadequacies in the langauge $\mathscr{L}$.

Pseudo-Classes. The effect of some quantification over sets of propositions can be introduced into DV either by means of class abstracts or, more directly, by means of rigid conditions or what I shall call pseudo-classes. Their introduction then allows Conjunctive Closure to be absorbed into Comprehension. To understand what pseudo-classes are, suppose that φ is a formula of $\mathscr{L}^s$ in which all set variables are free and occur in atomic contexts of the form $\varrho \in X$. Fix on a variable a that does not occur in φ, and let φ' be the result of replacing the atomic formulas $\varrho \in X$ in φ by formulas of the form $T(\psi, a) \wedge E(\varrho, a)$, where the free variables of ψ, other than ϱ, do not already occur in φ. Then each formula ψ, as used above, is a *pseudo-class*, the variable ϱ its *argument*, and the free variables of ψ, other than ϱ, its *parameters*. The formulas φ' themselves are said to be *regular in a*. (But note: the pseudo-classes and their parameters may depend upon the underlying formula φ of $\mathscr{L}^s$.)

The extension of Comprehension and Conjunctive Closure may now be stated:

General   Comprehension: $\Box\forall\sigma_1\Box\ldots\Box\forall\sigma_m\Box\forall a\forall\varrho_1\ldots\varrho_n\exists\sigma\Big[Wa$

$$\supset\Big(\Box(T\sigma\equiv\varphi')\wedge\Box\Big(E\sigma\equiv\Big(E\varrho_1\wedge\ldots\wedge E\varrho_n\wedge\bigwedge_{i=1}^{1}\varPi\tau i\big((T(\psi_i(\tau_i),\tau)\wedge E(\tau_i,\tau)\big)$$

$\supset E\tau_i\big)\big)\big)\Big]$, where $\sigma_1,\ldots,\sigma_m,\varrho_1,\ldots,\varrho_n$ are exactly the free variables of φ', each distinct from the two variables σ and τ, and where φ is a formula regular in a in which $\psi_1,\ldots,\psi_l$ are the pseudo-classes with respective arguments $\tau_1,\ldots,\tau_l$ and, collectively, parameters $\sigma_1,\ldots,\sigma_m$.

Let DV^+ be the result of replacing Comprehension and Conjunctive Closure in DV with General Comprehension. Then it readily shown that:

THEOREM 36.   *All theorems of DV are also theorems of DV^+.*

PROOF: Comprehension is obtained by supposing that φ' contains no pseudo-classes, and Closure is obtained by letting φ be $\varPi\varrho(\varrho\in X\supset T\varrho)$ and φ' be $\varPi\varrho\big((T(\psi,a)\wedge E(\varrho,a))\supset T\varrho\big)$.

I suspect that the converse of Theorem 36 is not true and, indeed, that DV^+ cannot be obtained by adding finitely many axioms to DV. Let ψ be the sentence $\Box\forall\sigma\Box\exists a\exists\tau\big[Wa\wedge\Box\big(T\tau\equiv\varPi\varrho(\chi\supset\exists\varrho'(\Diamond T\varrho'\wedge\wedge\Box(T\varrho'\supset T\varrho))\big)\big]$, where is the formula $T\big(\Diamond T\varrho\wedge\Box(T\varrho\supset T\sigma)\wedge\forall\varrho_1((\Diamond\Diamond T\varrho_1\wedge\Box(T\varrho_1\supset T\sigma)\supset\Box(T\varrho\supset T\varrho_1)),a\big)\wedge E(\varrho,a)$. Then I conjecture, in particular, that ψ is not a theorem of DV.

Let DV^s — be the result of requiring of the formulas in the abstraction and comprehension schemes of DV^s that all of the variable-occurrences to the right of ε be free. (No internal quantification over sets.) To each formula φ of $\mathscr{L}$ may be associated a formula φ^s of $\mathscr{L}^s$ in the obvious way, be relativizing all bound variables to propositions. It may now be shown that:

THEOREM 37.   *A sentence φ of $\mathscr{L}$ is a theorem of DV^+ iff φ^s is a theorem of DV^s—.*

PROOF: $\Rightarrow$. Proceeding semantically (via the completeness proof for modal logic), it must be shown that any structure $\mathfrak{A}$ for DV^s — verifies φ^s for each axiom of DV^+. If φ is World-Proposition, then it must be shown, for each world w of $\mathfrak{A}$, that $(\mathfrak{A},w)\vDash\exists\varrho W\varrho$. By (the verification of) Abstraction, there is a set f in $\bar{A}_w$ such that $(\mathfrak{A},w)\vDash e\in f=Te$ for each proposition e in $\bar{A}_w$. By Comprehension^s, there is a proposition g in $\bar{A}_w$ such that $\mathfrak{A}\vDash\Box\big(Tg\equiv\varPi\varrho(\varrho\in f\supset T\varrho)\big)$. By the rigidity and existence axioms, $(\mathfrak{A},v)\vDash\varPi\varrho(\varrho\in f\supset T\varrho)$ iff for all $e\in\bar{A}_w$, $(\mathfrak{A},v)\vDash Te$ whenever $(\mathfrak{A},w)\vDash Te$. But then $(\mathfrak{A},w)\vDash Wg$.

For General Comprehension, the crucial point is to show that for any formula $\psi(a, \sigma_1, \ldots, \sigma_k)$ of $\mathscr{L}$, world w, and propositions $e_1, \ldots, e_k$ in $\mathfrak{A}$, there is a set f in $\mathfrak{A}$ such that $\mathfrak{A} \vDash g \in f$ iff $w \vDash \psi(g, e_1, \ldots, e_k) \wedge Eg$, and $f \in \bar{A}_v$ iff $g \in \bar{A}_v$ for all g such that $w \vDash \psi(g, e_1, \ldots, e_k) \wedge Eg$. But the existence of such a g follows from the set-theoretic axioms.

$\Leftarrow$. Suppose φ is not a theorem of DV^+. Then there is a structure $\mathfrak{A} = (W, \bar{A}, t)$ for DV^+ which fails to verify φ. We may suppose, without loss of generality, that the members of A are not sets. Say that a subset B of A is *definable* if for some formula $\varphi(\varrho)$ of $\mathscr{L}_A$ and world w, $B = \{e \in \bar{A}_w : (\mathfrak{A}, w) \vDash \varphi(e)\}$. Now define a structure $\mathfrak{B}$ for the extended language $\mathscr{L}^s$ in the obvious way by letting the worlds be W, the propositions in w be $\bar{A}_w$, the sets in w be the definable sets $B \subseteq \bar{A}_w$, and membership be standard. Then it may be shown that $\mathfrak{B}$ is a structure for DV^s — which fails to verify φ^s. In order to verify the restricted forms of Comprehension^s and Abstraction, it is necessary to use pseudo-classes in place of sets.

From theorems 36 and 37, it follows that:

COROLLARY 38. *Every theorem of DV is a theorem of DV^s.*

I conjecture that the full system DV^s is not finitely axiomatizable relative to DV^s — and that it is not a conservative extension of DC^+. In the light of such conjectures, it should be of some interest to gauge the effects of placing different restrictions on the formula φ in the schemes of Comprehension^s and Abstraction.

Other Extensions. Further axioms may be added to DV to reflect various philosophic viewpoints. Rather than discuss such extensions systematically, let me merely give some examples. First, even on an objectualist conception of propositions, it might be supposed that each proposition was necessarily equivalent to a purely general one. The sentence $\Box \forall \varrho \exists \sigma(\sigma \approx_i \varrho \wedge \Box E\sigma)$ should then be adopted as an axiom. Secondly, various assumptions might be made about the cardinality of propositions. For example, if it is supposed that there are infinitely many possible worlds (or, equivalently, infinitely many possibly existing propositions), then the sentence $\Diamond \exists \varrho_1 \ldots \Diamond \exists \varrho_n \left(\bigwedge_{1 \leqslant i < j \leqslant n} \varrho_i \neq \varrho_j \right)$ should be adopted as an axiom for $n = 1, 2, \ldots$. Thirdly, actualist doctrines give rise to axioms that go beyond Covering. These will be considered towards the end of the next section. Finally, the use of sets along with structuralist news on propositions leads to various Russel-type paradoxes. Their solution, though, is a large topic and shall not be considered.

8. Truth and actualism

According to one form of actualism, mere possibles do not have any genuine properties and do not enter into any genuine relations, either among themselves or with actuals. Since mere possibles do have properties (being non-actual, for example) and do enter into relations, it is natural

for the actualist of the sort described to attempt to explain these properties and relations in terms of the genuine ones. There are various forms such explanation may take, but perhaps the simplest is through definition. To take a typical case, a property of mere possibles is defined in terms of the genuine properties that the possibles would have were they actual.

The request for such explanation leads, then, to the demand that all relations should be defined in terms of those that are actualist, i.e. to those that only hold, in each possible world, of the actuals of that world. A more general discussion of this demand is given in [10]. In application to the present paper, it means that the truth-property of propositions should be defined in terms of actualist relations alone.

Given a relation R, let R^+ be its actualist restriction, i.e. that actualist relation which agrees with R on the actuals of each possible world. Then it is natural to attempt to define R^+ in terms of R. Sometimes this can easily be done. For example, $x \in y$ can be defined as $\Diamond(x \in^+ y)$. (See p. 133 of [5].) However, no definition of T can be given in terms of T^+ alone.

To make this result precise, let $\mathscr{L}^+$ be the language with the two predicates T and T^+. Extend the truth-definition for a propositional structure $\mathfrak{A} = (W, \overline{A}, \tau)$ to the language $\mathscr{L}^+$ by adding the clause:

$$w \vDash T^+ e \quad \text{iff} \quad \langle w, e \rangle \in t \quad \text{and} \quad e \in \overline{A}_w.$$

We may mark the distinction between the two truth-predicates by saying that a proposition may be true *of* (T) or true *in* (T$^+$) a world. Thus the proposition that Socrates does not exist is true of worlds in which Socrates does not exist, but not true in such worlds.

Say that T is *definable* in the structure $\mathfrak{A}$ if there is a formula $\varphi(\varrho)$ which lacks T and for which $\mathfrak{A} \vDash (\square)[T\varrho \equiv \psi(\varrho)]$. Then it may be shown that:

THEOREM 39. *If the sentence* $\varphi^* = (\square)\big[\big(\square(T^+ \varrho \equiv T^+ \sigma) \wedge \square\ (E\varrho \equiv E\sigma)\big) \supset \square(T\varrho \equiv T\sigma)\big]$ *is false in the structure* $\mathfrak{A}$, *then* T *is not definable in* $\mathfrak{A}$.

PROOF: If φ^* is false in $\mathfrak{A}$, there are $e, f \in A$ and $w \in W$ for which $\mathfrak{A} \vDash \square(T^+ e \equiv T^+ f) \wedge \square(Ee \wedge Ef)$ and yet $(\mathfrak{A}, w) \vDash -(Te \equiv Tf)$. It is readily shown by an appropriate induction that $(\mathfrak{A}, w) \vDash \varphi(e) \equiv \varphi(f)$ for each T-free formula $\varphi(\varrho)$ of $\mathscr{L}^+$. But then if (*) is satisfied, $(\mathfrak{A}, w) \vDash Te \equiv Tf$ — which is a contradiction. Although the above result has only been stated for the predicate T of $\mathscr{L}^+$, it admits of an obvious extension to an arbitrary predicate of any language.

For structures verifying Comprehension, a particularly simple necessary and sufficient condition for the definability of T may be given:

THEOREM 40. *Suppose that* $\mathfrak{A}$ *verifies Comprehension (more exactly, its instances* $\Box \forall \varrho \; \exists \sigma \big(\Box (T\sigma \equiv -E\varrho) \wedge \Box (E\sigma \equiv E\varrho) \big)$ *and* $\Box \forall \sigma \; \exists \sigma \big(\Box (T\sigma \equiv (E\varrho \wedge -E\varrho)) \wedge \Box (E\sigma \equiv E\varrho) \big)$. *Then* T *is definable in* $\mathfrak{A}$ *iff* $\mathfrak{A} \vDash \Box \forall \varrho \; \Box E\varrho$.

PROOF: The right-to-left direction is trivial, since then $T\varrho$ may be defined as $T^{+}\varrho$. Now suppose that $\mathfrak{A} \not\vDash \Box \forall \varrho \; \Box E\varrho$, so that for some $w \in W$ and $g \in A$, $w \not\vDash Eg$. By $\mathfrak{A}$'s verifying the two instances of Comprehension, there are e and f in A for which $\mathfrak{A} \vDash \Box (Te \equiv -Eg)$, $\Box (Tf \equiv (Eg \wedge -Eg))$, $\Box (Ee \equiv Eg)$, $\Box (Ef \equiv Eg)$. Now $w \vDash Te$ and $w \vDash Tf$, and so e and f are distinct. Therefore they may be used to show that the sentence φ^{*} of theorem 39 is false in $\mathfrak{A}$.

In the light of this result, it would be of some interest to determine the T^{+}-fragments of the various systems which have Comprehension but not the Barcan formula.

Although T cannot be defined from T^{+} alone, it can be defined with the help of a new primitive for actualist strict implication, as long as the underlying modal structure is subject to certain actualist constraints. Let $\mathscr{L}^{*}$ be the language which contrains T^{+} and the two-place predicate $\Rightarrow^{+}$ as primitives. The predicate $\Rightarrow^{+}$ is subject to the following clause in the truth-definition:

$$ w \vDash e \Rightarrow^{+} f \quad \text{iff} \quad e, f \in \bar{A}_{w} \quad \text{and} \quad ts(e) \subseteq ts(f). $$

Note that the application of the predicate requires the existence of e and f, but the inclusion of their non-existential truth sets. The relation $\varrho \Rightarrow^{+} \sigma$ can be defined, in terms of T, as $E\varrho \wedge E\sigma \wedge \Box (T\varrho \supset T\sigma)$. But the definition $E\varrho \wedge E\sigma \wedge \Box (T^{+}\varrho \supset T^{+}\sigma)$ is clearly inadequate, and it will later follow from the positive result that no other definition of $\Rightarrow^{+}$ in terms of T^{+} will do.

Since the definition of T in the language $\mathscr{L}^{*}$ is rather complicated, let us, first of all, explain the underlying idea. Suppose that the proposition ϱ does not exist in the world w. Now ϱ exists in some world v. So ϱ will be constructed from certain individuals $a_{1}, a_{2}, \ldots, b_{1}, b_{2}, \ldots$, where some of them, $a_{1}, a_{2}, \ldots$, exist in both w and v, while the others, $b_{1}, b_{2}, \ldots$ exist in v alone. It is then plausible to suppose that there is a proposition τ which is constructed from $a_{1}, a_{2}, \ldots$ alone, which strictly implies ϱ when conjoined with the assumption that $b_{1}, b_{2}, \ldots$ do not exist, and which is true in w if ϱ is. Given that this is so, the non-existential truth of ϱ may be defined in terms of the existential truth of τ.

To convey this idea in the language $\mathscr{L}^{*}$, we must eliminate the reference to individuals. This requires some preliminary definitions, which are set out below. Each definition is accompanied by a statement of the relevant semantic facts concerning the defined notion. It should be as-

sumed, in order to establish these facts, that the given structure $\mathfrak{A}$ verifies all of the axioms of the system DV.

The original definition of world-proposition used T. An almost equivalent definition in T^+ is:

D1 $W^+\varrho$ for $T^+\varrho \wedge \forall\sigma(T^+\sigma \supset \varrho \Rightarrow^+ \sigma)$.

F1 $(\mathfrak{A}, w) \vDash We$ iff $e \in \bar{A}_w$ and $ts(e) = \{w\}$.

F1′ $\mathfrak{A} \vDash \square \exists \varrho (W^+\varrho)$.

There is no direct way of saying in the language $\mathscr{L}^*$ that a proposition ϱ and sentence φ have the same truth-set. To some extent, the same facts may be expressed by using the notions of a minimum and of a maximum proposition:

D2   (σ is the minimum proposition satisfying the condition $\varphi = \varphi(\varrho, \varrho_1, \ldots, \varrho_n)$): $\sigma \min_\varrho \varphi$ for $E\sigma \wedge \varphi(\sigma, \varrho_1, \ldots, \varrho_n) \wedge \forall\varrho(\varphi \supset \varrho \Rightarrow^+ \sigma)$, where σ is distinct from ϱ and is free for ϱ in φ.

F2 $(\mathfrak{A}, w) \vDash e \min_\varrho \varphi(\varrho, e_1, \ldots, e_n)$ iff $e \in \bar{A}_w$, $(\mathfrak{A}, w) \vDash \varphi(e, e_1, \ldots, e_n)$, and, for all $f \in \bar{A}_w$ for which $(\mathfrak{A}, w) \vDash \varphi(f, e_1, \ldots, e_n)$, $ts(e) \subseteq ts(f)$.

D3   (σ is the maximum proposition satisfying the condition $\varphi = \varphi(\varrho, \varrho_1, \ldots, \varrho_n)$): $\sigma \max_\varrho \varphi$ for $E\sigma \wedge \varphi(\sigma, \varrho_1, \ldots, \varrho_n) \wedge \forall\varrho(\varphi \supset \sigma \Rightarrow^+ \varrho)$, where σ is distinct from ϱ and is free for ϱ in φ.

F3 $(\mathfrak{A}, w) \vDash e \max_\varrho \varphi(\varrho, e_1, \ldots, e_n)$ iff $e \in \bar{A}_w$, $(\mathfrak{A}, w) \vDash \varphi(e, e_1, \ldots, e_n)$, and, for all $f \in \bar{A}_w$ for which $(\mathfrak{A}, w) \vDash \varphi(f, e_1, \ldots, e_n)$, $ts(e) \subseteq ts(f)$.

With the help of max, the relation and property of being an existential proposition can be defined:

D4   (ϱ an existential proposition of σ): $\varrho \, Ex^+ \sigma$ for $E\varrho \wedge E\sigma \wedge \square (E\varrho \supset E\sigma) \wedge \varrho \max_\tau \square (E\sigma \supset T^+\tau)$.

F4 $(\mathfrak{A}, w) \vDash e \, Ex^+ f$ iff $e, f \in \bar{A}_w$ and $ts(e) = es(e) = es(f)$.

F4′ $\mathfrak{A}, \vDash \square \forall\sigma \exists \varrho(\varrho \, Ex^+ \sigma)$.

D5   (ϱ is an existential proposition): $Ex^+\varrho$ for $\exists\sigma(\varrho \, Ex^+ \sigma)$.

The conjunction and negation of propositions are not directly definable in $\mathscr{L}^*$. However, we may give lattice-theoretic definitions of when a conjunctive proposition implies and a negative proposition is implied.

D6 $\varrho, \sigma \Rightarrow^+ \tau$ for $E\varrho \wedge E\sigma \wedge E\tau \wedge \forall\zeta[(\zeta \Rightarrow^+ \varrho \wedge \zeta \Rightarrow^+ \sigma) \supset \zeta \Rightarrow^+ \tau]$.

F5 $(\mathfrak{A}, w) \vDash e, f \Rightarrow^+ g$ iff $e, f, g \in \bar{A}_w$ and $ts(e) \cap ts(f) \subseteq ts(g)$.

D7 $\varrho \Rightarrow^+ F\sigma$ for $E\varrho \wedge E\sigma \wedge \forall\tau(\varrho, \sigma \Rightarrow^+ \tau)$.

(In this definition, the group of symbols ′$\Rightarrow^+ F$′ must, of course, be taken as a whole.)

F7 $(\mathfrak{A}, w) \vDash e \Rightarrow^+ Ff$ iff $e, f \in \bar{A}_w$ and $ts(e) \cap ts(f) = \psi$.

We shall need a rather special notion, which expresses that σ is true just when all of the actual propositions which do not exist in a specified world do not, in fact, exist:

D8 σDa (σ is a distinguishing proposition for a) for $\sigma \min_\tau \forall \varrho \big((Ex^+\varrho \wedge \\ \wedge \Box (T^+a \supset -E\varrho)) \supset \tau \Rightarrow^+ F\varrho \big)$.

F7   Given that $(\mathfrak{A}, w) \vDash Wf$, then $(\mathfrak{A}, w) \vDash eDf$ iff $e \in \bar{A}_w$ and $ts(e)$ $= \{u \in W : \text{for all } g \in \bar{A}_w - \bar{A}_v,\ g \notin \bar{A}_u\}$.

F7' $\mathfrak{A} \vDash \Box \forall a (Wa \supset \Box \exists \sigma (\sigma\, Da))$.

(Note: in the proof, Conjunctive Closure must be applied to the condition $Ex^+\varrho \wedge \Box(T^+a \supset -E\varrho)$. Since a may not exist in the world in question, there is no obvious way of using Comprehension instead.)

The final definition is of truth:

D9 $T^o\varrho$ for $\exists a \Big[W^+ a \wedge \Diamond \exists b \big(W^+ b \wedge E\varrho \vee \exists \sigma (\sigma\, Da \wedge E\tau (\sigma, \tau \Rightarrow^+ \varrho \wedge \\ \wedge \Box(T^+a \supset T^+\tau))) \big) \Big]$.

The adequacy of the definition may now be proved:

THEOREM 41. *Suppose that $\mathfrak{A}$ is a structure for the system DV and verifies, in addition, the sentence:*

$\varphi^A = (\Box) \Big[\big(\Diamond W^+ a \wedge \Box(T^+a \supset T\varrho) \wedge E\varrho \wedge \sigma Da \big) \supset \exists \tau \big(\sigma, \tau \Rightarrow^+ \varrho \wedge \Box \\ (T^+a \supset T^+\tau) \big) \Big]$.

Then $(\mathfrak{A}, w) \vDash T_e^o$ iff $w \in ts(e)$.

PROOF: $\Rightarrow$. Suppose $(\mathfrak{A}, w) = T_e^o$. Then By F1, there is a v in W, an e_w in $\bar{A}_w$ and e_v, f, g in $\bar{A}_v$ such that $ts(e) = \{w\}$, $ts(e_v) = \{v\}$, $(\mathfrak{A}, v)$ $\vDash fDe_w \wedge f, g \Rightarrow^+ e \wedge \Box(T^+ e_w \supset T^+ g)$. Since $(\mathfrak{A}, v) \vDash \Box(T^+ e_w \supset T^+ g), w \in ts(g)$ by F1. Also $w \in ts(f)$ by F8. But $(\mathfrak{A}, v) \vDash f, g \Rightarrow^+ e$, and therefore $w \in ts(e)$.

$\Leftarrow$. Suppose that $w \in ts(e)$. By F1', there is an $e_w \in \bar{A}_w$ for which $w \vDash W^+ e_w$. Suppose that $e \in \bar{A}_v$. Then by F1 again, there is an $e_v \in \bar{A}_v$ for which $v \vDash W^+ e_v \wedge Ee$. By F8', there is an $f \in \bar{A}_v$ for which $v \vDash f\, Da$. Now $v \vDash \Diamond W^+ e_w \wedge \Box(T^+ e_w \supset Te) \wedge Ee \wedge f\, Da$. So by the verification of φ^A, $v \vDash \exists \tau (f, \tau \Rightarrow^+ e \wedge \Box(T^+ e_w \supset T^+\tau))$. But then $w \vDash Te$.

The above proof of the definability of T rests upon $\mathfrak{A}$ verifying certain axioms of DV, which themselves contain T. It is not clear, though I have no proof, whether the same or otherwise adequate conditions on $\mathfrak{A}$ can be formulated without the aid of T. Take, as an example, the sentence $\Box \forall \sigma \forall \tau \exists \varrho \big(\Box (T\varrho \equiv (T\sigma \wedge T\tau)) \big)$ expressing closure under conjunction. In the language $\mathscr{L}^*$ we can say $\Box \forall \sigma \forall \tau \exists \varrho (\varrho \min_\zeta (\zeta \Rightarrow^+ \sigma \wedge \zeta \Rightarrow^+ \tau))$. But this merely asserts the existence of a minimum implier of two propositions, which may not have the same truth-conditions as their conjunction.

The sentence φ^A is, of course, ad hoc, and so it is of some interest that it is equivalent to the following more natural postulate:

Actualist Distinction. $(\Box)$ $\big[\big(Qa \wedge Qb \wedge Qc \wedge E(\varrho, a) \wedge T(\varrho, b) \wedge -T(\varrho, c) \wedge$
$\wedge \Box \forall \sigma \big((E(\sigma, a) \wedge E(\sigma, b)) \equiv (E(\sigma, a) \wedge E(\sigma, c))\big)\big) \supset \Diamond \exists \tau \big(E(\tau, a) \wedge E(\tau, b) \wedge$
$\wedge T(\tau, b) \wedge -T(\tau, c)\big)$.

THEOREM 42. *In DV, AD is provably equivalent to φ^A.*

PROOF: $\Rightarrow$. Proceeding semantically, suppose that $w \vDash W^+ e_w$ and $v \vDash \Box(T^+ e_w \supset Te) \wedge Ee \wedge fDe_w$. By Conjunctive Closure, there is a $g \in \bar{A}_v$ for which $v \vDash g \ conj_\varrho \ (E(\varrho, e_w) \vee T(\varrho, e_w))$. Clearly, $v \vDash \Box(T^+ e_w \supset T^+ g)$. Now suppose $v \cdot \vDash f$, $g \Rightarrow^+ e$. Then by F6, for some u, $u \in ts(f) \cap ts(g)$ but $u \notin ts(e)$. Now by F8, $\bar{A}_u \cap \bar{A}_w \subseteq \bar{A}_v \cap \bar{A}_w$. If the inclusion is proper, there is an h in $\bar{A}_v \cap \bar{A}_w - \bar{A}_u$ and so, by Comprehension, there is a g' (with $ts(g') = es(g') = es(h)$) in $\bar{A}_v \cap \bar{A}_w$ for which $v \vDash Tg'$ and $u \vDash Tg'$. If the inclusion is improper, then by AD, there is again a g' in $\bar{A}_v \cap \bar{A}_w$ for which $v \vDash Tg'$ and $u \vDash Tg'$. But in either case there is a conflict with the defining property of g.

$\Leftarrow$. Suppose that $e \in \bar{A}_w$, $v \in ts(e)$, $u \notin ts(e)$ and $\bar{A}_w \cap \bar{A}_v = \bar{A}_w \cap \bar{A}_u$. By F1$'$, there is an $e_v \in \bar{A}_v$ for which $v \vDash W^+ e_v$ and, by F8$'$, there is an f in $\bar{A}_w$ for which $w \vDash f De_v$. By φ^A, there is a g in $\bar{A}_w$ for which $w \vDash f$, $g \Rightarrow^+ e \wedge \Box(T^+ e_v \supset T^+ g)$. From F8, it follows that $u \vDash f$. Since $u \notin ts(e)$ and $w \vDash f$, $g \Rightarrow^+ e$; $u \notin ts(g)$ by F5. But since $w \vDash \Box(T^+ e_v \supset T^+ g)$, $v \in ts(g)$ and $g \in \bar{A}_v$ by F1, and the proof is complete.

The assumption AD is itself quite plausible. It says that if two worlds can be externally distinguished, i.e. by a proposition that exists in a third or external world, then they may be internally distinguished, i.e. by a proposition which exists in the external world and also in one of the two given worlds. The assumption AD only deals with the case in which the two worlds share the same propositions from the external world. But the other cases are easily taken care of, for then a proposition e exists in the external world but in only one of the two given worlds, and so the distinguishing proposition can be to the effect that e exists.

The truth of AD can be derived from properties of the underlying individual structure. Let AD^* be the condition:

if $\alpha = \langle \alpha_1, \alpha_2 \rangle$ is an automorphism of $\mathfrak{I}$ for which $\alpha_1(v) = u$ and $J \subseteq I$ a set for which $\bar{I}_v \cap J = \bar{I}_u \cap J = \varphi$, then there is an automorphism $\beta = \langle \beta_1, \beta_2 \rangle$ such that $\alpha_2 | \bar{I}_v = \beta_2 | \bar{I}_v$, β_2 is fixed on J, and $\beta_1(v) = u$. It can then be shown that:

THEOREM 43. *Given that $\mathfrak{I}$ underlies $\mathfrak{A}$ and satisfies AD^*, $\mathfrak{A}$ verifies AD.*

PROOF: Suppose that $e \in \bar{A}_w$, $v \in ts(e)$ and $u \notin ts(e)$, and that $\bar{A}_w \cap \bar{A}_v = \bar{A}_w \cap \bar{A}_u$. It follows that $\bar{I}_w \cap \bar{I}_v = \bar{I}_w \cap \bar{I}_u$. For suppose otherwise,

say $i \in \bar{I}_w \cap \bar{I}_v - \bar{I}_w \cap \bar{I}_u$. Let $U = \{t \in W \colon i \in \bar{I}_t\}$. Then $\{i\}$ determines U. So for some $f \in A$, $mv(f) = \langle U, U \rangle$. But then $f \in \bar{A}_w \cap \bar{A}_v - \bar{A}_w \cap \bar{A}_u$, contrary to supposition.

Let $V = \{t \in W \colon a_1(v) = t$ for some automorphism $\langle a_1, a_2 \rangle$ of $\mathfrak{I}$ which is fixed on $\bar{I}_w \cap \bar{I}_v = \bar{I}_w \cap \bar{I}_u\}$. Clearly, $v \in V$. However, $u \notin V$. For suppose otherwise. Then for some automorphism $\langle a_1, a_2 \rangle$ fixed on $\bar{I}_w \cap \bar{I}_v$, $a_1(v) = u$. Let $J = \bar{I}_w - \bar{I}_w \cap \bar{I}_v$. By AD^*, there is an automorphism $\beta = \langle \beta_1, \beta_2 \rangle$ of $\mathfrak{I}$ such that $a_2 | \bar{I}_v = \beta_2 | \bar{I}_v$, β_2 is fixed on J and $\beta_1(v) = u$. Since a_2 is fixed on $\bar{I}_w \cap \bar{I}_v$, β_2 is fixed on $\bar{I}_w$. Now $e \in \bar{A}_w$; so $\beta_2(e) = e$; and so $\beta_2(es(e)) = es(e)$. But then either both or neither of v and u are members of $es(e)$, contrary to supposition.

Clearly, V is determined by $\bar{I}_w \cap \bar{I}_v$. Therefore there is an f in A such that $ts(f) = V$ and $es(f) = es(\bar{I}_w \cap \bar{I}_v) \supseteq \{w, v\}$.

The condition AD^* itself follows from the Strong Extendibility Condition on page 156 of [6]. It is thus reassuring that the actualist's demands on the definition of truth should only have required postulates that were independently plausible from the actualist point of view. A characteristic case in which Strong Extendibility holds is when Extendibility and the Falsehood Principle hold. If Extendibility holds, then whenever there is a proposition e of w which distinguishes between v and u, there is a non-modal one, i.e. one with the same truth-set as a non-modal sentence φ of the infinitary language with constants from $\bar{I}_w$. Given that the Falsehood Principle holds, there is then a simple and uniform way of obtaining an internal distinguishing proposition f. For suppose that φ is of the form $\varphi(i_1, i_2, \ldots)$, where $i_1, i_2, \ldots$ are the individuals of $\bar{I}_w$ which appear in φ. Let φ^* be the result of replacing each atomic sentence of φ in which one of the individuals belongs to $\bar{I}_w - \bar{I}_v$ by a standard necessary falsehood, say $\exists \varrho(-\varrho = \varrho)$. Then one may let f be the proposition expressed by φ^*.

Theorem 41 is rather particular. It states that truth (T) is definable from T^+ and $\Rightarrow^+$ when the underlying structure satisfies some rather special conditions. It would be of interest to have more general information on which combinations of primitive and conditions permit a definition of T.

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