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FIRST-ORDER MODAL THEORIES III – FACTS

This paper forms the third part of a series on the development and study of first-order modal theories. It was not originally intended for this issue, but is relevant to Prior's work in two main ways. First, it does not treat modal logic as a mere technical exercise, but attempts to relate it to common philosophical concerns. This was an approach that Prior himself adopted and perhaps did more than anyone else to foster. Secondly, the paper deals with the more specific topic of facts. This was a matter upon which Prior had definite views and upon which he had written extensively – in relation to the definition of necessity ([25]), the semantics for the modal system Q ([26]), and the correspondence theory of truth ([27] and [29]). I have found all of these writings useful and, although I have disagreed with him on several points, the influence of his views on my own should be evident.

It is therefore with respect and affection that I dedicate this paper to his memory.

The paper falls into two main parts, one philosophical and the other technical. Either may be read independently of the other, but both are required for an all-round view. The first part is in two sections. One attempts to show that a modal first-order theory of facts is viable, and the other discusses its principles and their bearing on various philosophical issues. The second part is in six sections, which fall into three groups. Those of the first group (§§3–4) deal with the modal theory of possible worlds, both in itself and in its application to other subject-matter. Since I regard worlds as very big facts, it is only natural that they should be considered in this paper. The next section (§5) deals with the theory of facts under the anti-objectualist assumption that they contain no individual constituents. The sections of the last group (§§6–8) deal with facts under objectualist assumptions and include a statement of the appropriate objectualist conditions, a proof of their equivalence to the corresponding conditions for propositions, and an account of the resulting theories. It will be helpful, and

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sometimes essential, to have the earlier parts of the series ([15] and [16]) at hand.

In the technical part of this paper, I have concentrated on the question of finding a correct essentialist theory of facts. As far as I know, very little work has been done in this direction, although there is a start in [46]. On the other hand, there is now a fair amount of material on facts as a subject, not of object-theory, but of semantical metatheory (see [45], [21] and [43], for example). I do not wish to dispute the interest of this material, either for logic or the philosophy of language; but it will not fall within the purview of the paper.

1. PRELIMINARIES

Before the task of formalizing the theory of facts is begun, it will be helpful to stress some preliminary points concerning the means by which the various principles are to be formalized. The first point is that there is singular or nominal reference to facts; the second is that facts are not true propositions, but a distinctive kind of entity; and the last is that the obtaining of a fact is merely a form of existence. Let us consider each point in turn.

Nominal reference is reference by means of an expression in nominal position. It is either definite, with a name referring to a particular object, or indefinite, with a variable ranging over a class of objects. Now on the face of it, there is nominal reference to facts of both varieties in ordinary usage. When we say "the fact that the Watergate bugging took place is disgraceful" or "whatever fact he uncovers is unlikely to help," we seem to be using "the fact that the Watergate incident took place" as a name for a particular fact and the phrase "whatever fact" as part of the nominal quantificational apparatus for referring to facts; and similarly for a multitude of other expressions.

Even locutions that might appear to suggest an alternative view conform readily to the nominalist theory. With the construction 'it is a fact that *p*' for example, it seems most natural to follow the transformational grammarians and to take it as a transform of 'that *p* is a fact', and then to suppose, not too recklessly one hopes, that the phrase 'that *p*' does indeed refer to a fact when the whole statement is true.

This doctrine of singular reference stands in opposition to two other views. One is that we have reference of sorts to facts but that it is carried by expressions in sentential position, either sentences themselves, in the case of definite reference, or what are effectively sentential variables, in the case of indefinite reference. The other view is that there is no reference of any sort to facts, either nominal or sentential, definite or indefinite.

Of these two opposing views, the latter is most implausible. Surely when we say that not all of the relevant facts have been considered or that the recently discovered fact will prove critical, there is reference of some sort to facts, a reference that will show up either in the use of nominal or sentential variables.

The other view is more plausible and, ever since it was propounded by Russell [36], has found favor with a large number of philosophers.[1] But it has never been made clear what, other than a philosophical prejudice against facts, stands in the way of taking the linguistic data at its face value, as evincing singular reference. The view is also subject to difficulties of its own. If the quantifiers for facts were sentential, one would expect the sentence 'whatever fact Holmes discovered, Watson will be surprised that it' to be grammatical, and not with just 'it obtains' in place of 'it'. One might adopt the ingenious pro-sentential theory of Belnap et al. [4] to overcome this particular difficulty. I am not sure this helps altogether; but there are other difficulties. We do not say 'some facts might be false', even though on the sentential theory it should be true, and we say such things as 'He considered the books on the subject but not the recently discovered facts', requiring a syntactic ambiguity of 'He considered -----' as predicate and connective. Given that there is such a simple and natural alternative, it seems better to drop a view that is so unnatural and complicated and that has only a certain philosophical prejudice in its favour.

This is not to say, though, that the sentential theory may not serve various philosophical purposes. I have no objection, as such, to denominalizing fact talk as a way of getting rid of an ontological commitment to facts. But I distinguish sharply between the task of discerning our ordinary commitments and the task of retrenching on those commitments (see §A2 of [14]). My criticisms are of the sentential theory as an account of what gets retrenched, not as a retrenchment.

Although I shall emphasize this and subsequent linguistic points, it should not be thought that the feasibility of my project of formalising the theory of facts depends completely on them. Let it be granted that my most adamant opponent is right and that there is no reference of any sort to facts in ordinary discourse. Then I would be in the same position as a philosopher whose native language lacked even any alleged reference to facts. So my theory could not possess the interest or rationale of dealing with entities with which we are already familiar. I would prefer my theory to have that interest, and this is why I have stressed the linguistic points. But the theory may have another interest altogether. For we may suppose that facts have been *introduced* by philosophers to satisfy certain theoretical demands. Perhaps it is thought that it is only by positing certain entities that the notion of truth can properly be explained, regardless of whether we ordinarily refer to such entities. We can then take our theory of facts to be about the entities that these philosophers have in mind.

It is in the same spirit that the mathematician may introduce sets into his discourse. He is certainly not going to be bothered by the ordinary usage of the term 'set'; and perhaps no more should the philosopher, in his more theoretical moments, he bothered about the ordinary usage of the term 'fact'.

In what follows, I shall assume that my linguistic points are correct, though the possibility of an alternative interpretation may always be borne in mind. Let it be granted, then, that ordinary language sanctions the interpretation of the individual variables of our first-order modal theory as ranging over facts. Now if the resulting theory is to be of independent interest, it is necessary that facts not be instances of entities that are already well understood. But it has been suggested (by Ducasse [9], Carnap [6] and others) that facts are merely true propositions. If this were so, the theory of facts could simply be absorbed into a theory of propositions, as developed in Fine [15] or elsewhere.

There are, however, compelling arguments against the proposed identification. Some of them are summarized in Clark [8]. But even within the confines of a simple modal language, with existence as its sole predicate, it is possible to show that facts are not propositions. For consider, the proposition p that Carter was president in 1979 and the fact f that Carter was president in 1979. Then the proposition p exists even should Carter not be president, but the fact f does not. Therefore p and f are distinct. (This argument may be traced back to

Moore [22], p. 308; it also appears in Sprigge [40], p. 83, and Slote [39], p. 99.)

Of the different arguments for the non-identity of facts and propositions, this one is especially noteworthy; for it relies on one of the most significant features of facts, viz. the conditions under which they exist. As we shall see, reference to this feature is essential to the formulation of the most fundamental principles concerning facts. Thus, it is not even as if, for the limited purposes of our theory, we could suppose that facts and propositions were the same.

Our argument for non-identity has been put in terms of counterfactuals. It may perhaps be put more perspicuously in terms of possible worlds. Let the *existence-set* of an entity be the class of possible worlds in which it exists. Given a proposition ρ and a fact f, say that f *corresponds to* ρ if, for some sentence ϕ, ρ is the proposition that ϕ and f is the fact that ϕ[2] (cf. Moore [22], pp. 256–7). Then the point of the argument is that the existence-sets of a proposition and its corresponding fact will not, in general, be the same.

Thus the argument is one in which two entities are shown to be distinct in terms of the counterfactual conditions for their existence. What is of interest about such arguments is that they do not depend upon the features of the entities *within* any one world but upon what one might call their transworld features. Such arguments are of importance in establishing the distinctness of other allegedly identical entities (Kripke's argument [18] against the identity thesis is a case in point) and would often seem to constitute the only means of establishing such claims.

Of course, our particular argument does not establish that *each* fact is distinct from a proposition to which it corresponds or, indeed, that any fact is distinct from *all* propositions. But the plausibility of the proposed identification is of a peculiar sort. For it will only be plausible to hold that a fact is identical to a true proposition in case it is identical to the proposition to which it corresponds; and this will be plausible in one case, only if it is in all other cases. It therefore suffices, in showing that no fact is identical to a proposition, to provide a single instance of a fact which is not identical to the corresponding proposition.

Indeed, the argument is of even greater generality than this further consideration might suggest; for I have merely argued against the identity of *actual* facts to propositions. But a closely related argument applies against the identity of *possible* facts to propositions. Suppose, for reductio, that the possible fact f were identical to the proposition ρ. Now any possible fact is possibly an actual fact. Quite generally, we may say, it lies in the nature of a possible ϕ-er that it should possibly be an actual ϕ-er. So take a world w in which f is an actual fact. Then f is still identical to ρ in w, and since ρ is essentially a proposition, f is identical to a proposition in w. But presumably our earlier claim concerning the non-identity of actual facts holds necessarily; necessarily, whatever the actual facts, they are distinct from the propositions. Therefore the actual fact f in w is not a proposition in w, contrary to our previous conclusion.

A similar argument applies against other attempted conflations of the possible.[3] A possible man, for example, is not an abstract object; for a possible man can possibly be an actual man and hence be concrete, but no abstract can possibly be concrete. Another example, of significance to our later concerns, is that of possible worlds. A possible world is not any sort of abstract object, such as a proposition or set; for a possible world is possibly actual and is not then a proposition or the like. Possible worlds are merely a species of possible objects,[4] with the peculiarity that the actual instances are world-bound and necessarily number one.

There has been a philosophical tradition that has allowed a distinctive role for actual facts but has taken possible facts to be propositions; and, more generally, there has been a philosophical tradition that has allowed a distinctive role for actual or concrete entities of a certain kind but has taken the corresponding possible entities to be abstract. But if I am right, both of these traditions rest upon a simple modal mistake.

Just as my previous arguments did not tell against the reduction of nominal reference for facts to sentential reference, so the present arguments do not tell against the reduction of fact talk to propositional talk. It is commonly recognized that the reduction of Y's to X's does not require the identity of Y's with X's. But my arguments were merely directed against the claim of identity, not of reducibility.

Even so, the intuitive distinctness of propositions and facts will have a direct bearing on how a reduction is to be achieved. For suppose that we require a *proxy reduction*, one in which each entity y of Y is identified with a proxy entity y' of X. (See Quine [29] and

[32].) Let us also suppose that the entities Y of the reduced theory include the entities X of the reducing theory and that it is part of the reduced theory that some element y_0 of Y is not an X. Then it follows that the reduction is not one in which each element of X can be identified with itself. For $y_0 \neq x$ holds for each element x of X in the reduced theory; so, under self-identification of the X's, we would have that the contradiction $y_0' \neq y_0'$ holds in the reducing theory.[5]

Thus given that it is part of the intuitive theory of facts that facts are not propositions, reduction cannot be achieved by identifying facts with propositions but leaving the propositions themselves alone. This is not to say that a proxy reduction cannot be achieved at all. There will be more or less artificial ways of finding proxies for propositions that will preserve the relevant distinctness claims. But such a reduction will not be as simple as the original identification of facts with true propositions would suggest.

It is also important to emphasize that these strictures only apply to proxy reductions, ones in which each entity to be reduced can be assigned a proxy from among the entities that reduce. But not all reductions are of this sort.[6] An example from modal logic is the reduction of possibles to actuals (see [10], especially p. 131). An example from classical logic is the reduction of pairs of individuals, in which each individual pair quantifier $\forall p$ gives way to a pair $\forall x \forall y$ of individual quantifiers. Later, in sections 5 and 8, we shall also specify some non-proxy reductions of fact to proposition theories, obtained by eliminating the offending identities prior to apply a standard proxy reduction.

Having distinguished facts from true propositions, we must now consider the role of existence in a theory of facts. One naturally talks of facts obtaining, not existing, and so the question arises as to the relationship between the two. There are various opinions one might have here. One is that facts exist, *regardless* of whether they obtain or not. This is a view that might be held by someone who took facts to be true propositions; to say that they obtain would then just be another way of saying that they were true. Another view is that facts exist just in case they obtain; 'obtain' here is a variant on 'exist', not 'true'. A final view is that facts do not exist at all. This position then divides according as to whether it is of ontological significance that facts do not exist. On the one hand, one might argue with Moore [22], pp. 295, 372) that the difference in usage between 'exists' and 'obtains' or between other such pairs of terms reflects, not an ontological distinction, a distinction of *being*, but a difference in the *nature* of the subjects to which the terms apply. On the other hand, we might argue that there is an ontological point to the difference of usage.

Of these three opinions, the first strikes me as being definitely wrong. Indeed, if facts could exist regardless of whether they obtained then my arguments against the identity of facts and propositions would fail. However, it seems clear that we only say *there is* a certain fact if the fact obtains. But surely the fact cannot exist unless there is the fact, i.e., unless it obtains.

Between the other two opinions, the ordinary use of 'exist' does not seem settled enough to decide. However, I am inclined to think that there is an important ontological distinction between what one might call *existence in the narrow sense* and *being in the broad sense*, and it may then be true that particulars exist but facts have only being. But whether this be so,[7] the obtaining of facts will still be a form of being, a form that is attributed to propositions when we say 'there is a proposition such that ---'. Using a term for such being, then, in place of 'exists', our previous argument against the identity of facts and propositions may stand.

For the purposes of this paper, the distinction between a narrower and broader notion of being will not be important, and I have therefore found it convenient to use the term 'exists' for the broader notion. However, if my arguments are correct, nothing of substance will turn on this usage.

Although I have concentrated my discussion on facts, I would want exactly similar considerations to apply to worlds. Thus I would want to say that there is singular reference to worlds (something that has not been seriously doubted by those who think there is reference at all), that worlds are not propositions, and that the obtaining of a world consists in some kind of existence. Indeed, if worlds were facts, these conclusions would follow from the previous ones. But I would want them to hold independently of any claim of subsumption.

In the case of worlds, a rather special consideration arises in the formulation of the modal theory. For we shall want to make essentialist claims about worlds, to say that necessarily they have certain features and so forth. But it has been supposed that this is illegitimate and depends upon an illicit union of the semantical meta-theory with its object-language.

On this question, Prior had no doubt. There was no basis, in his philosophy, for a demarcation between world and modal talk and, indeed, his favoured way for talking of worlds was in a modal context (see [28]). But whether or not we accept Prior's general philosophy of modality, it seems clear that necessity, like identity and other logical notions, is a universal concept. We may intelligibly ask of *any* proposition whether or not it is necessary, even if it is a proposition that itself concerns necessity or possible worlds.

Moreover, it is often of great significance to make essentialist claims about possible worlds. For example, an important difference between worlds and instants is that it is essential to a world that what hold in it hold in it, but not to an instant. But this is a distinction that cannot even be formulated without a modal language for worlds.

We shall therefore suppose that a first-order modal theory in which the variables range over worlds is an object of both interest and legitimate study.

2. PRINCIPLES

We may now state the various modal principles that govern facts. But first, it will be necessary to distinguish between two conceptions of facts, the propositional and the worldly; and afterwards, it will be helpful, if not essential, to discuss their bearing on an important philosophical doctrine, the correspondence theory of truth.

The Distinction

Facts may be propositional or worldly, derivative or autonomous. This distinction may be variously explained, but perhaps most profitably in terms of how the identity of facts is to be accounted for, of what one might call their *ontological genesis*.[8] On the propositional conception, the facts will, in a certain sense, be derivative from or posterior to the propositions. The proposition will enter, in a certain essential way, into the explanation of the identity of the fact to which it corresponds. Thus we will explain what the fact that ϕ *is* by means of its relation to the proposition that ϕ.

The exact nature of the explanation will depend upon the type of propositional conception in question. On one view, there will be a simple fact-forming operation, F, applying to any true proposition ρ

to yield the corresponding fact $F(\rho)$. On another, and perhaps more plausible, view, there will be a simple operation C of concretization, that applies to a property P and an individual x had by P to yield a new object $C(P, x)$, the *P-ness of x*. The facts will then result from applying this operation to the property of truth and a proposition. Thus, on this view, the facts will belong to a broader category of objects, that include possibilities, necessities, likelihoods and certainties, all obtained by 'concretizing' the appropriate property.[9]

In coming to understand ontological genesis, it is helpful to think in terms of *canonical descriptions*. Say that certain objects *generate* another object if they are used to explain its identity. The canonical description of an object then displays its ontological genesis, the objects from which it is generated and the manner of generation.

Under this approach, the canonical description of a set of individuals would assume the form '$\{x_1, x_2, \ldots\}$', where the outer braces signify the set-builder. For a fact, on the other hand, the canonical description would either be of the form '$F(\rho)$', where 'F' signifies the fact-former or else of the form '$C(T, \rho)$', where 'C' signifies the concretizer.

As far as I know, the first philosopher to propound the view of facts as the truth of propositions was Moore ([22], pp. 261–2). More recently, it has been put forward by Slote [39]. It is, of course, essential to distinguish between this view and the view that facts are true propositions. Truths are derived from true propositions, not identical to them; the truth of a proposition is no more a true proposition than the wisdom of a man is a wise man.

On the worldly conception of facts, their identity will not be explained in terms of propositions. It will either be taken as primitive or, more plausibly, will independently be explained. On the most natural view of this sort, facts will be structured entities or complexes, built up in certain characteristic ways from their constituents.

Although propositions will not enter into the analysis of facts on this view, there will normally be a correspondence between the structure of facts and of propositions, at least on a structural conception of propositions. For the operations used to construct the facts will correspond to operations of help in constructing the propositions. Thus just as each subject-predicate fact will be the result $I(x, P)$ of applying an operation I of inherence to an individual x and property P, so each subject-predicate proposition will be the result $i(x, P)$ of applying a like operation i to x and P.

There may well be other differences, concomitant with, if not consequent to, the distinction as I have drawn it. There is the question of whether facts are *in* the world, *part of* it (see [2] and [41]). It is perhaps difficult to know what such talk amounts to. But on one natural use of the phrase, propositions are not in the world, but descriptive of it. Since facts, on the propositional view, are even further removed from reality, they too would seem to enjoy an extra-worldly status. This point becomes especially clear if we think of facts, or truths, as belonging to the same general category as falsehoods, with the falsehood property replacing the truth property in the application of the concretizer. For there is hardly any temptation to regard falsehoods as in the world. On the non-propositional view, by contrast, there would not be the same grounds for denying intra-worldly status to facts, and, although there may be other grounds, it seems most plausible in this case to suppose that the facts are in the world.

Connected to this question of worldliness is that of determinacy or specificity. Roughly, facts are *determinate* if they cannot obtain in different ways.[10] (Cf. Moore's distinction in [23], p. 67, between non-general and general facts.) It then seems reasonable, if facts are to be in the world, that they should be determinate. We will acknowledge the fact that the sense-datum is of a determinate shade of red, but not the fact that it is red or blue or that it is not orange. On the propositional conception of facts, however, there is no difficulty in accepting a fact for any true proposition. We may, of course, still distinguish the determinate facts; but they will then form a sub-class of the whole, not in the outer limits of a category.

The nature of the relation of correspondence will be different on the two conceptions. In both cases, the relation will be *internal*, depending, as it does, only upon the internal structures of the entities in question. But in the one case, there will not be any inner analysis of the proposition but only a simple and direct relationship of it to the fact; while in the other case, both proposition and fact must be analysed. This becomes especially clear from the canonical descriptions. For the description of the propositions will be directly embedded in that of the propositional fact, but merely reflected in the description of the worldly fact. Taking the subject-predicate case as an example, the one fact will have the description $F(i(x, p))$, while the other will have the description $I(x, P)$.

We may put the difference thus: on the propositional conception, the analysis of the fact will be *parasitic* upon that of the corresponding proposition; on the non-propositional view, the analysis will be *parallel* to that of the proposition.

On the worldly conception, there may also exist a relation of *underlying* between a proposition and a fact. (I use the term 'underly' rather than the more familiar 'verify' in order to emphasize that the relation, like correspondence, is a purely structural one.) The way the relation works may be gathered from examples. Thus whereas there may be no fact corresponding to the true disjunction of ρ and σ, there will be a fact which underlies the disjunction, a fact that will either correspond to one of ρ and σ or will itself underly one of them.[11]

The significance of the relation of underlying is this. Whereas, on the worldly conception, there may be no fact corresponding to each true proposition, it may be held that to each true proposition there is an underlying fact that verifies it, should it be true. Thus the facts will be determinative of reality, even if they are not in correspondence with the true propositions.

It would certainly be of interest to explicate the nature of this relation more fully. This is a task that was begun by the logical atomists and that has been pursued, in more recent times, by Van Fraassen [45]. However, it is not a question that I shall take up in this paper.

I have talked of two conceptions of facts. But do we have two views on a single category of facts, or two categories? My view is that we have two categories of objects here. Thus, given an individual x and a property P had by x, there will be two fact-type entities that may be distinguished, one, $F(i(x, P))$ or $C(T, i(x, P))$, parasitic upon the proposition, and the other, $I(x, P)$, not.

But what then comes of my claim that the facts of the theory are the ones to which we ordinarily refer? Do we refer to propositional facts, to worldly ones, or to some other type of fact altogether? Ordinary usage, I think, is usually suggestive of the propositional conception, but is not sufficiently definite to exclude the other. If that usage allows an interpretation of the term 'fact' under which it is problematic that there are disjunctive or negative facts, then some sense other than the propositional one would seem to be indicated.[12]

Nor should this referential indeterminacy, crossing categories, trouble us. There are many other terms that in ordinary usage are ambiguous between categories. The term 'word', for example, can refer either to word-types of word-tokens. Depending on the context, the question 'How many words appear in the book?' could concern either the one or the other.

If I am right about the duality of reference, then several theses taken to concern facts as a whole should be taken to concern one category rather than another. An obvious example is the pair of contrasting theses that facts are truths and that facts are in the world. These should be taken to relate to the propositional and to the worldly conceptions respectively.

A less obvious example arises from the familiar philosophical query as to whether there are disjunctive or negative facts. On the propositional interpretation, this question is unproblematic. So what can be meant? One possibility is that it is being asked whether we can get by without disjunctive or negative facts in explaining which propositions are true. (Cf. the discussion in Russell [36].) But such an interpretation does not do justice to our inclination simply to deny that there are any disjunctive or negative facts. Another possibility is that a special ontological sense of 'there are' is in question. But ontological considerations usually cut across categories, not through them. Thus it is unclear why one should grant ontological status to some of the facts in the given category, but not to the others. A better interpretation, that avoids the above problems, is one in which the question relates to facts on the worldly, not the propositional, conception.

Yet another example, to be discussed later, is provided by criticisms of the correspondence theory of truth.

In what follows, it will be helpful to have separate terms for the facts from the two categories. For reasons that should already be clear, I shall use the term 'truth' for facts from the propositional category, though I do not think that it is invariably used with this sense. For the facts from the worldly category, I shall use the term 'circumstance'. In case the category has been fixed or is not important, I shall stick to the term 'fact'.

Identity Principles

Having distinguished the two types of fact, let us now consider the essentialist principles appropriate to each. As with the formulation of many other modal theories, these principles are best approached through the study of the existence- and identity-conditions for the objects in question. The principles will then be ones that impose the conditions on the objects.

The principles may be internal or external, relative or absolute.[13] Internal principles are intra-worldly. In regard to existence, they state which objects exist within each world; and in regard to identity, they state when two objects are identical within a world. The external principles are trans-worldly. For existence, they state when an object from one world exists in another; and for identity, they state when an object from one world is identical to an object from another. In stating the existence- or identity-conditions for a given category of objects, the objects from another category of objects may be presupposed. In this case, the principles are said to be *relative to* that category; and otherwise they are said to be *absolute.*

Each distinction may be illustrated by the case of sets. The ordinary axioms of set theory (excepting Extensionality) are internal existence principles. That (necessarily) a set exists iff its members do is an external existence principle. The extensionality axiom itself is an internal identity principle, which, in conjunction with the rigidity of a set's members from world to world, becomes an external identity principle. The existence and identity of a set is explained in terms of the existence and identity of its members, respectively, which are explained in terms of the existence and identity of their members, and so on, until ultimately the existence and identity of the original set is explained in terms of individuals (*urelements*). Thus the various principles are relative to the category of individuals.

Given this division in the principles and our two-fold conception of facts, there will be four separate sets of questions to consider. They will be treated in turn, beginning with the question of the identity principles for truths.

These principles may be stated in relative fashion, in terms of the propositions to which the facts correspond. Recall that, for the propositional conception, it was suggested that the fact f should correspond to the proposition ρ just in case it was of the form $C(T, \rho)$ or, perhaps, of the form $F(\rho)$. If such an account is correct, then from the mere fact that C or F or the like is a function, it will follow that distinct facts cannot correspond to the same proposition. But it also seems reasonable that the same facts should not correspond to distinct propositions, that $C(T, \rho)$ or $F(\rho)$ should vary with ρ. We are thus led to what might be called the *one-one thesis*, viz. that the relation of correspondence between propositions and facts is one-one.

This principle then reduces the question of the identity of facts, within a given world, to the identity of propositions. For under the assumption, which we shall later grant, that for each fact there is a proposition to which it corresponds, two facts will be identical iff the propositions to which they correspond are identical.

The question of the identity of propositions is, of course, an independent matter. But on a broadly structural conception of propositions, we may suppose that their identity is explained in terms of their structure. The identity of facts might then also be explained in structural terms, without the detour through propositions; for the fact will inherit whatever structure is possessed by the proposition. But there is something non-fundamental about such an account, since the attribution of structure to the fact must itself be explained in terms of the corresponding proposition.

A formulation of the internal identity principles for circumstances may also be given in terms of the correspondence with propositions. Recall that, on the non-propositional conception, it seems most plausible to suppose that fact and proposition correspond just in case they have parallel structure. Now a priori, this is compatible with the relation of correspondence being many-one, one-many or even many-many. Thus it may be that the facts $I(x, P)$ and $I(y, Q)$ are distinct, even when the propositions $i(x, P)$ and $i(y, Q)$ are the same. It seems most reasonable, though, to suppose that the relevant structure of both facts and propositions is unique or, at least, variable in the same way. Under this assumption, the relation of correspondence between propositions and facts will again be one-one.

However, in this case, the formulation of the identity conditions in terms of correspondence will not be fundamental; for the relation of correspondence must itself be explained in terms of the structure possessed by facts and propositions. Since facts now have a structure on their own account, the most direct formulation of the identity conditions will be in terms of that structure, with facts being the same, at least under the simplest assumptions, just in case their structure is the same.

Given the internal identity principles, the formulation of the external principles is a straightforward matter. For the relation of correspondence or the constitutive structure of a fact is rigid; it does not vary from world to world. Thus a fact may be identified across worlds in terms of the corresponding proposition or constitutive structure, and the principles that guarantee this will be ones that express the rigidity of correspondence or of those relations that are definitive of structure.

Although the account of the identity conditions for facts in terms of propositions or propositional structure is very natural, there is an opposing, and quite different, account in terms of what one might call empirical content. Under this account, two facts will be identical when they necessarily co-exist, i.e. when it is necessary that the one exist just in case the other does. This new principle might then be applied to either truths or circumstances.

In application to truths, at least, it will usually diverge from the previous propositional criterion. There is one circumstance in which the two criteria will agree. For let it be supposed that propositions are also subject to an empirical criterion of identity, with necessarily equivalent propositions being the same, and let it further be supposed that distinct facts cannot correspond to the same proposition. Then given the reasonable assumption that (necessarily) a fact exists iff the (or a) corresponding proposition is true, the present empirical criterion of identity for facts is readily seen to follow.

Assuming, however, a structural criterion of identity for propositions, there will be facts corresponding to distinct, though necessarily equivalent, propositions; and so the present criterion will run into conflict with the previous assertion of the one-one thesis.

The present criterion, once it is detached from questions of propositional identity, is best seen as arising from the demand that facts be empirical entities. Roughly speaking, we may say that empirical entities are ones that are empirically distinguishable, where empirical distinctions are ones that make a possible difference to the world. How the difference shows up depends upon the nature of the entities in question. Two (simple) individuals will be empirically distinguishable if it is possible for the one to exist and the other not. Thus necessarily existent individuals will be empirically indistinguishable and there will be at most one empirical necessary existent individual. On the other hand, two properties of individuals will be empirically distinguishable if it is possible for the one to apply to an individual empirically distinguishable from all of the individuals to which the other property applies. Thus necessarily co-extensive properties will be empirically indistinguishable and only one empirical property will have a given extension in each possible world.

Applying the simplest of these criteria to facts then yields the present criterion, that facts should be the same when they necessarily coexist.

Such a conception of facts was upheld by Ramsey ([33], p. 146) and, in this form, criticized by Moore ([23], p. 175). However, Ramsey's view differs on two counts from the one being proposed here. First, he would have held a similar view on propositions and so still would have maintained that the correspondence relation was one-one (had he been prepared to think in these terms). Second, his necessity was analytical or logical, whereas mine is metaphysical.

As with the previous distinction between the propositional and worldly conceptions, we may ask whether the present distinction between the empirical and structural conceptions reflects a genuine difference of view or merely of subject-matter. Again, I am inclined to think that two distinct categories of fact-type entities can be distinguished, one satisfying an empirical and the other a quasi-structural criterion of identity. Moreover, this classification, at least in principle, will cut across the previous one, resulting in a potentially fourfold division of entities.

However, it is not clear that ordinary usage is neutral, in this case, on the question of whether facts are to be subject to an empirical or structural criterion of identity. As I shall later suggest, circumstances already are empirical entities and so no separate question arises here. But when it comes to truths, ordinary usage seems to favour the structural criterion. Thus we acknowledge that someone may be surprised at the truth of Gödel's Incompleteness Theorem yet not at some simple arithmetical truth, even though both truths necessarily obtain.

There are, however, certain philosophical contexts that favour the empirical account. We ask: is there a fact (a truth to the effect) that two nations are at war in addition to a certain complex fact about people or is there but a single fact? Now this question may, rather misleadingly, be about the particular *circumstances* in which two nations are at war. In this case, it concerns the worldly conception and is not to the point. But it may also be about a highly general truth, one that can hold in a multiplicity of ways. In this case, there would appear to be two truths in the ordinary sense of the term; one about nations and the other about people. But the point of the question is whether there is anything more, *empirically*, to two nations being at war than a certain complex behaviour of people. Thus the question is one that takes the truths or facts to be empirical entities.

If I am right about ordinary usage, then this use of 'fact' is somewhat technical. Thus we have here a case in which a new class of entities is introduced in order to meet a certain theoretical demand – in this case, from ontological inquiry.

But whether or not empirical facts are objects of ordinary reference or a philosopher's invention, it will be important to inquire further into their nature and their relationship to facts as structurally conceived. For empirical truths (not truths that are contingent, but truths that are subject to an empirical criterion of identity), the only acceptable view appears to be that they are abstractions from the structural truths, in much the same way as directions are abstractions from lines. We note that different truths may have the same empirical content. Given that our interest is only in that empirical content, we then abstract from truths with the same empirical content to obtain the empirical truths.

On this view, the identity of the empirical truths will be explained in terms of propositions, but by means of a double operation. First, the truths, as structurally conceived, will be obtained from the propositions by means of some operation such as the fact-former or concretizer. Then the empirical truths will be obtained from structural ones by means of an operation of abstraction. Thus the empirical truths will be twice removed, or thrice removed if we include the genesis of the proposition, from the underlying reality.

On the other hand, in the case of circumstances, one may have a totally different view on the nature of the empirical facts. On the propositional conception, it seemed impossible that the facts should be subject both to a structural and to an empirical criterion of identity; for *each* true proposition was to correspond to a fact and so distinct, but necessarily equivalent propositions, would correspond to distinct facts. To take a simple example, the fact that $(p \wedge q) \vee (p \wedge -q)$ would be necessarily co-existent, for any true p, with the fact that p and yet distinct on any reasonable structuralist criterion.

However, on the worldly conception, the means for constructing the facts are severely constrained and so there would not be the same difficulties in reconciling the two criteria of identity. Let us grant, for example, that the means are limited to operations of inherence (I) and of "conjunction". Then although there would be interesting problems in determining what the underlying relations and particulars could be like, there would seem to be no essential difficulty in supposing the resulting facts to be subject to both the structuralist and empirical criteria of identity. Indeed, it might be taken to be the distinctive feature of facts in the world, as opposed to propositions or truths, that they combined the features of being empirical and of possessing proposition-like structure.

Such an account of circumstances might help to explain the special nature of necessity and its significance for metaphysics. We may ask: what are the ontological grounds for statements of metaphysical necessity? One answer is that such statements are true merely in virtue of one proposition being a logical consequence of other propositions. Thus on such a view, there would be no distinctive ontological ground for metaphysical necessities; all such necessities would be grounded in the relatively unproblematic case of logical implications. But an alternative answer is that such necessities rest, at least in part, on the identity of facts. Thus what makes it true that necessarily if people do certain things then two nations are at war is that there is but one fact in the world. The most obvious objection to any such account of ontological ground would be that it is circular; but no objection of this sort applies in the present case, since the identity of the facts may be independently explained in terms of their structure.

We might then see the interest of metaphysics in necessity as derivative. Ultimately, our interest is in what the facts are. We wish to know whether there are facts ('in the world') about nations in addition to facts about people, or facts about material things in addition to facts about experience. But given the empirical criterion of identity for facts, these questions of identity convert to questions of entailment, and hence give rise to our interest in necessity.

One might take this line of thought even further. For I have so far assumed that the worldly facts or circumstances are structured entities. But what if we drop this assumption and suppose that circumstances are subject *only* to the empirical criterion of identity (the facts would be empirical nuggets, devoid of internal structure). There would then be no difficulty in supposing that each true proposition corresponded to a circumstance, and so we could *define* necessity in terms of the identity of facts.

Such a suggestion is, in fact, to be found in the literature. It was first put forward by Malcolm [20]. Unfortunately, his actual definition is subject to difficulties, that were pointed out by Toms [44]. A correct definition was later given by Prior [25]; but it requires the notion of fact containment, which smacks too much of the notion of necessity to be completely helpful. A correct definition, that avoids fact containment and merely requires the fact – that operator $\exists$ in addition to standard logical notions, may be given by taking ϕ to be necessarily equivalent to ψ just in case there is a fact that ϕ and an identical fact that ψ *or* there is a fact that not-ϕ and an identical fact that $-\psi$ (in symbols, $\exists f(f = \exists\phi \wedge f = \exists\psi) \vee \exists f(f = \exists - \phi \wedge f = \exists - \psi)$).[14] Given the notion of necessary equivalence, the other modal notions may then be defined in the usual way.

Of course, the above definition is correct even when empirical facts are treated as abstractions from truths. But it is then circular, since the empirical facts themselves must be explained in terms of necessity. The present suggestion is that the empirical facts be given independently of necessity, by the world so to speak, and that the notion of necessity then be explained in terms of it.

Such an account would immediately justify our previous views on the ontological ground for necessities and on their significance for metaphysics; for the analysis of any statement involving necessities would be in terms of factual identities. Thus the claim that statements concerning nations were equivalent to statements concerning people would be analyzed in terms of the identity of facts concerning nations and people – and not conversely, as is commonly supposed.

But despite the attractions of this more extended view, it is not one that I can accept. For we are not justified in taking as primitive a whole host of facts corresponding to each true proposition and, even if we were, it would still not be clear how the application of the fact – that operator to sentences was to be understood in the absence of any structural identity criterion for facts or of any appeal to the notion of necessity. The most the world provides are circumstances corresponding to a limited class of propositions. The other facts are to be understood in terms of propositions and, if they are empirical, in terms of the notion of necessity as well.

Existence Principles

Let us now consider the existence principles for facts, first for truths and then for circumstances. What truths are there in each world? An answer may be given in terms of the corresponding propositions. For to each true proposition there corresponds a fact. Conversely, each fact corresponds to a true proposition. So the truths of each world are exactly those that correspond to the true propositions.

Given that we know what propositions there are in each world and when a proposition corresponds to a fact, these principles then enable us to determine what facts there are in each world. It reduces the internal existence question for facts to that for propositions.

The internal existence principles for circumstances may be stated in similar fashion, but in terms of the correspondence with a sub-class of true propositions, the definite ones. But this formulation is not the most basic, since the notion of a definite proposition must somehow be made out. It would be preferable to characterize what circumstances there are directly in terms of the means available for their construction, be they inherence and conjunction or some other class of operations.

The external principles of existence for truths and circumstances may also be given a common form. The question is: when does a fact from one world exist within another? And the answer, again in terms of the corresponding proposition, is: just in case the corresponding proposition is true. Thus we are led to the principle that (necessarily) a fact will exist iff the, or a, corresponding proposition is true.

Since the corresponding propositions will also have truth-conditions, the existence-conditions for the facts may be stated in terms of those truth-conditions, without the detour through propositions. Thus in case the fact is atomic (subject-predicate), we may say that it exists iff the given particulars exemplify the relation; and in case the fact is conjunctive, we may say that it exists iff each of the conjunct facts exists. Proceeding in this way, we may then give a recursive characterization of the existence conditions for any fact. However, whereas this characterization will be the most basic for the circumstances, it will be less basic for the truths, arising as it does, from the combination of the existence-conditions in terms of propositions and the truth-conditions for those propositions.

The external existence conditions are indicative of one of the most distinctive features of facts, viz. the intimate connection between factual existence and propositional truth. However, it is hard to state exactly what is so special about this connection. It is not that the existence conditions of a fact are the truth-conditions of some proposition. For given any object whatever, its existence conditions will be the truth-conditions of a proposition, viz. the proposition that the object exists. In the case of facts, though, the proposition can be stated without circularity, not as the proposition that the fact exists, but as the proposition that gives the inner content of the fact. Moreover, the fact and the corresponding proposition, that gives its existence-conditions, will stand in a peculiarly intimate relationship. For there will be a purely internal or structural connection between the two. It will be possible to read off the proposition from the fact; and so the fact, unlike other entities, might be said to bear its existence-conditions on its face. Finally, for truths though not for circumstances, we may note that there is a generality in the connection between factual existence and propositional truth. For although the existence conditions of any entity are the truth-conditions of some proposition, a converse relationship holds for truths, with each true proposition being the existence-conditions of some truth. And in this respect, the category of facts, and its cognates, would appear to be quite special.

Although the above existence principles may appear to be very reasonable, they are open to a subtle, yet severe, objection. For consider the proposition that Socrates does not exist. In some possible world, this proposition is true. But is there, *in that world*, a fact that Socrates does not exist?

I think not; for how can the fact exist when the individual does not? The principle "no individual, no fact" has a great deal of intuitive appeal. But it can also be taken to derive from the more general structuralist principle "no constituent, no complex". In other words, when an object, the complex, is constructed from other objects, its constituents, then the complex cannot exist unless the constituents do. Let it now be granted that facts are complexes and that the objects which they are in the relevant sense about are their constituents. Then the principle of no fact without the individual will follow.

It is essential to this piece of reasoning, though, that we take an objectualist stand on facts, according to which individuals, as opposed to their concepts, can actually be constituents of facts. I think that such a stand is justified for propositions, and for *circumstances* it would appear to be even more plausible; for in so far as facts relate to the world and not to our conception of it, there would not be the same reasons for supposing that their constituents were purely conceptual. For *truths*, the validity of the objectualist stand is perhaps more problematic. If propositions are actually constituents of the corresponding truths, then objectualism propositions will imply objectualism for truths. But although propositions are used to explain the identity of truths, it might be held that they are not constituents of those truths. In this case, objectualism for the truths might well fail. However, we may then give an alternative justification of the principle "no individual, no fact". For we may suppose that the general structuralist principle of no complex without a constituent is replaced by the more general ontological principle that the identity of any existent object, when it stands in need of explanation, is to be explained in terms of other objects that exist. The previous structuralist principle would then be a consequence of this ontological principle, granted that the identity of a complex must always be explained in terms of its constituents. But we may suppose that the ontological principle also applies in cases in which the explanatory objects are not actually constituents of the object to be explained. Given that the propositions and hence, ultimately, their individual constituents were required to explain the identity of truths, the required principle would then follow.

As far as I know, Prior [26] was the first modern philosopher to insist on these principles for the existence of facts or propositions and to stress their importance for metaphysics and logic; though whether he would have accepted our structuralist or more general ontological arguments for them I do not know.

If the objectualist requirement on the existence of facts is accepted, then both the internal and external existence principles must be modified. In stating the external condition, two requirements on the existence of the fact should be laid down, one deriving from its propositional and the other from its objectualist content.[15] First, it should be required that the corresponding proposition be true, and second, that the objects which it is about exist. In stating the internal principles, on the other hand, it should be required that the given proposition both exist and be true for there to be a corresponding fact.

In many cases, the objectualist requirement will be included in the propositional one. This will be true, for example, of any subject-predicate fact in which the relation only applies to existents. Indeed, it might be argued that there must always be this inclusion of requirements for circumstances, so that, in this case, no reformulation of the existence principle need be given. However, for truths it is clear that the two requirements can come apart. It is necessary for the existence of the proposition that Socrates or Plato is a philosopher, for example, that both Socrates and Plato exist, though not necessary for its truth.

Fortunately, there is a convenient way of combining both requirements in the formulation of the various principles. Let us distinguish between the *inner* and the *outer* truth of a proposition. A proposition has outer truth if it is simply true and it has inner truth if, in addition, it exists. Thus the proposition that Socrates does not exist has outer truth in a world in which Socrates does not exist, but not inner truth. We may now retain the original formulation of our principles, but substitute inner for outer truth. Thus the external existence principle will now say that the existence set of a fact is the inner, not outer, truth-set of a corresponding proposition; and similarly for internal existence.

Although the difference between the objectual and anti-objectual approaches may appear slight, the consequences for the theory of facts will be great. On the technical side, we may note that a large part of the interest of the formal work will arise from the adoption of the objectualist requirement. As in the case of propositions (see [15]), it will be necessary to formulate adequate semantic conditions on models in terms of automorphisms and not merely in straightforward set-theoretic terms, and , once this is done, the status of the resulting theories will be drastically altered, from decidable to undecidable and from axiomatizable to non-axiomatizable.

On the philosophical side, we may note, first of all, that objectualism for facts leads to a new perspective on the existence of propositions. Call a construction (*existentially*) *transparent* if no more is required for an object constructed by its means than the existence of the objects to which it applies; and otherwise call the construction *opaque*. Then before it was supposed that all propositions were obtained by means of transparent constructions, so that, ultimately, the existence of the proposition would only depend upon the individuals from which it was constructed. But the fact-former F or concretizer C are opaque constructions. The existence of $F(\rho)$ or of $C(T, \rho)$ not only depends upon the existence of ρ, but also on its truth. Therefore if facts themselves enter into the construction of propositions, there will be no guarantee that the existence of the propositions can ultimately be traced back to the existence of their individual constituents.

This point may then go some way towards explaining why the means for constructing circumstances should so severely be constrained. Why is there not 'in the world' a disjunctive or a negative fact? One possible answer is this. Suppose there were a disjunctive fact. Then it would be the result of applying a disjunctive operation to the two disjunct facts. So by general structural considerations, the existence of the two disjunct facts would be necessary (and presumably also sufficient) for the existence of the disjunctive fact. But then the disjunctive fact would be indistinguishable in its existence-conditions from the corresponding conjunctive fact and so, by the empirical criterion of identity, would be identical to it. A similar argument applies to negative facts. And more generally, it should be clear why, from this point of view, the only truth-functional mode for constructing facts should be that corresponding to conjunction.

The Correspondence Theory

The various existence principles have an obvious bearing on the correspondence theory of truth and on attempts at reducing discourse concerning facts. We cannot give a thorough account of all of the issues that arise, but enough can be said to put the remaining formal work into some sort of philosophical perspective.

Put roughly, the correspondence theory states that the truth of a proposition is to be explained in terms of its relation to the facts. It will be important, however, to distinguish between two fundamentally different forms of the theory, one oriented towards the concept of truth and the other oriented towards instances of truth. The first provides an analysis of the general concept of truth; it says what it is, in general, for a proposition to be true. The other attempts to articulate the ontological ground for particular truths, to describe in the most fundamental terms what it is in the world that accounts for those truths.

The two versions of theory will assume subtly different forms, locating both what is explained, what explains, and the explanatory link in different places. The concept-oriented theory will analyse the general (open) statement to the effect that a given proposition is true, and the analysandum, at least in standard cases, will be to the effect that the proposition correspond to or be verified by a fact. The instance-oriented theory will attempt to explain how a particular proposition is true, and the explicandum will be to the effect that a certain fact exists. This fact, in standard cases, will be one that underlies or corresponds to the proposition. But *that* it does will be no part of the explanandum. Of course, the explanation may be generalized. We may say that for any true proposition, its ontological ground is that a certain fact exists, where this is the fact underlying or corresponding to the proposition. But we have, then, a general form of explanation, not an explanation in general form.

The distinction becomes particularly sharp once we note that the concept of truth is not even required for the formulation of the instance-oriented theory. For taking a particular truth, say that the cat is on the mat, we might state its ontological ground to be that there is a fact that the cat is on the mat. Or if we wish to be more general, we may assert, as a sentential scheme, that, for any true sentence ϕ, the ontological ground for ϕ is that there is a fact that ϕ. On the other hand, for the concept-oriented theory, the concept of truth is essential; since otherwise there is nothing to explain.

Of the two forms of the theory, it is the instance-oriented one that is most fundamental. Yet, for the most part, philosophers have attended to the concept-oriented theory, and this has introduced distortions into the ensuing discussions. This is no clearer than in some of the criticisms of the correspondence theory. For it is often wondered whether the subjects of truth or the concept of truth can legitimately be made out. But whatever validity these points may have against the concept-oriented theory, they have no bearing on the instance-oriented version, since neither the subjects nor the concept of truth are required for its formulation.

Is our theory of facts committed to either form of the correspondence theory? It might appear to be. Within the theory of truths, we may derive, from the internal existence principles, that (necessarily) a proposition is true iff it corresponds to a fact; and with the additional help of the external existence principles, it may then be shown that (necessarily) to each true proposition there corresponds a fact for which it is necessary that the fact exists iff the proposition is true. From the theory of circumstances, the same principle will not be derivable; since to each true proposition there will not correspond a fact. But from suitable assumptions for the relation of underlying, it may be shown that (necessarily) to each true proposition there is an underlying fact for which it is necessary that if the fact exists the proposition is true. Thus a similar principle may be derived, but with a less direct structural connection between the proposition and the fact and a weaker modal connection.

Now although these principles are highly suggestive of the correspondence theory, neither is constitutive of it. For it is essential to the theory not merely that such connections be postulated but also they be, in the appropriate sense, explanatory. For the concept-oriented form, one requires not merely that a proposition be true iff it corresponds to a fact, but also that the concept of truth thereby be analysed. For the instance-oriented theory, one requires not merely that each true proposition be made true by a fact in the sense that necessarily if the fact exists the proposition is true, but also that the truth of the particular proposition thereby be explained.

The principles of the theory give what one might call the modal content of the respective forms of the correspondence theory. They state what modal truths must be accepted by an upholder of the theory. But they do not, conversely, commit the upholder to the theories from which they might be derived.

In the same way, one who accepts the Leibnizian thesis that a proposition is necessary iff it is true in all possible worlds does not thereby commit himself to the possibilist's *analysis* of necessity in terms of possible worlds. He might, like myself, accept the principle and yet analyse possible worlds in terms of necessity in such a way that the truth of the principle can still be maintained. No matter how appealing a particular analysis, the mere assertion of an underlying equivalence does not commit one to the claim that it is an analysis.

In order to distinguish the underlying modal principles from the correspondence theory, let us call them correspondence theses. Even if the theses themselves do not constitute a theory, the question still arises as to when they can be of explanatory value. It is helpful here to distinguish between truths and circumstances. For truths, the proposed explanations will all be circular. For we explain what it is for a fact to exist, say $F(p)$ or $C(T, p)$ in terms of the truth of the corresponding proposition. This means that the explanatory link between the truth of a particular proposition and the existence of a fact will run in exactly the opposite direction to that required by the correspondence theory; for it is the truth of the proposition that will explain the existence of the fact, not vice versa. In the same way, it is an object's being such and such that will explain its membership in a set, not the other way round. Moreover, on the concretizer view, the relation of correspondence will be explained in terms of the concept of truth, with f corresponding to p just in case $f = C(T, p)$. So in this case, there will be a double circularity in the definition of truth as correspondence to fact, with one circularity arising from the attribution of existence to a fact and the other from reference to the correspondence relation.

Several other philosophers have also argued against the explanatory value of the various correspondence theses. But they have usually argued against this on the basis of some simple-minded reduction of discourse concerning facts. Some have supposed that facts are to be identified with true propositions. This then makes it natural to interpret the obtaining of a fact as the truth of a proposition and the correspondence of fact and proposition as identity; so that it becomes a simple logical truth that to each true proposition there is a corresponding fact. Others have treated the fact quantifiers as substitutional quantifiers over sentences and then have interpreted the other parts of speech in such a way that the various theses are again truths of a broadly logical kind. In either case, the application of the reduction to a correspondence thesis dispels any appearance of explanatory value that it may possess.

Now I am not against reduction as such, even for circumstances; and it seems likely that any reduction will eliminate the explanatory value of the correspondence theses. For once the proposed explanans is reduced, the reference to facts will appear to be redundant. However, I do object to the idea that the reduction can proceed through any simple-minded identification of facts with true propositions or of fact-terms with true sentences. There is the previous difficulty over identity, but more important is the point that facts themselves may figure in propositions.

The difficulty may be illustrated by a simple example. Take any true proposition p (that exists). Let f be the corresponding fact and $\bar{p}$ the proposition that f exists. Then it is clear that the existence-set of $\bar{p}$ coincides with its truth-set and with the inner truth-set of ρ. We may call a proposition *coincident* when its truth- and existence-set coincide. Then what we have established is that for each true proposition ρ there is a coincident proposition $\bar{p}$ with the same inner truth-set.

Now this is a conclusion lying entirely within the theory of propositions. Yet its truth arises from the admission of facts; for there would be no reason, in general, to suppose that each fact coincided in its existence-condition with some constellation of individuals. But this means that no reduction will be correct if, like the standard ones, it leaves the pure theory of propositions alone.

An acceptable reduction must proceed along different lines. It is most natural first to expand the domain of propositions so as to include those with facts as constituents, and then to reduce facts to the resulting propositions and those propositions to the original ones – with the whole process being re-iterated once structuralist, as well as modal, considerations are taken into account. The standard attempts at reduction are inspired by the idea of a simple grammatical transformation. It is hoped that this, rather than an inner probing into the structure of the propositions and facts, might secure the desired result. But this ignores what one might call the creative aspect of facts, with the need not only to push out the domain once, but repeatedly in response to the application of the different constructions.

In this respect, facts might be compared to sets and properties or to other "logical" entities of this kind. All of them enjoy a suitable form of abstraction principle (with the principle for facts being that, for each true sentence ϕ, there is a fact such that it is necessary that the fact exists iff ϕ and the objects of ϕ exist.) If attention is confined to simple abstracts, defined on expressions from the base language, then they may be eliminated by means of a simple grammatical transformation. Once quantification over arbitrary, yet non-iterative, entities is introduced, this can no longer be done, although there may be a simple "grammatical" reduction to entities of another kind. But once unrestricted quantification over entities from different categories is allowed, the simple character of the reduction is perforce lost, since the results of applying one construction will now figure as the arguments for the application of another construction.

When we turn to circumstances, the status of the correspondence theory is rather different. We may note first that an exact form of the theory can longer be retained, since to each true proposition there will not correspond a circumstance. Rather the exact match between true proposition and fact must give way to the less direct relation of underlying. Thus if a correspondence theory of truth is to be plausible, it is the verification by facts, not the correspondence with propositions, that is to be the key relation.

However, even this modified form of the theory is subject to difficulties. These centre not so much on the circularity of the proposed explanations; for in contrast to truths, the existence of circumstances and the relation of underlying are most directly explained in structuralist terms, without the reference to propositions. Rather, the difficulties concern the irrelevance of facts to the proposed explanation. For always we may ask, when facts are used to explain truth, why the relationships among the constituents of the facts should not be used in their place. Consider subject-predicate propositions of the form x and P. Why should we not explain the truth of such propositions in terms of the particular exemplifying the property or even deny that any non-trivial explanation can or need be given? What does the explanation in terms of facts add to the explanation that avoids them?

One possible answer is that it is only through the reference to facts that we can give expression to some form of realism. But it is necessary here to distinguish between a general realist position in regard to a given subject-matter and a more distinctive realist position in regard to the corresponding facts. Often the *point* of stating the correspondence theory is to suggest a general realist position. Thus one finds philosophers denying that there are moral facts as a way of expressing moral anti-realism. But the use of facts to this end is completely idle. For there is a perfectly innocuous sense in which we may talk of the moral facts, or the facts of any other subject-matter, even if we are anti-realists in the given area. And if we insist that the facts are to be ones acceptable to a realist, then we can equally well insist that the particulars and properties, in their relationships of exemplification, are to be acceptable to the realist. Whether we give an account of truth in terms of facts or in terms of their constituents, there is nothing in the accounts as such to suggest either a realist or anti-realist position. Whatever the explanation of the distinction between the two positions, it is presumably one that lies elsewhere.[16]

But although facts are of no help in expressing realism in regard to a given subject-matter, there may be a point, once such a realism is presupposed, in espousing a distinctive realism in regard to the facts. For suppose we ask how a concrete individual comes to exemplify a given property, either in general or in a particular case. Now one answer is that there is nothing to explain here, that this is a place where explanation must stop. But from another point of view, it is highly mysterious that a particular should exemplify a property; for the property is an abstract object, divorced as such from any particular, and the particular is a concrete object, divorced as such from any property. How then can they come together in the distinctive manner required by exemplification?

If the demand for an explanation is accepted, then another possible reply to our question is that this coming together should be explained in terms of a unity, a fact to which both constituents belong. It is not that the existence of the fact is to be explained in terms of the particular exemplifying the relation, since that leaves the nature of the relationship unexplained. Rather the exemplification itself must be explained in terms of the fact.

If this were so, then both the instance- and the concept-oriented versions of the correspondence theory would require the appeal to facts. For the fundamental ontological ground for any true proposition would presumably consist, either wholly or in part, in the existence of certain atomic facts, and definitions of truth for propositions would presumably lead back to logical relations like exemplification, which would then have to be analysed in terms of facts.

But whether any form of fact realism is reasonable is hard to say. The question is an instance of the more general issue of holism, of whether properties and relations of parts are to be analysed in terms of the wholes to which they belong, or vice versa. In some cases, it seems clear whether or not a holistic or non-holistic form of explanation is called for. It seems clear, for example, that what it is for two people to be married is to be explained in terms of various socio-legal relationships and not in terms of their belonging to a unifying entity, the marriage. On the other hand, it seems plausible that physical instants and points should be analysed in terms of larger physical units.

But whatever the answer in the present case, it is important to insist on the legitimacy of the issue. It has too readily been assumed that because facts are redundant for ordinary purposes they are valueless for the formulation of philosophical doctrine. But let it be granted that reference to facts can be eliminated from all ordinary contexts. It would still be in order for the philosopher to insist that the reduction respect his intuitions on ontological ground and to reject it if it broke the explanatory link between the truth of a proposition and the existence of a fact. The philosopher need not be so self-effacing that he cannot make demands on the adequacy of a reduction that arise solely from the needs of his own discipline.

3. PURE WORLD THEORIES

We shall prove some simple results on the pure modal theory of worlds.

It will be helpful, at the outset, to display some of the languages we shall be considering in the rest of the paper. Recall from [11] that a language is specified by its set of non-logical predicates. The formulas of the language are then constructed from the non-logical predicates, the 'logical' predicates E and $=$, and the logical operations in the usual way.

There are five main languages in all:

(i) $\mathscr{L}^w$, the pure language of worlds. There are *no* non-logical predicates, and the intended range of the variables $w_1, w_2, \ldots$ is worlds.

(ii) $\mathscr{L}^p$, the pure language of propositions. The sole non-logical predicate is T (for truth), and the variables $\rho_1, \rho_2, \ldots$ are to range over propositions;

(iii) $\mathscr{L}^f$, the pure language of facts. There are again *no* non-logical predicates. The variables $f_1, f_2, \ldots$ are to range over facts.

(iv) $\mathscr{L}^{p,f}$, the language of propositions and facts. The predicates are T (for truth) and P (for is a proposition). The variables $x_1, x_2, \ldots$ are to range over facts and propositions. The symbols $\rho_1, \rho_2, \ldots$ will be used for variables relativized to propositions, and the symbols $f_1, f_2, \ldots$ for variables relativized to facts (non-propositions).

(v) $\mathscr{L}^{p,f,c}$, the language of propositions and facts, with correspondence. This is like the language $\mathscr{L}^{p,f}$ above, but for the addition of a two-place predicate C (for correspondence).

Occasionally, we shall admit languages of the same general sort and languages of different sorts; but these will be specially indicated as the need arises.

Note that we have departed from the official policy in [11] of using 'x_1', 'x_2', ... for variables. Nothing turns on this; it is merely to help the reader recall the intended interpretation.

I shall sometimes use the same symbols, both as object-language variables and as meta-linguistic variables for elements of a model. No confusion should arise from this practice; but where there is danger that it may, I shall underscore the object-language variables.

The language $\mathscr{L}^w$ for worlds is the simplest kind of modal language, completely devoid of any special predicates. Its only predicates are the 'universal' ones for existence and identity, and, for this reason, such languages, and the theories based upon them, may be said to be ones of *pure existence and identity*. In principle, though not permitted by our definitions, such a language might omit either the existence or the identity predicate, and in these cases we talk, respectively, of *pure identity* and *pure existence* languages.

In classical logic, languages and theories of such expressive paucity have little interest, either as objects of technical study or as the means for formalizing a given subject-matter. For all that can be expressed within them are simple facts, statable in obvious normal form, concerning the cardinality of the universe. In modal logic, on the other hand, such languages are of far greater interest. For despite their meagre means, they permit the expression of highly complex facts, statable in no obvious normal form, concerning the distribution of existents from one world to another, and they thereby allow for the formulation of theories of both technical and philosophical interest.

The structures $\mathfrak{A}$ for the language $\mathscr{L}^w$ will simply be of the form $(W, \bar{A})$, where W(worlds) is a non-empty set and $\bar{A}_w$ (individuals of w) is non-empty for at least one w. The truth-definition will then proceed in the usual way. But note: in accord with the actualist intuitions of §2, the variables will range, in each world, not over all worlds, but only over that world.

Let W-Th be the theory whose axioms are as follows:

World Existence $\Box \exists w E w$;

World Identity $\Box \forall w \forall v\ (w = v)$.

According to World Existence, necessarily there is an actual world; and according to World Identity, necessarily there is at most one actual world. It is these axioms that justify our talk of *the* actual world or of what would, counterfactually, be *the* actual world. We could have added a constant w_0 for the actual world to our language. But for most purposes such a constant would be redundant, since each formula $\phi(w_0)$ could be replaced by $\exists w \phi(w)$.

A word on nomenclature. We shall be referring to theories and sets of conditions, each pertaining to one of the languages $\mathscr{L}^w$, $\mathscr{L}^p$, $\mathscr{L}^f$, $\mathscr{L}^{p,f}$ or $\mathscr{L}^{p,f,c}$. The language in question will be indicated by a suitable prefix—W, P, F, etc., and whether the object is a theory or a set of conditions will be indicated by the use of 'Th' or 'Cond.'. Thus W-Th is a theory of worlds, while P-Cond is a set of conditions for propositions. Sometimes the prefixes will be used to distinguish axioms for the different languages that bear the same name and other variants on the notation may be introduced in an obvious manner.

Let us briefly indicate some syntactic results for the theory W-Th. First, it is readily shown that the two axioms of the theory are independent. Secondly, it may be shown that the sentence $\Box\forall w\Box\forall v[w = v \equiv \Diamond(Ew \wedge Ev)]$ is a theorem. Hence identity may be defined in terms of modality and existence. It may be wondered, more generally: when can identity be defined within a model theory T, i.e. when can one find an identity-free formula ϕ such that $(\Box)$ $[x = y \equiv \phi]$ is a theorem of T. Say that two individuals $e, f \in A$ of a modal structure $\mathfrak{A}$ are *absolutely indiscernible* if $w \models Re_1e_2 \ldots e_n$ iff $w \models Rf_1f_2 \ldots f_n$ whenever $w \in W$, $e_1, \ldots, e_n$, $f_1, \ldots, f_n \in A$ and $e_i = f_i$ or $e_i, f_i \in \{e, f\}$ for all $i = 1, 2, \ldots, n$. A similar definition may be given in the classical case. Then it is a consequence of Beth's Definability Theorem, in the case of classical logic, that identity is definable in a theory iff the theory admits of no models with distinct absolute indiscernibles. It is natural to suppose that the corresponding result holds for modal logic. However, it is shown in [12] that Beth's Definability fails in quantified S5, and I conjecture that the result would also fail for the special case in which identity is to be defined.

Let *World Completeness* be the scheme:

$$(\Box)[(Ew \wedge \phi) \supset \Box(Ew \supset \phi)].$$

Then, finally, we may note that this scheme is a theorem of W-Th for any formula ϕ. Indeed, in *any* theory, the scheme is equivalent to those of its instances in which ϕ is an atomic formula whose predicate is either non-logical or else E. Thus in the present case, the scheme is simply equivalent to $\Box\forall w\Box\forall v(Ew \supset \Box(Ew \supset Ev))$, which is a consequence of identity.

The principles of World Existence and World Completeness have already appeared in [16], where they may be obtained from the formulas under clauses (i) and (ii) of lemma 4.4 by letting Wx be the predication Ex. Hence all of the relevant results of that section also apply to the theory W-Th. In particular, the possibilist quantifier Πw, ranging over all worlds in each world, may be defined in W-Th, and the notion of truth-in-a-world defined by means of the formula $\Box(Ew \supset \phi)$.

Semantically, the theory W-Th is rather simple. It is readily shown that:

LEMMA 1. $\mathfrak{A} = (W, \bar{A})$ is a structure for W-Th iff $\bar{A}_w$ is singleton for each $w \in W$.

Of course, not all such $\mathfrak{A}$ will be intended models for the theory, even when W is the set of all possible worlds. So let us say that a structure $\mathfrak{A} = (W, \bar{A})$ is *natural* if $\bar{A}_w = \{w\}$ for each w. Recall that a structure $\mathfrak{A}$ is *differentiated* if $\mathfrak{A}_w = \mathfrak{A}_v$ implies $w = v$ for all $w, v \in W$. Then it is again readily shown that:

LEMMA 2. Each differentiated structure $\mathfrak{A}$ for W-Th is isomorphic to a natural structure (in the sense that there are separate one-one correspondences for worlds and individuals).

Let us use W-Cond. for the condition of being natural. Then since each structure is equivalent to a differentiated structure, it follows from the completeness theorem for modal logic that:

THEOREM 3. The theory W-Th is sound and complete for the condition W-Cond., i.e. a sentence is a theorem of W-Th iff it is true in all models satisfying W-Cond.

Strong completeness may similarly be proved.

The theory W-Th has the finite model property and hence is decidable. For it may be shown by an easy induction on the complexity of $\phi(w_1, w_2, \ldots, w_n)$ that:

LEMMA 4. Suppose that $\phi(w_1, w_2, \ldots, w_n)$ is a formula of $\mathscr{L}^w$ containing at most n occurrences of the symbol $\Box$. Let $\mathfrak{M} = (\mathfrak{A}, w_0)$ and $\mathfrak{N} = (\mathscr{C}, v_0)$ be natural models of W-Th, and $w_1, w_2, \ldots, w_n$ and

$v_1, v_2, \ldots, v_n$ elements of W and V respectively for which:

(i) $w_i = w_j$ iff $v_i = v_j$

for $0 \leq i < j \leq n$;

(ii) $$\min(n, \mathrm{card}(W - \{w_0, w_1, \ldots, w_n\})) = \min(n, \mathrm{card}(V - \{v_0, v_1, \ldots, v_n\})).$$

Then $(\mathfrak{A}, w_0) \models \phi(w_1, w_2, \ldots, w_n)$ iff $(\mathscr{C}, v_0) \models \phi(v_1, v_2, \ldots, v_n)$.
So it follows that:

THEOREM 5. The theory W-Th has the finite model property and is decidable.

As should be clear from this proof, the theory W-Th bears an intimate relationship to the pure classical theory of identity. There is a translation, outlined in section 2 of [11], from any modal language and theory into a classical language and theory. Under this translation, we may suppose that E*, the enlargement of the existence predicate, is simply identity. Conversely, $x_i = x_j$ in the classical language may be replaced by $\Diamond(\mathrm{E}w_i \wedge \mathrm{E}w_j)$ and $\exists x_i$ by $\Diamond\exists w_i$. More precisely, if * is the one translation and ′ the other, then it may be shown that if the classical (modal) formula ϕ is a theorem of the classical (modal) system then ϕ' (ϕ^*) is a theorem of the modal (classical) system, and that $\phi \equiv \phi^{*\prime}$ is a theorem of the modal system for each modal formula ϕ. Corresponding semantical results may also be established.

It is of interest to consider the properties of some of the theories weaker than W-Th. Let W-Th$^-$ be the theory whose axioms are World Existence and World Completeness. Such a theory would be of interest to someone who accepted the (necessary) existence of worlds but denied their uniqueness. Then by techniques similar to those used in establishing theorem 5, it may be shown that:

THEOREM 6. The theory $T \cdot W^-$ has the finite model property and is decidable.

Now let W-Th′ be the theory whose sole axiom is World Existence and W-Th^0 the theory with no axioms at all.
Then:

THEOREM 7. The theories W-Th′ and W-Th^0 are undecidable.

The proof proceeds by embedding the classical theory of a reflexive and symmetric relation into the modal theory W-Th′ through the definition of the relation as $\Diamond(\mathrm{E}w \wedge \mathrm{E}v)$. (The undecidability of the theory of a reflexive and symmetric relation is established in Rogers [34] and the technique for reducing the modal theory to this case derives from Kripke [17] and Slomson [38].)

The theory $W\text{-Th}^0$ is, of course, merely the pure modal theory of identity. Thus the undecidability result for this theory stands in contrast to the classical case; for there, as follows from earlier observations or from Ackermann ([1], Chapter 3), the pure theory of identity is decidable.

4. APPLIED WORLD THEORIES

All of the world theories considered in §3 were pure; they contained no extra-logical vocabulary. But we may also consider applied world theories, obtained from a given theory by adding a standard world component. Let T be the given theory, with language $\mathscr{L}$. Then the *world expansion* T^+ *of* T is defined as follows. The language $\mathscr{L}^+$ of T^+ is a two-sorted one, consisting of the individual variables $x_1, x_2, \ldots$ of $\mathscr{L}$, the new world variables $w_1, w_2, \ldots$, and the predicates of $\mathscr{L}$. In the formation rules it is stipulated that the non-logical predicates of $\mathscr{L}$ only apply to individual variables. However, the existence- and identity-predicates may apply indiscriminately to either sort of variable. The logical axioms of T^+ are like those of the one-sorted theories, but with the appropriate sortal restrictions. The non-logical axioms of T^+ are those of T (in the individual variables), those of W-Th (in the world variables), and all instances of the World Completeness scheme. Note that the instances will come from the expanded language $\mathscr{L}^+$. By our earlier remark, the scheme will be equivalent to those of its instances in which ϕ is an atomic formula of the original language $\mathscr{L}$. Thus if T is finitely axiomatized and has a finite language, then the same will be true of T^+.

Various extensions to T^+ can be given. In case the language $\mathscr{L}$ is finite, one may make the important assumption that $\mathscr{L}$ is capable of differentiating between distinct worlds. Suppose, for example, that $\mathscr{L}$ consists of a one-place predicate P and a two-place predicate R. Then the Differentiation Assumption would take the form:

$$\Box\forall w\Box\forall v(w\neq v \supset \Diamond\exists x\Diamond\exists y[-(\Box(Ew \supset Px) \equiv \Box(Ev \supset Px)) \vee -(\Box(Ew \supset Rxy) \equiv \Box(Ev \supset Rxy))].$$

One may also make various assumptions about the identity or distinctness of worlds and individuals. It may be assumed, for example, that the two are disjoint ($\Box\forall w\Box\forall x(w \neq x)$) or that the worlds are among the individuals ($\Box\forall w\Diamond\exists x(w = x)$), in which case the previous Differentiation Assumption would be a consequence.

The world-expansion T^+ bears an intimate connection to the classical two-sorted analogue T^{fs} of T. Indeed, the interest of the world-expansion largely derived from its being a modal intermediary between the orthodox model theory T and the fully classical theory T^{fs}. The language $\mathscr{L}^{fs}$ may be defined as was $\mathscr{L}^s$ in the last subsection (*Classical Analogues*) of §5 of [11], with the single exception that identity may now be flanked by any two variables. For convenience, we shall use the same symbols for world- and individual-variables in $\mathscr{L}^{fs}$ as in $\mathscr{L}^+$. There is a standard translation, described in [11], of each formula ϕ of $\mathscr{L}$ into a formula ϕ^s of $\mathscr{L}^{fs}$. The non-logical axioms of T^{fs} are then simply the translates ϕ^s of the non-logical axioms ϕ of T.

There are extensions to the theory T^{fs}, analogous to those considered for the theory T^+, that need not be separately considered.

As one would expect, the theories T^+ and T^{fs} are mutually interpretable. The translation of $\mathscr{L}^+$ into $\mathscr{L}^{fs}$ is obtained from the translation of $\mathscr{L}$ into $\mathscr{L}^s$ (given in [11]) by adding the clauses:

(i) $(Ev)_w = w = v$
(ii) $(s = t)_w = s = t$, for s and t any variables;
(iii) $(\forall v\phi)_w = (\phi)_w$

On the other hand, the translation of $\mathscr{L}^{fs}$ into $\mathscr{L}^+$ is defined with the help of the following clauses:

(i) $(R^*wx_1 \ldots x_n)^+ = \Box(Ew \supset Rx_1 \ldots x_n)$
(ii) $(E^*wx)^+ = \Box(Ew \supset Ex)$
(iii) $(s = t)^+ = s = t$
(iv) $(-\phi)^+ = -(\phi)^+$
(v) $((\phi \vee \psi))^+ = ((\phi)^+ \vee (\psi)^+)$
(vi) $(\forall x\phi)^+ = \Box\forall x\Box(\phi)^+$
(vii) $(\forall w\phi)^+ = \Box\forall w\Box(\phi)^+$

Suppose that ω is the actual world constant in $\mathscr{L}^{fs}$ (formerly called w_0^*). Then the translate $(\phi)^+$ of a sentence ϕ of $\mathscr{L}^{fs}$ containing ω is $\exists w(\phi(w))^+$, where $\phi(w)$ is the result of substituting a new world variable w for ω in ϕ.

It may now be shown that:

THEOREM 8 (i) For any sentence ϕ of $\mathscr{L}^+$, $T^+ \vdash \phi$ implies $T^{fs} \vdash \phi^s$ and $T^+ \vdash \phi \equiv (\phi^s)^+$;

(ii) for any sentence ϕ of $\mathscr{L}^{fs}$, $T^{fs} \vdash \phi$ implies $T^+ \vdash (\phi)^+$ and $T^{fs} \vdash \phi \equiv (\phi)^{+s}$.

Proof. Either semantically, using the natural correspondence between the classical models of T^{fs} and the differentiated models of T^+, or else syntactically.

The result may be extended to the various strengthenings of T^+ and T^{fs} in the obvious way.

In regard to the provability within T^+ of $\phi \equiv (\phi^s)^+$, we may note that the combined translation $((\phi)_w)^+$ provides a modelling of the possible worlds semantics within T^+ of the sort depicted in corollary 7 of Fine [16]. Indeed, part of the value of the theorem is that it allows one to simplify the proof that the classical theory T^{fs} is interpretable within T by establishing interpretability for the modal theory T^+, that is so much closer in form to T.

Usually, the classical theory T^{fs} or, equivalently, the world expansion T^+ will have greater expressive power than the original modal theory T. Therefore considerable interest attaches to positive results which establish the interpretability within T of its own semantics T^{fs} and to negative results which show that T and T^{fs} are far apart.

On the positive side, we have the interpretability results of [15] and [16]. However, such results need to be stated with great care. For in the earlier interpretations, the individuals of T^+ were identified with themselves, and the worlds of T^+ with a restricted class of the individuals. In effect, each formula ϕ of $\mathscr{L}^+$ was translated into a formula ϕ^0 of $\mathscr{L}$ by replacing each subformula Ew of ϕ with Ww and each subformula $\forall w\psi$ of ϕ with the relativization $\forall x(Wx \supset \psi)$ (changing variables to avoid conflict). Now although it may be true, for suitable Wx, that:

(i) if $T^+ \vdash \phi$ then $T \vdash \phi^0$;

it will not in this case also be true that:

(ii) if $T \vdash \phi^0$ then $T^+ \vdash \phi$;

unless T and hence T^+ are inconsistent. For let ϕ be the differentiation-type formula $\Box\forall v\Box\forall w[\Diamond(Ew \wedge -Ev) \supset$

$\exists x(\Box(Ew \supset Wx) \vee -\Box(Ev \supset Wx))]$. Then as long as the one direction (i) of the interpretability result is provable, ϕ^0 will be a theorem of T, since it will follow by letting "x" be the world "w". On the other hand, ϕ itself will not be a theorem of T^+, unless T is inconsistent; for otherwise there will be a model $\mathfrak{M}$ for T, and therefore a model $\mathfrak{M}^+$ for T^+ and $-\phi$, obtained by associating distinct worlds with a given individual structure $\mathfrak{A}_w$ of $\mathfrak{M}$.

The best that can be established for the previous translation is the following:

THEOREM 9. Suppose that T satisfies the conditions (i) and (ii) of lemma 4.4 of [16], i.e. there is a formula Wx of T with one free variable x such that T proves $\Box\exists xWx$ and $(\Box)[(Wx \wedge \phi) \supset \Box(Wx \supset \phi)]$, for ϕ any atomic formula of $\mathscr{L}$ not containing x. Let ϕ be any sentence of $\mathscr{L}^+$ in which each atomic subformula containing a world-variable occurs in the context $\Box(Ew \supset \psi)$, for ψ an atomic formula of $\mathscr{L}$. Then:

(i) $T^+ \vdash \phi$ iff $T \vdash \phi^0$;
(ii) $T^+ \vdash \phi \equiv \phi^0$.

Proof. Semantically in terms of the obvious correspondence between the worlds and the individuals satisfying Wx in any model of T^+.

Although the restriction on ϕ may seem rather strange, it corresponds, in $\mathscr{L}^s$, to the natural condition that no world-variables should flank identity. Thus, as should be expected, the translation ϕ^0 only preserves the truths and falsehoods of the semantical metatheory in so far as they do not concern the identity of worlds.

This restriction on ϕ is not really acceptable, since it means that certain questions from $\mathscr{L}^s$ may be improperly answered in T. One would like, first of all, to allow atomic subformulas Ew to occur in ϕ or, equivalently, identities between world-variables.[17] There are then various ways in which the interpretability of T^+ within T might still be established. One possibility is to change the nature of the translation, so that the individuals of T^+ are now associated with a subclass of the individuals of T; but in the nature of the case it would be difficult to do this in such a way as to secure exact interpretability. Another possibility is to add new axioms to T^+. In case the language

$\mathscr{L}^+$ is finite, we may add the Differentiation Assumption as an axiom to T^+. Theorem 8 can then be proved without restriction on ϕ, as long as T also proves $(\Box)[\Diamond(Wx \wedge Wy) \supset x = y]$ or as long as, in the translation ϕ^0, the identities $w = v$ are replaced with $\Box(Wx \equiv Wy)$.

Whether such additions to T^+ are legitimate depends upon the theory under consideration. The question is: does the vocabulary of T enable us to distinguish between what, intuitively, are distinct possible worlds? In the case of modal set theory, it does not; for distinct worlds may have the same sets. In the case of modal proposition theory, though, it does; since any two worlds will differ in their true propositions. Thus although the reductions of the world-based theories to the modal theories are on a formal par for sets and propositions, there is an important intuitive distinction between the two. For the set-theoretic reduction requires us to conflate intuitively distinct worlds, whereas the propositional reduction does not. Although such a difference does not matter for reductions from a restricted language, it does matter for extensions of the reduction in which the identity of worlds can explicitly be discussed.

In addition to identities between world-variables, we should allow identities between world- and individual-variables within the scope of our reduction. But then further problems arise. For let ϕ be the sentence $\forall w \exists x(w = x)$. Then on the standard reductions, ϕ^0 will be a theorem of T and yet, unless T is inconsistent, ϕ will not be a theorem of T. Again, this difficulty may be removed either by changing the nature of the translation, which would appear to be difficult, or by adding the Inclusion Assumption $\forall w \exists x(w = x)$ to T^+. And again, whether this is legitimate depends upon whether worlds are, intuitively, individuals of the given theory T. For the modal theories of propositions and sets, they are not. But for the modal theory of facts, they are. Thus we see that the latter theory has a big advantage over the others; it allows a reduction of the full classical theory of worlds, with no bars on the expressible identities. If the other theories also permit a full reduction, it is not in any straightforward way.

In theorem 8 and its envisaged extensions, I have considered a very simple and particular reduction, ϕ^0, from T^+ to T; and it is therefore natural to wonder if there are essentially different ways in which a reduction might be achieved. One natural idea is to expand on the number of variables associated with the formula Wx. So let us suppose that with each world-variable w are associated new and distinct individual-variables $x_1, x_2, \ldots, x_n$, with no single individuable-variable associated in this way with two distinct world-variables. Let $\chi(x_1, x_2, \ldots, x_n)$ be a particular formula in the free variables $x_1, x_2, \ldots, x_n$. Now set up a translation ϕ^1 from $\mathcal{L}^+$ to $\mathcal{L}$ in the usual way, but with Ew being replaced by $\chi(x_1, x_2, \ldots, x_n)$ and $\forall w\psi$ being replaced by $\forall x_1 x_2 \ldots x_n(\chi(x_1, x_2, \ldots, x_n) \supset \psi^1)$. Such a translation ϕ^1 might be called *standard*. Then it can be shown that ϕ^1 is a reduction, satisfying (i) and (ii) of theorem 9, as long as T proves $\Box\exists x_1 x_2 \ldots x_n \chi(x_1, x_2, \ldots, x_n)$ and $(\Box)[\chi(x_1, x_2, \ldots, x_n) \wedge \phi \supset \Box(\chi(x_1, x_2, \ldots, x_n) \supset \phi)]$, for all atomic ϕ not containing $x_1, x_2, \ldots, x_n$.

It may also shown that, for each $n = 2, 3, \ldots$, there is a theory T_n that admits a standard reduction for a formula $\chi(x_1, x_2, \ldots, x_n)$ of n free variables, but not for a formula of fewer free variables. To illustrate how the construction T_n goes, let T_2 be the theory with no non-logical constants and with non-logical axioms $\Box\exists x \exists y(x \neq y \wedge \forall z(z = x \vee z = y))$ and $\Box\forall x \Box\forall y(x \neq y \supset \Diamond(Ex \wedge Ey))$. Then T_2 permits a standard reduction of T_2^+ with $\chi(x_1, x_2)$ as $x_1 \neq x_2 \wedge Ex_1 \wedge Ex_2$. But since any permutation on the individuals of a model $\mathfrak{M}$ of T_2 induces an automorphism on $\mathfrak{M}$, it may be shown that no reduction can be effected with a single free-variable formula in place of $\chi(x_1, x_2)$. The result for T_2 is of some philosophical interest, since it indicates how a reduction from T^+ to T might be achieved without there being a definable proxy function.

Are there any other reductions of T^+? One might consider a translation in which each subformula $\forall w\psi$ was replaced by a fixed and elementarily definable function of ψ (I need not be exact about details). I would then conjecture, though with not much confidence, that any theory that permitted a reduction of this more general sort also permitted a standard reduction.

Let us now consider the negative question of how far short the modal theory T may fall from its classical counterpart T^s. Corresponding to the different restrictions on $\mathcal{L}^+$ are three different classical theories: T^{rs}, with no world-variables flanking identities; T^s, with no world- and individual-variable flanking an identity; T^{fs}, with no restriction on identities. The successive gaps between T and T^{rs}, T^{rs} and T^s, and T^s and T^{fs} may then be considered.

In regard to completeness, it should be clear that the one theory can be complete when its successor is not, because of the difference in expressive power. The only case that would seem to cause any difficulty is that of T and T^{rs}. To deal with this case, we may modify an example given in the proof of theorem 3.2 of [12]. Let T be the modal theory with no non-logical predicates and with axioms:

$$\exists x_1 \exists x_2 \dots \exists x_n \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j;$$

$$\Box \forall x \Diamond - Ex; \text{ and}$$

$$(\Box)(-Ex \wedge -Ey) \supset x = y].$$

Then:

THEOREM 10. The theory T is complete, although the theory T^{rs} is not.

Proof. Let $\mathfrak{A}$ and $\mathfrak{B}$ be any structures for T. Then it may be shown by a simple induction on ϕ that:

If (a) $e_i = e_j$ iff $f_i = f_j$ and (b) $e_i \in \bar{A}_w$ iff $f_i \in \bar{B}_v$ for $i, j = 1, 2, \dots, n$, then $(\mathfrak{A}, w) \models \phi[e_1, \dots, e_n]$ iff $(\mathfrak{B}, v) \models \phi[f_1, \dots, f_n]$.

Since any two models of T are elementarily equivalent, T is complete.

Now let $\mathfrak{A} = (W, \bar{A})$ be the structure in which $W = \{0, 1, \dots\}$, $A = \{1, 2, \dots\}$ and $\bar{A}_n = A - \{n\}$ for $n = 0, 1, \dots$; let $\mathfrak{B}$ be the restriction of $\mathfrak{A}$ to $V = \{1, 2, \dots\}$; and let ϕ be the sentence $\exists w \forall x E^* wx$ of $\mathscr{L}^{rs}$. Then $\mathfrak{A}$ and $\mathfrak{B}$ are both models for T. But ϕ is true in $\mathfrak{B}$ but not in $\mathfrak{A}$, and therefore ϕ goes undecided in T.

More interesting than completeness is the gap in decidability between two theories. The most important case is that in which the modal theory T is decidable but the weaker classical theory T^{rs} is not. Let T be the pure theory whose non-logical axioms are:

$$\Box \exists x_1 \exists x_1 \dots \exists x_n \Big(\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \Big); \text{ and}$$

$$(\Box) \Big[\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j$$
$$\supset \Diamond (Ex_1 \wedge \cdots \wedge Ex_m \wedge -Ex_{m+1} \wedge \cdots \wedge -Ex_n) \Big], \quad \text{for all } m, n = 1, 2, \dots \text{ and } m \leq n.$$

Then:

THEOREM 11. T is decidable, while T^{rs} is not.

Proof. We show that T is decidable by showing that it is complete. To this end, it suffices to show that any two models $\mathfrak{M}$ and $\mathfrak{N}$ of T are elementarily equivalent. But, as with the proof of theorem 10, we may show by a simple induction on $\phi(x_1, x_2, \dots, x_n)$ that:

(1) Suppose that $\mathfrak{M} = (\mathfrak{A}, w)$ and $\mathfrak{N} = (\mathscr{C}, v)$ are models of T, and that $e_1, e_2, \ldots, e_n$ and $f_1, f_2, \ldots, f_n$ are elements of A and B respectively such that (a) $e_i = e_j$ iff $f_i = f_j$ and (b) $e_i \in \bar{A}_w$ iff $f_i \in \bar{B}_v$ for all $i, j = 1, 2, \ldots, n$. Then:

$$(\mathfrak{A}, w) \models \phi[e_1, e_2, \ldots, e_n] \text{ iff}$$
$$(\mathfrak{A}, w) \models \phi[f_1, f_2, \ldots, f_n].$$

The required result on elementary equivalence now follows.

The classical theory T^c of a partial ordering R is undecidable (see Tarski [42]). We show that T is undecidable by interpreting T^c within T^{rs}. Set a world-variable w_0 aside. Translate each formula ϕ of T^c into a formula ϕ' of T^{rs} by replacing each atomic subformula Rx_ix_j with $\forall x(E^*w_ix \supset E^*w_jx)$ and each quantified subformula $\forall x_i\psi$ with $\forall w_i(\forall x(E^*w_0x \supset E^*w_ix) \wedge -\forall x(E^*w_ix \supset E^*w_0x) \supset \psi')$, for $i, j = 1, 2, \ldots$. It may now be shown that:

(2) for each sentence ϕ of $\mathscr{L}^c$, $T^c \vdash \phi$ iff $T \vdash \forall w_0\phi'$.

The difficult part of the proof is from right to left. So suppose $T^c \nvdash \phi$. Then for some model $\mathfrak{M} = \langle A, r\rangle$ of T^c (where r gives the extension of R), $\mathfrak{M} \not\models \phi$. Define a model $\mathfrak{N} = \langle W, B, e^*\rangle$ of $\mathscr{L}^{rs}$ (where e^* gives the extension of E^*), by letting:

$B = A \cup C_0$, for C_0 some infinite set disjoint from A;

$W = \{C_0 \cup \{b : \langle b, a\rangle \in r\} : a \in A\} \cup \{C \subseteq A : C$ is infinite, yet $C \cap C_0$ is finite$\}$;

$e^* = \{\langle w, b\rangle \in W \times B : b \in W\}$.

Clearly, η is a model for T^s. Now let f be the map from A into W for which:

$$f(a) = C_0 \cup \{b : \langle b, a\rangle \in r\}.$$

Then f is an isomorphic embedding of $\langle A, r\rangle$ into $\langle W, \subseteq\rangle$, from which it readily follows that $\mathfrak{N} \not\models \forall w_0\phi'$.

To take care of the gap between T^{rs} and T^s, let T be the previously considered pure theory with axioms $\Box\exists x\exists y(x \neq y \wedge \forall z(z = x \vee z = y))$ and $\Box\forall x\Box\forall y(x \neq y \supset \Diamond(Ex \wedge Ey))$.

THEOREM 12. T^{rs} is decidable, but T^r is not.

Proof. To show that T^{rs} is decidable, associate with each world-variable w_i a pair x_i, y_i of distinct individual-variables. Translate each formula ϕ of T^{rs} into another formula ϕ' of T^{rs} by replacing each atomic subformula E^*w_ix with $x = x_i \vee x = y_i$ and each quantified subformula $\forall w_i\psi$ with $\forall x_i, y_i(x_i \neq y_i \supset \psi')$. Then it may be shown that:

(1) for each sentence ϕ of $\mathscr{L}^{rs}$, $T \vdash \phi$ iff $\exists x \exists y(x \neq y) \supset \phi'$ is a theorem of the pure classical theory T^I of identity.

The decidability of T^{rs} then follows from the decidability of T^I.

To show that T^r is undecidable, we may embed the classical theory of a symmetric and irreflexive relation into T^r. The essential idea of the proof is to define the relation Rxy by $x \neq y \wedge \exists w \exists v(w \neq v \wedge E^*wx \wedge E^*wy \wedge E^*vx \wedge E^*vy)$.

Finally, the gap between T^s and T^{fs} can be handled by means of the modal theory T of pairs of individuals. T contains one non-logical predicate R of degree 3. Let Px(x is a pair) abbreviate $\exists y, z(y, zRx)$ and Ix(x is an individual) abbreviate $-Px$. The non-logical axioms of T are $\Box\forall x\Box\exists x$, $\Box\forall x\forall y \quad (Ix \wedge Iy \supset \exists! z(x, yRz))$, $\Box\forall x, y, z, s, t(x, yRz \wedge s, tRz \supset x \in s \wedge y = t)$, $\Box\forall x(Px \supset - \exists y, z(x, yRz \wedge y, xRz))$, $\Box\forall x, y, z(x, yRz \subset \Box x, yRz)$.

THEOREM 13. T^s is decidable and T^{fs} undecidable.

Proof. The theory T^s can be reduced to the pure classical two-sorted theory of identity by interpreting quantification $\forall x(Px \supset \ldots)$ over pairs as multiple quantification $\forall x \forall y \ldots$. The theory T^{fs} is undecidable since the classical theory of an arbitrary two-place relation S can be interpreted with T^{fs} upon defining Sxy as $\exists w \exists z(w = z \wedge Pz \wedge x, yRz)$.

5. ANTI-OBJECTUALIST FACTS

This section gives an account of fact theories under the anti-objectualist approach. We begin with truths and follow with circumstances. To a large extent, our treatment will parallel that in §2 of [15] for propositions; and so, for that reason, we may be brief.

The structures for $\mathscr{L}^f$ will be of the form $(W, \bar{F})$, where W (worlds) is a non-empty set and $\bar{F}_w$ (facts of W) is a set for each $w \in W$, with $F = \bigcup_{w \in W} \bar{F}_w$ non-empty. Structures of this sort will be called *fact structures*, or *F-structures* for short. Recall that it is our intention that the variables range over the actual, not the possible, facts of each world; and so we should not expect $\bar{F}$ to be a constant function.

It will be helpful to appropriate some of the terminology for existence-sets from §1 of [15]. Accordingly, given a structure $\mathfrak{B} =$

$(W, \bar{F})$, a world w in W, an element f of F, and a subset G of F, let:

$$es(f) = \{w \in W : f \in \bar{F}_w\};$$
$$es(G) = \{w \in W : g \in \bar{F}_w \text{ for all } g \in G\} = \bigcap_{g \in G} es(g);$$
$$ES_w = \{es(e) : e \in \bar{F}_w\};$$
$$ES = \bigcup_{w \in W} ES_w = \{es(e) : e \in F\}.$$

When necessary, the given structure $\mathfrak{B}$ may be indicated by a superscript.

When an anti-objectualist stance is combined with (a) a conception of facts as derivative, (b) an empirical criterion for their identity, and (c) a Platonic attitude towards propositions, as in §2 of [15], we are led to the following two conditions, $F(A)$-Cond, on a model $\mathfrak{N} = (\mathfrak{B}, w)$:

(i) *Platonism*: for each non-empty $V \subseteq W$, there is an $f \in F$ such that $es(f) = V$;

(ii) *Empirical Identity*: $es(f) = es(g)$ implies $f = g$ for all $f, g \in F$.

In order to facilitate the formulation of the theory for the two conditions, we shall use the following two abbreviations:

$$e \approx_e f \text{ for } \Box(Ee \equiv Ef);$$
$$We \text{ for } \forall f \Box(Ee \supset Ef).$$

According to the first, e and f have the same existence conditions; and according to the second, e is a *world fact*, one whose existence requires the existence of all other (actual) facts. The axioms of the theory $F(A)$-Th are then:

Naive Abstraction: $(\Box)[\phi \supset \exists f \Box(Ef \equiv \phi)]$, where ϕ is any formula and f is a variable not free in ϕ.

Empirical Identity: $(\Box)(e \approx_e f \supset e = f)$.

World Fact: $\Box \exists f Wf$.

Naive Abstraction says that (necessarily) to each sentence there corresponds a fact whose existence conditions are the truth conditions of the sentence. Empirical Identity says that facts with the same empirical content are the same. And World Fact says that necessarily there is a world fact.

As the axioms have been stated, there is a necessary fact, one that necessarily exists. Without much difference to the formal development, the existence of a necessary fact could either be left open or else excluded. In the first case, the correspondence axiom should assume the form:

$$(\Box)[(\phi \wedge \Diamond - \phi) \supset \exists f \Box (Ef \equiv \phi)];$$

and in the second case, the axiom:

$$- \Diamond \exists f \Box Ef$$

should also be added. For the most part, though, I shall follow the simpler course of postulating a necessary fact.

Let us relabel the anti-objectualist propositional theory *MC* in §2 of [15], $P(A)$-Th. Recall that this theory is complete for the conditions, which we may now label $P(A)$-Cond:

Constant Domain: For any $e \in P$, $es(e) = W$;

Empirical Identity: $ts(e) = ts(f) \Rightarrow e = f$;

Platonism: For each $V \subseteq W$, $V \in TS^{\mathfrak{A}}$.

Between the anti-objectualist theories $F(A)$-Th and $P(A)$-Th, for propositions and facts, there is a rather obvious correspondence. For, in the one direction, we may identify facts with true propositions and, in the other direction, we may identify propositions with possible facts. With the latter identification, there are some difficulties, though; for, first, there are impossible propositions, but no impossible facts, and, second, there is only actual quantification over facts, but possibilist quantification over propositions. However, there are technical tricks for removing these difficulties.

The correspondence has both a semantical and a syntactical side. Recall from [15], that a *propositional* or *P-structure* $\mathfrak{A}$ for the language $\mathscr{L}^p$ is a triple $(W, \bar{P}, t)$, where W(worlds) is a non-empty set, $\bar{P}_w$(propositions), for each $w \in W$, is a set, with $P = \bigcup_{w \in W} \bar{P}_w$ non-empty, and t(truth) is a subset of $W \times P$. On the semantical side, we may first associate with each P-structure $\mathfrak{A} = (W, \bar{P}, t)$ an F-structure $\mathfrak{A}^f = (W^f, \bar{F})$, for which:

(i) $W^f = W$; and

(ii) $\bar{F}_w = \{e \epsilon \bar{P}_w : \langle w, e \rangle \in t\}$.

Conversely, with each F-structure $\mathfrak{B} = (W, \bar{F})$, we may associate a P-structure $\mathfrak{B}^p = (W^p, \bar{P}, t)$ for which:

(i) $W^p = W$;

(ii) $\bar{P}_w = F \cup \{\wedge\}$, for each $w \in W$, with $\wedge$ an element foreign to F; and

(iii) $t = \{\langle w, e \rangle \epsilon W \times P : e \in \bar{F}_w\}$.

The first transformation in effect identifies each fact with a true proposition, and the second transformation in effect associates each proposition, the impossible one excepted, with a possible fact.

Note that $(\mathfrak{B}^p)^f = \mathfrak{B}$ for any F-structure $\mathscr{C}$ and that $(\mathfrak{A}^f)^p$ is isomorphic to $\mathfrak{A}$ for any P-structure subject to $P(A)$-Cond.

Each transformation of structures is matched by a translation of languages, though in the opposite direction. In going from $\mathscr{L}^f$ to $\mathscr{L}^p$, the translate ψ^p of any formula ψ of $\mathscr{L}^f$ is obtained, first by replacing each atomic sub-occurrence Ef_i with $T^+\rho_i$, and then by replacing each subformula $\forall f_i\psi$ with $\forall\rho_i(T\rho_i \supset \psi)$. Recall from [15], §8, that $T^+\rho$ abbreviates the expression $T\rho \wedge E\rho$ for inner truth. In the anti-objectualist theories, $T^+\rho$ and $T\rho$ are equivalent; but the use of $T^+\rho$ in the translation will later be important for the objectualist theories.

The transition from $\mathscr{L}^p$ to $\mathscr{L}^f$ is more complicated. It will be helpful to suppose that the language $\mathscr{L}^p$ is enriched with an individual constant $\underline{\wedge}$ (for the impossible proposition). Call the new language $\mathscr{L}^{p,\wedge}$. Recall that $\Pi f\phi$ (for the possibilist quantifier) abbreviates $\exists g\Box\forall f\Box(Eg \supset \phi)$, where g is a variable distinct from f and not free in ϕ. Then the translate ϕ^f of a formula ϕ *of* $\mathscr{L}^{p,\wedge}$ is obtained, first by replacing each sub-formula $\forall\rho_i\psi(\rho_i)$ with $\Pi f_i\psi(\rho_i) \wedge \psi(\underline{\wedge})$, and then by making the following replacements for atomic sub-occurrences:

$\top$ for $E\rho_i$ or $E\wedge$;
Ef_i for $T\rho_i$;
$\bot$ for $T\wedge$;
$\bot$ for $\wedge = \wedge$; and
$\bot$ for $\wedge = \rho_i$ or $\rho_i = \wedge$.

The relationship between the transformations and the translations is given by the following results:

LEMMA 14. Let $\mathfrak{A}$ be any P-structure, $f_1, f_2, \ldots, f_n$ individuals from $\mathfrak{A}^f$, and $\psi(f_1, f_2, \ldots, f_n)$ any formula of $\mathscr{L}^f$ with free variables $f_1, f_2, \ldots, f_n$. Then:

$$(\mathfrak{A}^f, w) \models \psi[f_1, f_2, \ldots, f_n] \text{ iff } (\mathfrak{A}, w) \models \psi^p[f_1, f_2, \ldots, f_n];$$

LEMMA 15. Let $\mathfrak{B}$ be any F-structure, $e_1, e_2, \ldots, e_n$ individuals from $\mathfrak{B}^p$, and $\phi(\rho_1, \rho_2, \ldots, \rho_n)$ any formula of the expanded language $\mathscr{L}^{p,\wedge}$

with free variables $\rho_1, \rho_2, \ldots, \rho_n$. Then:

$$(\mathfrak{B}^p, w) \models \phi[e_1, e_2, \ldots, e_n] \text{ iff } (\mathfrak{B}, w) \models \phi^f[e_1, e_2, \ldots, e_n].$$

The proofs are by a straightforward induction on the complexity of the formula ϕ or ψ.

On the syntactic side, we have that the different translations preserve provability in the respective theories:

THEOREM 16. The theories $P(A)$-Th and $F(A)$-Th are equivalent in the sense that for each sentence ϕ of $\mathscr{L}^p$ and ψ of $\mathscr{L}^f$:

(i) $P(A)\text{-Th} \vdash \phi \Rightarrow F(A)\text{-Th} \vdash \phi^f$;
(ii) $F(A)\text{-Th} \vdash \psi \Rightarrow P(A)\text{-Th} \vdash \psi^p$; and
(iii) $P(A)\text{-Th} \vdash \phi \equiv (\phi^f)^p$ and $F(A)\text{-Th} \vdash \psi \equiv (\psi^p)^f$.

Proof. By a simple induction either on the length of proofs or the complexity of formulas (noting that some of the clauses (i)–(iii) are redundant).

One aspect of these translations may be regarded in a more general light. Let T be any arbitrary theory. Let P be a one-place predicate not in the language $\mathscr{L}$ of T, and let $\mathscr{L}^P$ be the result of adding the predicate P to $\mathscr{L}$. Finally, let ϕ^P, for ϕ a formula of $\mathscr{L}$, be the result of relativizing the quantifiers to P and of replacing each occurrence of Ex in ϕ with Px, and let T^P be the theory with language $\mathscr{L}^P$, whose axioms are $\Box\forall x\Box Ex$, $\Box\forall x\Diamond Px$ and all translates ϕ^P of axioms ϕ of T. Then it may be shown that, for each sentence ϕ of $\mathscr{L}$, $T \vdash \phi$ iff $T^P \vdash \phi^P$. Thus the translation ϕ^P gives an embedding of an arbitrary theory T into a theory T^P with the formula $\Box\forall x\Box Ex$ for constant domain.

If we now think of truth T as the special predicate P, then the above construction gives an embedding of the fact theory $F(A)$-Th into the theory of *possible* propositions. From this, the theory of all propositions may be obtained by making explicit allowance for the impossible proposition.

Standard meta-theorems may be established for $F(A)$-Th. From the above results and Corollaries 2.2 and 2.3 of [15] we obtain:

COROLLARY 17. The theory $F(A)$-Th is sound and complete for the conditions $F(A)$-Cond;

COROLLARY 18. Let Inf be the sentences $\Diamond\exists fEf$, $\Diamond\exists f\Diamond\exists g(f\neq g),\ldots$. Then the theory that results from adding those sentences as axioms to $F(A)$-Th is negation-complete, i.e. contains either ϕ or $\sim\phi$ as a theorem for each sentence ϕ of $\mathcal{L}^f$.

It may also be shown that:

THEOREM 19. The axioms of $F(A)$-Th are independent.

Proof. Use the relevant portions of theorem 33 of [15].

Similar results may be established for anti-objectualist theories of facts and propositions, with or without a relation of correspondence. It may be supposed that the structures for $\mathcal{L}^{p,f}$, the *PF-structures*, are of the form $(W, \bar{P}, \bar{F}, t)$, where W(worlds) is a non-empty set, $\bar{P}_w$(propositions) and $\bar{F}_w$(propositions) are sets, with $P = \bigcup_{w\in W} \bar{P}_w$ and $F = \bigcup_{w\in W} \bar{F}_w$ non-empty, and t(truth) is a subset of $W \times P$. The domains $\bar{P}$ will be used to interpret the predicate $\underset{\sim}{P}$, so that: $w\models \underset{\sim}{P}e$ iff $e \in P$. Clearly, structures of this kind will be equivalent to structures of the more orthodox kind, containing combined domains $\bar{P}_w \cup \bar{F}_w$ for propositions and facts and a separate valuation for the predicate $\underset{\sim}{P}$.

With each PF-structure $\mathfrak{C} = (W, \bar{P}, \bar{F}, t)$ may be associated the P-structure $\mathscr{C}^p = (W, \bar{P}, t)$ and the F-structure $\mathscr{C}^f = (W, \bar{F})$. Let us establish a notation for the *inner* truth-sets of P-structures and their extensions by setting:

$$its(e) = ts(e) \cap es(e);$$
$$ITS_w = \{its(e) : e \in \bar{P}_w\}$$
$$ITS = \{its(e) : e \in P\}.$$

Then the following conditions may be imposed upon a PF-structure $\mathscr{C}$:

P(A)-Adequacy $\mathscr{C}^p$ satisfies $P(A)$-Cond;

F(A)-Adequacy: $\mathscr{C}^f$ satisfies $F(A)$-Cond;

PF-Correspondence: There is a one-one correspondence γ between $\{e \in P : \text{its } (e) \neq \phi\}$ and F for which $es(\gamma(e)) = its(e)$ whenever $e \in$ domain (γ);

Distinctness: $e \in P\ \&\ f \in F \Rightarrow e \neq f$;

Nontruth: $\langle w, e\rangle \in t \Rightarrow e \in P$.

The correspondence condition states that there is a one-one correspondence between propositions with non-empty inner truth-set and facts with identical existence-set. This correspondence will be unique under the empirical criteria of identity for facts and propositions, and, given that it exists, the condition of F-adequacy will be a consequence of P-adequacy. The conditions of Distinctness and Nontruth state, respectively, that propositions and facts are distinct and that facts are never true. The various justifications for these conditions have already been given and need not further be discussed.

Let us call the set of these conditions $PF(A)$-Cond. Then the natural theory $PF(A)$-Th for these conditions will have axioms that divide into three parts:

(i) the propositional axioms from $P(A)$-Th, with variables relativized to P;

(ii) the factual axioms from $F(A)$-Th, with variables relativized to F;

(iii) the mixing axioms:

P-Rigidity: $\Box\forall x(Px \supset \Box Px)$

Nontruth: $(\Box)\text{-}Tf$

Distinctness: $(\Box)\text{-}(\rho = f)$.

However, we should allow, in the abstraction schemes for propositions and facts, that the formulas range over those from the full language $\mathscr{L}^{p,f}$ and not merely from the respective languages $\mathscr{L}^{p}$ and $\mathscr{L}^{f}$.

To some extent, the above axioms can be simplified. In the presence of Naive P-Abstraction, Naive F-Abstraction can be replaced by the correspondence thesis:

PF-Correspondence: $\Box\forall\rho(T\rho \supset \exists f\Box(Ef \equiv T^{+}\rho))$.

For arguing intuitively, each ϕ will yield an appropriate ρ, which, by the above principle, will then yield an appropriate f. Conversely, in the presence of Naive F-Abstraction, Naive P-Abstraction may be replaced by:

FP-Correspondence: $\Box\forall f\exists\rho\Box(T^{+}\rho \equiv Ef)$; and

Impossible Proposition: $\exists\rho\Box - T\rho$.

Also, World Fact may be dropped, for PF- and FP-Correspondence will give a world fact for any world proposition.

In a way, the present theory of facts and propositions is not significantly different from the previous theory of propositions; for the part that concerns facts merely duplicates one aspect of the part that concerns propositions. We may, in fact, show that the two theories are mutually interpretable. On the semantic side, we may associate with each PF-structure $\mathscr{C}$ the P-structure $\mathscr{C}^p$, and with each P-structure $\mathfrak{A}$ the PF-structure $\mathfrak{A}^{p,f}$, obtained by combining $\mathfrak{A}$ with a copy of $\mathfrak{A}^f$ in which the facts F are disjoint from the propositions P. It may then be shown that $\mathscr{C}^p$ satisfies $P(A)$-Cond if $\mathscr{C}$ satisfies $PF(A)$-Cond and that each $\mathscr{C}$ satisfying $PF(A)$-Cond is of the form $\mathfrak{A}^{p,f}$ for some $\mathfrak{A}$ satisfying $P(A)$-Cond.

The translation ψ^0 from $\mathscr{L}^p$ to $\mathscr{L}^{p,f}$ is readily given. For ψ^0 may be obtained from ψ by replacing quantification $\forall \rho \chi(e)$ over propositions with the relativization $\forall x(Px \supset \chi(x))$. The translation from $\mathscr{L}^{p,f}$ to $\mathscr{L}^p$ is a little more complicated. It will not do simply to identify facts with true propositions, as in the previous reduction of a purely factual language. For our theory is one in which the propositions and facts are assumed to be distinct and so, for the reasons given in §2, the theory will not admit of any kind of proxy reduction.

However, the only problem with a proxy reduction arises from the presence of identities between facts and propositions, and so an alternative form of reduction may be given by eliminating the identities prior to applying the standard reduction. To be exact, in each formula ϕ of $\mathscr{L}^{p,f}$, we may first replace each subformula $\forall x \psi$ with $\forall x(Px \supset \psi) \wedge \forall x(Fx \supset \psi)$ to obtain the formula ϕ'. Clearly:

(1) $\phi \equiv \phi'$ is logically valid.

Now suppose that all of the variables of $\mathscr{L}^{p,f}$ come from one of the two lists $e_1, e_2, \ldots$ and $f_1, f_2, \ldots$ and that the e_i's are always relativized to propositions and the f_i's to facts. (This is also to apply to the bound variables that figure in the modal-quantifier prefix ($\Box$).) Let ϕ^+ be the result of replacing each occurrence of $e_i = f_j$ or $f_j = e_i$ in ϕ' by $\bot$. Then it is shown by an easy induction on ϕ that:

(2) $(\Box)(\phi' \equiv \phi^+)$ is a logical consequence of the axioms of PF-Distinctness and P-Rigidity.

Finally, let ϕ^p be the result of identifying facts with true propositions. Thus $\forall \rho_i(P\rho_i \supset \psi)$ is replaced with $\forall \rho_i \psi$, $\forall f_i(Ff_i \supset \psi)$ with $\forall \sigma_i(T^+_{\sigma_i} \supset \psi)$ (for $\sigma_1, \sigma_2, \ldots$ new variables, distinct from $f_1, f_2, \ldots$), Ef_i with $T^+\sigma_i$, and Pf_i or Tf_i with $\bot$. In all essential respects, ϕ^p is an extension of the previous translation ϕ^p from $\mathscr{L}^f$ to $\mathscr{L}^p$ and hence the use of the same notation.

It can now be shown that the theories $P(A)$-Th and $PF(A)$-Th are equivalent with respect to the translations ϕ^p and ψ^0:

THEOREM 20. For each sentence ψ of $\mathscr{L}^p$ and ϕ of $\mathscr{L}^f$:

(i) $P(A)\text{-Th} \vdash \psi \Rightarrow PF(A)\text{-Th} \vdash \psi^0$;

(ii) $PF(A)\text{-Th} \vdash \phi \Rightarrow P(A)\text{-Th} \vdash \phi^p$; and

(iii) $P(A)\text{-Th} \vdash \psi \equiv (\psi^0)^p$ and $PF(A)\text{-Th} \vdash \phi \equiv (\phi^p)^0$.

A comparable result for models may also be established, but we will not give details.

Part (iii) is established with the help of the following result, which will also be useful later. Let us use $\phi^{p,0}$ for $(\phi^p)^0$. Then:

LEMMA 21. For $\phi' = \phi'(\rho_1, \ldots, \rho_m . f_1, \ldots, f_n)$ a formula of $\mathscr{L}^{p,f}$ with free variables among $\rho_1, \ldots, \rho_m, f_1, \ldots, f_n$, the sentence $(\Box)\left[\bigwedge_{j=1}^{n} \Box(Ef_j \equiv T^+\sigma_j) \supset \phi'(\rho_1, \ldots, \rho_m, f_1, \ldots, f_n) \equiv \phi^{p,0}(\rho_1, \ldots, \rho_m, \sigma_1, \ldots, \sigma_n)\right]$ is a logical consequence of Distinctness, P-Rigidity, Non-truth and PF- and FP-Correspondence.

Proof. By induction on ϕ.

We may note that the lemma enables us to make a further simplification to the system $PF(A)$-Th. For given the assumptions of the lemma as axioms, Naive P-Abstraction may be restricted to the case in which the formula ϕ belongs to $\mathscr{L}^p$. For when the formula is $\phi'(\rho_1, \ldots, \rho_m, f_1, \ldots, f_n)$, P-Abstraction applied to $\phi^p(\rho_1, \ldots, \rho_m, \sigma_1, \ldots, \sigma_n)$ will yield the desired proposition.

From theorem 20 and the completeness of $P(A)$-Th, it is readily established that:

THEOREM 22. The theory $PF(A)$-Th is sound and complete for $PF(A)$-Cond.

The theory and semantics for the language $\mathscr{L}^{p,f,c}$ is essentially no different from that for the language $\mathscr{L}^{p,f}$; for since the one-one correspondence postulated in the Correspondence Condition is unique, xCy may be given the definition $Px \wedge Fy \wedge \Box(T^+x \equiv Ey)$. In the objectualist systems to be considered later, the relation C will have independent interest and it will then be given more detailed attention.

Let us now consider an interpretation of the language $\mathscr{L}^f$ in which the variables range over circumstances. In this case, it will not be important to distinguish between the objectualist or anti-objectualist approaches, since neither will lead to any special requirements on the systems or their semantics. As before, the F-structures $\mathfrak{B}$ may be subject to the condition:

Empirical Identity: $es(f) = es(g) \Rightarrow f = g$

Given that circumstances are determinate, it is plausible to replace the Platonic condition with:

Conjunctive Closure: If $V_i \in ES^{\mathfrak{A}}$, $i \in I$, and $\bigcap V_i$ is non-empty, then $\bigcap V_i \in ES^{\mathfrak{A}}$.

We allow that I be empty; so it follows, in particular, that $W \in ES^{\mathfrak{A}}$. However, no other sets need belong to $ES^{\mathfrak{A}}$, and so it should be separately stated that there are world circumstances:

World Fact: $\{w\} \in ES^{\mathfrak{A}}$ for each $w \in W$.

Alternatively, in the presence of Conjunctive Closure, World Fact may be replaced by the condition that for distinct $w, v \in W$ there is a $V \in ES^{\mathfrak{A}}$ for which $v \in V$ but $v \notin V$. The two conditions together may be labelled C-Cond.

On the syntactic side, the abstraction axiom may be replaced by two of its consequences. One is

Determination: $(\Box)[\phi \supset \exists f \Box(Ef \supset \phi)]$.

In this axiom, the biconditional in comprehension is replaced by a conditional. The other consequence is an axiom of conjunctive closure. Following the definition of arbitrary conjunction in §5 of [15], let us put:

$$g \text{ Conj}_f \psi \text{ for } \forall a[Wa \supset \Box(Eg \equiv (\Pi f \Box(Ea \supset \psi) \supset Ef))],$$

Then the axiom takes the form:

Conjunctive Closure: $\exists g(g \text{ Conj}_f \psi)$, for g a variable distinct from f and not free in ψ.

As with the case of propositions, it is important to distinguish between the general fact that $\forall f \psi(f)$ and the conjunction of all facts satisfying $\psi(f)$. Thus in case $\psi(f)$ is the condition Ef, the general fact is the necessary circumstance that $\forall f Ef$, while the conjunction is a world-circumstance.

The axiom of Determination may be replaced by a single one of its instances:

Covering: $\Box \forall f \Box(-Ef \supset \exists g(\Box Eg \supset -Ef))$.

The axiom in its full generality may then be proved from World Fact by an easy induction. It would be interesting to know if Conjunctive Closure could be replaced by finitely many of its instances, since then the whole theory could be finitely axiomatized.

The new theory is deprived of many of the consequences of Comprehension. It cannot be proved, for example, that the disjunction of arbitrary facts is a fact ($\Box\forall f\forall g\exists h\Box(Eh \equiv (Ef \vee Eg))$) or that the complement of a non-necessary fact is a fact ($\Box\forall f\Box(-Ef \supset \exists g\Box(Eg \equiv -Ef))$).

Let C-Th (*circumstance theory*) consist of the axioms of Empirical Identity, World Fact, Covering and Conjunctive Closure. Then the standard meta-theorems on soundness and independence may be established:

THEOREM 23. The theorems of C-Th are true in the structures satisfying C-Cond.

Proof. Straightforward.

THEOREM 24. The four axioms of C-Th are independent.

Proof. For Empirical Identity and World Fact, we may use the counter-models of theorem 19, with Abstraction replacing Closure. For Conjunctive Closure, one may use the structure $\mathfrak{B} = (W, \bar{F})$, where $W = \{0, 1, 2, 3\}$, $F = \{\{n\}: n = 0, 1, 2, 3\} \cup \{\{0, 1, 2\}, \{0, 1, 3\}\}$ and $\bar{F}_n = \{V \in F : n \in V\}$. Let $g = \{0, 1, 2\}$ and $h = \{0, 1, 3\}$. Then $\mathfrak{B}$ verifies the other axioms, but not $\exists f(f \text{ Conj } g, h)$. For Covering, let $\mathfrak{B} = (W, \bar{F})$, where $W = \{0, 1, 2, \ldots\}$, $F = \{\{m : m \geq n\} : n \in W\}$ and $\bar{F}_m = \{V \in F : m \in V\}$. Then $\mathscr{C}$ verifies the other axioms, but not Covering. For let $f = \{1, 2, \ldots\}$. Then $(\mathfrak{B}, 0) \models -Ef$, but there is no $g \in \bar{F}_0$ for which $\mathfrak{B} \models \Box(Eg \supset -Ef)$.

In regard to the decision problem for C-Th, we have the surprising result that:

THEOREM 25. The theory C-Th is undecidable.

Proof. The theory of a reflexive symmetric relation is undecidable. But this theory may be embedded in C-Th. For given a sentence ϕ of the classical theory, let ϕ^* be the result of replacing each atomic formula Rxy in ϕ by $\Diamond\exists g\Box(Eg \equiv (Ee \vee Ef))$, each identity formula $x = y$ by $e = f$, and each individual quantifier $\forall x$ by $\Box\forall e\Box(\Diamond We \supset \ldots)$. Then it may be shown that ϕ is a theorem of the classical theory iff ϕ^* is a theorem of C-Th.

Let C-Th* be the theory determined by the conditions C-Cond. Then for C-Th*, the situation is even worse.

THEOREM 26. The theory C-Th* is non-axiomatizable.

Proof. The second-order theory of a reflexive and symmetric relation (with quantification over arbitrary sets) is non-axiomatizable. Indeed, it is equivalent in undecidability to full second-order logic. Now this theory may be embedded in C-Th* by modifying the previous translation. Set on one side a fact variable e_0. Then the individual quantifier $\forall x$ now goes into $\Box\forall e\Box((\Diamond We \wedge (e \neq e_0)) \supset \ldots)$, the membership formula $x \in X$ is replaced by $\Box(Ee \supset Ee')$, and the set quantifier $\forall x$ by $\Box\forall x'\Box$. Let ϕ_0 be the formula $We_0 \wedge \Box\forall e((We \wedge e \neq e_0) \supset \Diamond\exists f\Box(Ef \equiv -Ee))$. Then it may be shown that the classical sentence ϕ is a theorem of the second-order theory iff its translate $\phi_0 \supset \phi^*$ is a theorem of C-Th*. (The antecedent ϕ_0 is introduced in order to gain the effect of quantification over arbitrary sets of individuals.)

Since identity is definable within C-Th*, we have here our first example of a pure existence theory that is of intuitive interest yet not axiomatizable.

Many other theories under the anti-objectualist approach may be considered. For example, factual counterparts to the propositional systems *PDC*, *PC** and *PC* of [15] may be set up, and comparable result proved. Or again, a relation of underlying or more explicitly structural devices may be introduced into either the theory of facts or of circumstances. However, I shall make no attempt to survey these various possibilities.

6. OBJECTUALIST CONDITIONS ON FACTS

We shall consider the conditions appropriate for structures under an objectualist theory of truths.

It will help to begin by reviewing and revising the objectualist conditions on propositional structures. Recall, from §3 of [15], that the three conditions of Diversity, World Actualism and Automorphism were imposed upon a P-structure $\mathfrak{A}$. We shall now impose a further condition on $\mathfrak{A}$:

Coincidence: For each non-empty $U \in ITS^{\mathfrak{A}}$, $\langle U, U\rangle \in MV^{\mathfrak{A}}$.

Call a proposition $e \in P$ *coincident* if $ts(e) = es(e)$. Then the above condition provides a coincident proposition for each inner truth-set. We may use P-Cond to refer to all four conditions.

As already explained in §2, the further condition will be justified if facts can appear as constituents in propositions and are subject to objectualist requirements; for given the proposition ρ with non-empty inner truth-set U, we may let the coincident proposition with modal value $\langle U, U\rangle$ be that the fact f, corresponding to e, exists. In the original conditions on P-structures, this further condition was not contemplated, since it was supposed that all propositions were constructured by existentially transparent means. But this supposition must be dropped since it is allowed that facts can occur as constituents in propositions.

Call a proposition *transparent* if it is constructed by existentially transparent means alone; and otherwise call it *opaque*. Then the previous conditions were adequate in their postulation of modal values for transparent propositions, as shown in the proof of theorem 11 in [15]. It may now be wondered whether the new conditions are adequate in their postulation of modal values for all propositions or, more exactly, whether for each P-structure $\mathfrak{A}$ satisfying the conditions there is an underlying individual structure $\mathcal{I}$ that generates propositions with the same modal values as $\mathfrak{A}$.

Fortunately, it is not necessary to be too specific about the opaque means available for constructing propositions in $\mathcal{I}$. Let $\rho = \rho(i_1, i_2, \ldots)$ be an arbitrary proposition with individual constituents $i_1, i_2, \ldots$. (The individual constituents may always be indicated in this manner.) Now:

(1) the proposition $\rho = \rho(i_1, i_2, \ldots)$ has the same truth-set as a transparent proposition $\rho'(i_1, i_2, \ldots)$;

for the use of opaque constructions should make no difference to the truth-sets that are ultimately expressible in terms of certain individuals. Also:

(2) the existence-set of $\rho = \rho(i_1, i_2, \ldots)$ is the truth-set of a transparent proposition $\sigma = \sigma(i_1, i_2, \ldots)$.

Indeed, (2) follows from (1); for the existence-set of the proposition $\rho = \rho(i_1, i_2, \ldots)$ will be the truth-set of the proposition that ρ exists, which will then be the truth-set of a transparent proposition $\sigma(i_1, i_2, \ldots)$.

Let U, V and V' be the sets of worlds for which the modal values of ρ, ρ' and σ, respectively, are $\langle U, V\rangle$, $\langle U, V'\rangle$ and $\langle V, V'\rangle$. Note that

V is non-empty and also that $V \subseteq V'$, since the existence of the individuals $i_1, i_2, \ldots$ is required for the existence of ρ. Given the adequacy of the original conditions, $\langle U, V'\rangle$, $\langle V, V'\rangle \in MV^{\mathfrak{A}}$. So $\langle V, V\rangle \in MV^{\mathfrak{A}}$, by the coincidence condition. Therefore $\langle W, V\rangle$ and $\langle U, V\rangle = \langle W \cap U, V \cap V'\rangle$ are members of $MV^{\mathfrak{A}}$ by simple applications of lemma 17 in [15].

Let us now consider the objectualist conditions appropriate for F-structures. Each of the conditions of Diversity, World Actualism and Automorphism for P-structures will correspond to a condition on F-structures.

Diversity now takes the form:

Given any $f \in F$, there are infinitely many g for which $es(g) = es(f)$. The justification for the condition is simply this. Let ρ be a proposition to which f corresponds. Then, as argued in Section 2 of [15], there are infinitely many propositions with the same truth- and existence-sets as ρ. But given that facts inherit the identity conditions of the propositions to which they correspond, there will be infinitely many facts with the same existence set as f.

We may say that $\mathfrak{B} = (W, \bar{F})$ is an ω-structure if $\{e \in F : es(e) = es(f)\}$ is always of cardinality ω. Intuitively, this strengthening of Diversity is not justified; but it will have certain technical uses.

World Actualism is the same as before:

$\bar{\mathfrak{B}}_w = \bar{\mathfrak{B}}_v$ implies $\mathfrak{B}_w = \mathfrak{B}_v$ for all $w, v \in W$.

However, the application of the condition is now to the new type of structure. The justification is immediate; for if two worlds are distinct, there must be a fact in one of the two worlds by which they are distinguished. Unless otherwise indicated, we shall assume that our structures are differentiated ($\mathfrak{B}_w = \mathfrak{B}_v$ implies $w = v$); so it will also be true that $\bar{\mathfrak{B}}_w = \bar{\mathfrak{B}}_v$ implies $w = v$.

The final condition is Automorphism. Recall the relevant definitions. The pair $\alpha = (\alpha_1, \alpha_2)$ is an *automorphism on* the F-structure $\mathscr{C} = (W, \bar{F})$ if:

(i) α_1 and α_2 are permutations on W and F respectively,

(ii) $e \in \bar{F}_w$ iff $\alpha_2(e) \in \bar{F}_{\alpha_1(w)}$ for all $e \in F$ and $w \in W$.

(The conditionals in the definition of [15] should, of course, have been biconditionals.) Given the F-structure $\mathfrak{B} = (W, \bar{F})$, the automorphism $\alpha = \langle \alpha_1, \alpha_2\rangle$ on $\mathfrak{B}$ is *fixed on* $G \subseteq F$ if $\alpha_2(f) = f$ for all $f \in G$, and the subset G of F *determines* the subset V of W if $\alpha_1[V] = V$ whenever the automorphism $\alpha = \langle \alpha_1, \alpha_2 \rangle$ is fixed on G. *Automorphism* now takes the form:

If the subset of F determines $V \subseteq W$ and $es(G) \cap V \neq \phi$, then $es(G) \cap V \in ES^{\mathfrak{B}}$.

The justification for the condition is roughly this. Given that G determines V, there is a proposition constructed from the elements of G by purely transparent means whose truth-set is V. The corresponding fact will then have existence set $es(G) \cap V$, assuming that the set is non-empty. The reader should consult [15] for further elaboration on this type of argument.

The above three conditions are the natural counterparts to the original conditions on P-structures. Moreover, there is no natural counterpart to the coincidence conditions, since there is no analogue to the distinction between truth- and existence-sets for facts. It therefore seems reasonable that the factual conditions should give the exact meaning of the propositional conditions for factual structures or, more exactly, that an F-structure $\mathfrak{B}$ should satisfy the factual conditions iff it is of the form $\mathfrak{A}^f$ for some P-structure satisfying the propositional conditions.

Such a supposition would be substantially correct if attention were confined to the modal values or existence sets represented in the different structures. But difficulties arise once the cardinality of facts with a given existence set is taken into account. For consider a fact with existence-set T. Each proposition with modal value $\langle U, V \rangle$, $U \cap V = T$, will correspond to a distinct fact with existence-set T. Moreover, each set of facts G will yield a proposition with modal value $\langle U, V \rangle$ if G determines U and the existence-set of G is V. We are therefore led to the following condition:

Cardinality: If $T \in ES^{\mathfrak{B}}$, then $\text{card}\{f \in F : es(f) = T\} \geq \text{card}\{\langle U, es(G) \rangle : G \text{ determines } U \text{ and } U \cap es(G) = T\}$.

All four conditions – Diversity, World Actualism, Automorphism and Cardinality – on F-structures may then be called F-Cond.

The separate conditions for P- and F-structures may readily be combined to give the appropriate conditions for PF- and PFC-structures. For the former, we adopt:

Diversity: Given any $e \in P$, there are infinitely many $f \in P$ for which $mv(f) = mv(e)$.

World Actualism: $\bar{\mathscr{C}}_w = \bar{\mathscr{C}}_v$ implies $\mathscr{C}_w = \mathscr{C}_v$.

Automorphism: Suppose that $B \subseteq A$ determines U and that $es(B)$ is non-empty. Then for some $e \in P$, $mv(e) = \langle U, es(B)\rangle$.

PF-Correspondence: There is a one-one correspondence γ between $\{e \in P : its(e) \neq \phi\}$ and F for which $es(\gamma(e)) = its(e)$ for all $e \in dm(\gamma)$,

Non-truth: $\langle w, e\rangle \in t \Rightarrow e \in P$.

Distinctness: $e \in P$ & $f \in F \Rightarrow e \neq f$.

In the automorphism condition, the automorphisms are now defined with respect to the total PF-structure $\mathscr{C}$. It is also allowed that arbitrary subsets of $A = P \cup F$ may determine a truth-set. Although it is not immediately obvious, the separate P- and F-conditions on the derived structures $\mathscr{C}^p$ and $\mathfrak{C}^f$ will then be a consequence.

The conditions for PFC-structures are the same as for PF-structures, but with γ now the particular relation from the structure. We use PF-Cond. and PFC-Cond. for the respective sets of conditions.

7. EQUIVALENCE

We now prove some results on the equivalence of the different objectualist conditions and of the circumstances in which the modal values of propositions can be recovered from the corresponding facts.

Before establishing the results, it will be helpful to state some elementary results on the existence of automorphisms. Most of these results will apply to all *monadic* structures, i.e. to all structures defined on one-place predicates alone. But for convenience, we shall only state them for the P-structures, and not for the F- or PF-structures as well.

Say that the pair $\alpha = \langle\alpha_1, \alpha_2\rangle$ is an *isomorphism from* the P-structure $\mathfrak{A} = (W, \bar{P}, t)$ *onto* the P-structure $\mathfrak{B} = (V, \bar{Q}, s)$ if α_1 and α_2 are one-one maps from W onto V and from F onto G respectively for which (i) $e \in \bar{P}_w$ iff $\alpha_2(e) \in \bar{P}_{\alpha_1(w)}$ and (ii) $\langle w, e\rangle \in t$ iff $\langle\alpha_1(w), \alpha_2(e)\rangle \in s$. Call α_1 a *world-isomorphism from* $\mathfrak{A}$ *onto* $\mathfrak{B}$ if it is extendible to an isomorphism $\alpha = \langle\alpha_1, \alpha_2\rangle$ from $\mathfrak{A}$ onto $\mathfrak{B}$. Then a necessary and sufficient condition for α_1 to be a world-isomorphism is given by:

LEMMA 27. If $\mathfrak{A}$ and $\mathfrak{B}$ are P-structures and α_1 is a one-one function from W onto V, then α_1 is a world isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$ iff it respects modal values and cardinalities, i.e. iff $\alpha_1(MV^{\mathfrak{A}}) = MV^{\mathfrak{B}}$ and

$\text{card}\{e \in P : mv^{\mathfrak{A}}(e) = \langle U, V\rangle\} = \text{card}\{e \in Q : mv^{\mathfrak{B}}(e) = \langle \alpha_1(U), \alpha_1(V)\rangle\}$ *for each* $\langle U, V\rangle \in MV^{\mathfrak{A}}$.

Proof. The left-to-right direction is straightforward; and for the other direction, we may define an extension of α_1 in the obvious way.

As a special case of lemma 27, we obtain:

COROLLARY 28(i): The one-one permutation α_1 on W is a world-automorphism on the P-structure $\mathfrak{A}$ iff $\alpha_1(MV^{\mathfrak{A}}) = MV^{\mathfrak{A}}$ and $\{e \in P : mv^{\mathfrak{A}}(e) = \langle U, V\rangle\} = \text{card}\{e \in P : mv^{\mathfrak{A}}(e) = \langle \alpha_1(U), \alpha_1(V)\rangle\}$.

Thus given that α_1 preserves modal values, all that need be got right for α_1 to be extendable to an automorphism is that it should preserve cardinalities. However, this last condition is not superfluous, since it is clear that the cardinality of the proposition with a given modal value and the cardinality of those with its image can differ.

Call a permutation pair $\alpha = \langle \alpha_1, \alpha_2\rangle$ on worlds and individuals *trivial* if $\alpha_1(w) = w$ for all $w \in W$ and $\alpha_2(e) = f$ only if $mv^{\mathfrak{A}}(e) = mv^{\mathfrak{A}}(f)$. Then another consequence of lemma 27 is:

COROLLARY 28(ii): All trivial permutation-pairs $\alpha = \langle \alpha_1, \alpha_2\rangle$ on a P-structure $\mathfrak{A}$ are automorphisms.

Proof. Since the conditions of the lemma are satisfied.

A rather useful criterion for when a set of propositions determines a truth-set can be stated. Given a subset Q of P in the P-structure $\mathfrak{A}$, let $mv[Q] = \{mv(e) : e \in Q\}$. An automorphism α on $\mathfrak{A}$ may be extended to modal values in the obvious way. We now have:

COROLLARY 29: Given the P-structure $\mathfrak{A}$, $Q \subseteq P$ and $V \subseteq W$, Q determines V iff $mv[Q]$ determines V.

Proof. $\Leftarrow$ Straightforward, since it is readily shown that any automorphism fixed on e is fixed on $mv(e)$.

$\Rightarrow$ Suppose Q determines V. Take any automorphism α fixed on $mv[Q]$. Note that α need not be fixed on Q, since it may take an element of Q into an element outside of Q but with the same modal value. Define a new automorphism $\beta = \langle \beta_1, \beta_2\rangle$ by:

$$\beta_2(w) = w \text{ for all } w \in W;$$
$$\beta_2(e) = e \text{ if } mv(e) \notin mv[Q];$$
$$\beta_2(e) = \alpha_2^{-1}(e) \text{ if } mv(e) \in mv[Q].$$

By corollary 28(ii), β is an automorphism on $\mathfrak{A}$. Therefore $\gamma = \langle\beta_1 \circ \alpha_1, \beta_2 \circ \alpha_2\rangle$ is an automorphism on $\mathfrak{A}$, and is fixed on Q, since, for $e \in Q$, $\beta_2 \circ \alpha_2(e) = \alpha_2^{-1} \circ \alpha_2(e) = e$. Since Q determines V, $\gamma_1(V) = V$. But $\gamma_1 = \alpha_1$; and so $\alpha_1(V) = V$, as required.

It is essential to this result that the concept of determine in '$mv[Q]$ determines V' be defined with respect to automorphisms on the full structure $\mathfrak{A}$. For, as we have noted, a permutation on worlds that merely respects the structure of the modal values $MV^{\mathfrak{A}}$ need not be extendible to an automorphism on $\mathfrak{A}$.

Let us now establish the equivalence of P-Cond. and F-Cond. for their respective structures. One direction requires the following result to connect the automorphisms in P- and F-structures:

LEMMA 30. If $\alpha = \langle\alpha_1, \alpha_2\rangle$ is an automorphism on the P-structure $\mathfrak{A}$, then $\beta = \langle\alpha_1, \alpha_2 \upharpoonright F\rangle$ is an automorphism on $\mathfrak{B} = \mathfrak{A}^f$.

Proof. Let β be as defined. Now $\alpha_1(its(e)) = its(\alpha_2(e))$; for $\alpha_1(its(e)) = \alpha_1(ts(e) \cap es(e)) = \alpha_1(ts(e)) \cap \alpha_1(es(e)) = ts(\alpha_2(e)) \cap es(\alpha_2(e)) = its(\alpha_2(e))$. It follows that the range of β_2 is F: for if $f \in F$, then $its(f) \neq \phi$, so $its(\alpha_2(f)) = \alpha_1(its(f)) \neq \phi$, and so $\alpha_2(f) \in F$; and, similarly, if $f \in F$ then $\alpha_2^{-1}(f) \in f$. It also follows, for all $f \in F$, that $f \in \bar{F}_w$ iff $\beta_2(f) \in \bar{F}_{\beta_1(w)}$; for the L.H.S. holds iff $w \in its(f)$, which holds iff $\alpha_1(w) \in \alpha_1(its(f)) = its(\alpha_2(f))$, which holds iff $\beta_1(w) \in es(\beta_2(f))$, i.e. iff $\beta_2(f) \in \bar{F}_{\beta_1(w)}$. Therefore the two conditions for β being an automorphism on $\mathfrak{B}$ are satisfied.

It can now be shown that:

LEMMA 31: If the P-structure $\mathfrak{A}$ satisfies P-Cond, then the associated F-structure $\mathfrak{A}^f$ satisfies F-Cond.

Proof. Assume that $\mathfrak{A}$ satisfies P-Cond. We show that $\mathfrak{B} = \mathfrak{A}^f$ satisfies each of the conditions of F-Cond. in turn.

Diversity: The justification is similar to the intuitive one. Take any $f \in F$. Then $f \in P$. So $G = \{g \in P : mv^{\mathfrak{A}}(g) = mv^{\mathfrak{A}}(f)\}$ is infinite, by P-Diversity. But each $g \in G$ is in F and has $es^{\mathfrak{B}}(g) = es^{\mathfrak{B}}(f)$. Therefore $\{g \in F : es^{\mathfrak{B}}(g) = es^{\mathfrak{B}}(f)\}$ is also infinite.

World Actualism: Suppose that $w \neq v$. From the proof of lemma

4.18 in [15], it follows that for each $w \in W$, there is an $e_w \in P$ such that $its^{\mathfrak{A}}(e_w) = \{w\}$. But then $e_w \in \bar{F}_w - \bar{F}_v$ and $\bar{\mathfrak{B}}_w \neq \bar{\mathfrak{B}}_v$, as required.

Automorphism: Suppose that $G \subseteq F$ determines $U \subseteq W$ w.r.t. the structure $\mathfrak{B}$ and that $es^{\mathscr{C}}(G) \cap U \neq \phi$. For each $f \in G$, it follows by the construction of $\mathfrak{B} = \mathfrak{A}^f$, that $its^{\mathfrak{A}}(f) = es^{\mathfrak{B}}(f)$; and so, by Coincidence, there is a coincident proposition f' of $\mathfrak{A}$ for which $ts^{\mathfrak{A}}(f') = es^{\mathfrak{B}}(f)$. Let $G' = \{f' : f \in G\}$. Then note that $es^{\mathfrak{A}}(G') = es^{\mathfrak{B}}(G') = es^{\mathfrak{B}}(G)$.

Since G determines U in $\mathfrak{B}$ and $\{es^{\mathfrak{B}}(f) : f \in G\} = \{ts^{\mathfrak{A}}(f') : f \in G\}$, it follows by corollary 29 and lemma 30 that G' determines U in $\mathfrak{A}$. Since $es^{\mathfrak{A}}(G') = es^{\mathfrak{B}}(G)$ is non-empty, it follows by P-Automorphism that $\langle U, es^{\mathfrak{A}}(G')\rangle \in MV^{\mathfrak{A}}$. Choose an $e \in P$ for which $mv^{\mathfrak{A}}(e) = \langle U, es^{\mathfrak{A}}(G')\rangle$. Now $U \cap es^{\mathfrak{A}}(G') = U \cap es^{\mathfrak{B}}(G)$ is non-empty. So $e \in F$ and has $es^{\mathfrak{B}}(e) = its^{\mathfrak{A}}(e) = U \cap es^{\mathscr{C}}(G)$, as required.

Cardinality: Take $T \in ES^{\mathfrak{B}}$, and consider $\langle U, V\rangle$ and $G \subseteq F$ for which: $es^{\mathfrak{B}}(G) = V$, G determines U in $\mathfrak{B}$, and $U \cap V = T$. Then it suffices to show that $\langle U, V\rangle \in MV^{\mathfrak{A}}$. Define G' as before. Then G' determines U in $\mathfrak{A}$ and $es^{\mathfrak{A}}(G') = es^{\mathfrak{B}}(G) = V$. Therefore by Automorphism for $\mathfrak{A}$, $\langle U, V\rangle \in MV^{\mathfrak{A}}$.

The proof of the other direction of the equivalence requires a preliminary definition and result. Given an F-structure $\mathfrak{B}$, define the P-structure $\mathfrak{A} = \mathfrak{B}^P$ by the following clauses:

(i) $W = V$;

(ii) $P = \{\langle U, es^{\mathfrak{B}}(G), \xi\rangle; G \subseteq F$ determines $U \subseteq W$ in $\mathfrak{B}$ and ξ is an ordinal $\leq \max(\omega, \text{card}\{f \in F : es^{\mathfrak{B}}(f) = U \cap es^{\mathfrak{B}}(G)\})\}$;

(iii) $\bar{P}_w = \{\langle U, V, \xi\rangle \in P : w \in W\}$;

(iv) $t = \{\langle w, \langle U, V, \xi\rangle\rangle \in W \times P : w \in U\}$.

LEMMA 32: If the F-structure $\mathfrak{B}$ satisfies F-Cond., then any world automorphism β_1 on $\mathfrak{B}$ is also a world automorphism on $\mathfrak{A} = \mathfrak{B}^P$.

Proof. Under the conditions of the lemma, suppose that β_1 is extendible to an automorphism $\beta = \langle \beta_1, \beta_2\rangle$ on $\mathfrak{B}$. By corollary 28, β_1 is a world automorphism on $\mathfrak{A} = \mathfrak{B}^P$ if it preserves modal values and cardinalities. Let us establish each condition in turn.

First, suppose that $\langle U, V\rangle \in MV^{\mathfrak{A}}$ (in order to show that $\langle \beta_1(U), \beta_2(V)\rangle \in MV^{\mathfrak{A}}$). By the construction of $\mathfrak{A}$, there is a $G \subseteq F$ for which G determines U in $\mathfrak{B}$ and $es^{\mathscr{C}}(G) = V$. It follows that $\beta_2(G)$ determines $\beta_1(U)$ in $\mathfrak{B}$. For suppose that the $\mathfrak{B}$-automorphism $\alpha =$

$\langle\alpha_1, \alpha_2\rangle$ is fixed on $\beta_2(G)$. Then $\alpha_2(\beta_2(f)) = \beta_2(f)$ for all $f \in G$. Therefore the $\mathfrak{B}$-automorphism $\langle\beta_1^{-1} \circ \alpha_1^{-1}, \beta_2^{-1} \circ \alpha_2^{-1}\rangle$ is fixed on G. But then $\beta_1^{-1} \circ \alpha_1^{-1}(U) = U$. So $U = \alpha_1(\beta_1(U))$; and $\beta_2(G)$ determines $\beta_1(U)$. Also, we have that $es^{\mathfrak{B}}(\beta_2(G)) = \beta_1(es^{\mathfrak{B}}(G)) = \beta_1(V)$. Hence $\langle\beta_1(U), \beta_1(V)\rangle \in MV^{\mathfrak{A}}$, as required.

Secondly, take any $\langle U, V\rangle \in MV^{\mathfrak{A}}$. Then $c = \text{card}\{e \in P : mv^{\mathfrak{A}}(e) = \langle U, V\rangle\} = \max(\omega, \text{card}\{f \in F : es^{\mathfrak{B}}(f) = U \cap V\})$ and, similarly, $d = \text{card}\{e \in P : mv^{\mathfrak{A}}(e) = \langle\beta_1(U), \beta_1(V)\rangle\} = \max(\omega, \text{card}\{f \in F : es^{\mathfrak{B}}(f) = \beta_1(U) \cap \beta_1(V)\})$. But since β_1 is a permutation on W, $\beta_1(U \cap V) = \beta(U_1) \cap \beta_2(B)$ and, so since β respects cardinalities in $\mathfrak{B}$, $c = d$.

It can now be shown that:

LEMMA 33: Given an F-structure $\mathfrak{B}$ satisfying F-Cond., there is a P-structure $\mathfrak{A}$ satisfying P-Cond for which $\mathfrak{A}^f = \mathfrak{B}$.

Proof. Let $\mathfrak{B}$ be an F-structure satisfying F-Cond. We show that $\mathfrak{A} = \mathfrak{B}^P$ is isomorphic to a P-structure of the required sort. First, let us go through the conditions in turn:

Diversity: It follows from the construction of $\mathfrak{A}$ that, for each $\langle U, V, \xi\rangle \in P$, $mv(\langle U, V, \xi\rangle) = \langle U, V\rangle$. But then, $mv(\langle U, V, n\rangle) = \langle U, V\rangle$ for $n = 0, 1, 2, \ldots$.

World Actualism: Suppose $w \neq v$. Then by World Actualism and Differentiation for $\mathfrak{B}$, there is an $f \in \bar{F}_w$, say, without $f \in \bar{F}_v$. Clearly, $G = \{f\}$ determines $U = es^{\mathfrak{B}}(f)$. Therefore $e = \langle U, U, o\rangle \in P$. But then $e \in \bar{P}_w$ and yet not $e \in \bar{P}_v$, as required.

Automorphism: Suppose that $Q \subseteq P$ determines $U \subseteq W$ in $\mathfrak{A}$ and that $es^{\mathfrak{A}}(Q) = V$ is non-empty. Then we must show that $\langle U, V\rangle \in MV^{\mathfrak{A}}$. Suppose, for each $e \in Q$, that $mv^{\mathfrak{A}}(e) = (U_e, V_e)$. Then for each such e there is a $G_e \subseteq F$ such that G_e determines U_e in $\mathfrak{B}$ and $es^{\mathfrak{B}}(G_e) = V_e$. Let $G = \cup_{e \in Q} Q_e$. Then $es^{\mathfrak{B}}(G) = \bigcup_{e \in Q} es^{\mathfrak{B}}(G_e) = \bigcup_{e \in O} V_e = es^{\mathfrak{A}}(Q) = V$.

Let $\mathscr{X} = mv^{\mathfrak{A}}(Q)$. Clearly, G determines each pair (U_e, V_e) of $\mathscr{X}$ in $\mathfrak{B}$. Also, since Q determines U in $\mathfrak{A}$, $\mathscr{X}$ determines U in $\mathfrak{A}$, by corollary 29; and so $\mathscr{X}$ determines U in $\mathfrak{B}$ by lemma 32. But then G determines U in $\mathfrak{B}$, and so $\langle U, es^{\mathfrak{B}}(G)\rangle = \langle U, V\rangle \in MV^{\mathfrak{A}}$.

Coincidence: Suppose $\langle U, V\rangle \in MV^{\mathfrak{A}}$ with $U \cap V$ non-empty. For some $G \subseteq F$, G determines U in $\mathfrak{B}$ and $es^{\mathfrak{B}}(G) = V$. Since $U \cap V$ is non-empty, $U \cap V \in ES^{\mathfrak{B}}$. Choose $f \in F$ with $es^{\mathfrak{B}}(f) = U \cap V$. Then $\{f\}$ determines $U \cap V$ in $\mathfrak{B}$. So $\langle U \cap V, U \cap V\rangle \in MV^{\mathfrak{A}}$, as required.

Finally, it must be shown that $\mathfrak{A}^f$ is isomorphic to $\mathfrak{B}$. In view of lemma 27, it suffices to show that (a) $ITS^{\mathfrak{A}} = ES^{\mathfrak{B}}$ and that (b) for each $T \in ITS^{\mathfrak{A}}$, $\text{card}\{e \in P : its^{\mathfrak{A}}(e) = T\} = \text{card}\{f \in F : es^{\mathfrak{B}}(f) = T\}$. As for (a), suppose first that $T \in ES^{\mathfrak{B}}$, with $es^{\mathfrak{B}}(e) = T$. Then since $\{e\}$ determines U in $\mathfrak{B}$, $\langle U, U\rangle \in MV^{\mathfrak{A}}$ and hence $U \in ITS^{\mathfrak{A}}$. Now suppose that $T \in ITS^{\mathfrak{A}}$ with $its(e) = T$ and $mv(e) = \langle U, V\rangle$. Then for some $G \subseteq F$, $es^{\mathfrak{B}}(G) = V$ and G determines U in $\mathfrak{B}$. Therefore by Automorphism for $\mathfrak{B}$, $T = U \cap V \in ES^{\mathfrak{B}}$.

As for (b), let $c = \text{card}\{e \in P : its^{\mathfrak{A}}(e) = T\}$ and $d = \text{card}\{f \in F : es^{\mathfrak{B}}(f) = T\}$. Then $c = \text{card}\{\langle U, V, \xi\rangle : U \cap V = T, \xi \leq d,$ and for some $G \subseteq F$, $es^{\mathfrak{B}}(G) = V$ and G determines $U\} = \text{card}\{\langle U, es(G)\rangle : U \cap es(G) = T$ and G determines $U\}.d$. But by Cardinality for $\mathfrak{B}$, $\kappa = \text{card}\{\langle U, es(G)\rangle : U \cap es(G) = T$ and G determines $U\} \leq d$. So $c = \kappa . d = d$.

Putting lemmas 31 and 33 together gives the equivalence of the two sets of conditions:

THEOREM 34: An F-structure $\mathfrak{B}$ satisfies F-Cond. iff it is of the form $\mathfrak{A}^f$ for some P-structure $\mathfrak{A}$ that satisfies P-Cond.

Similar results can be proved for the conditions imposed on PF- and PFC-structures. It is, first of all, a trivial matter to show:

LEMMA 35: If $\mathfrak{C}$ is any PF-structure satisfying PF-Cond., then $(\mathfrak{C}^p)^f$ is isomorphic to $\mathfrak{C}^f$.

We now have:

LEMMA 36: If $\mathfrak{C}$ satisfies PF- or PFC-Cond., then $\mathfrak{C}^p$ satisfies P-Cond. and $\mathfrak{C}^f$ satisfies F-Cond.

Proof. Given lemmas 31 and 35, it suffices to prove the result for $\mathfrak{C}^p$. So supposing that $\mathfrak{A}$ satisfies PF-Cond., let us consider each of the conditions of P-Cond. in turn.

Diversity: Trivial.

World Actualism: Use the fact $\{w\}$ will belong to $TS^{\mathfrak{C}}_w$.

Automorphism: Suppose $Q \subseteq P$ determines $V \subseteq W$ in $\mathfrak{C}^p$. Then Q determines V in $\mathfrak{C}$, since any automorphism on $\mathfrak{C}$ restricts to an automorphism on $\mathfrak{C}^p$. The rest then follows by Automorphism for $\mathfrak{C}$.

Coincidence: Suppose that $e \in P$ and that $its(e) = V$ is non-empty. Let $f = \gamma(e)$ for a correspondence γ from the conditions on $\mathfrak{C}$. Then $es(f) = its(e)$. Clearly, $\{f\}$ determines V. So by Automorphism on $\mathscr{C}$, $\langle V, V\rangle \in MV^{\mathfrak{C}}$, as required.

The other direction of the equivalence is given by:

LEMMA 37: (i) Any structure $\mathfrak{A}$ satisfying P-Cond. is of the form $\mathfrak{C}^p$ for some $\mathfrak{C}$ satisfying PF-Cond.;

(ii) any structure $\mathscr{C}$ satisfying F-Cond. is of the form $\mathfrak{C}^f$ for some $\mathfrak{C}$ satisfying PF-Cond.

Proof. Again, given lemmas 33 and 35, it suffices to prove case (i). So take $\mathfrak{A} = (W, \bar{P}, t)$ satisfying P-Cond. Let $\mathfrak{C} = \mathfrak{A}^{pf}$, i.e. for $\mathscr{C}$ a copy $(W, \bar{F})$ of $\mathfrak{A}^f$, F disjoint from P, $\mathfrak{C} = (W, \bar{P}, \bar{F}, t)$. Clearly $\mathfrak{C}^p = \mathfrak{A}$. So it remains to show that $\mathfrak{C}$ satisfies PF-Cond.

Of the various conditions, only Automorphism is not completely trivial. Suppose, then, that $B \subseteq A$ determines U and that $es(B)$ is non-empty. Let $B' = (B \cap P) \cup \{e \in P : ts(e) = es(e) = es(f)$ for $f \in B \cap F\}$. By corollary 29, B' also determines U in $\mathfrak{C}$; and by lemma 30, any automorphism on $\mathfrak{A}$ can be extended to an automorphism on $\mathfrak{C}$. Therefore B' determines U in $\mathfrak{A}$. Now, $es^{\mathfrak{C}}(B') = es^{\mathfrak{C}}(B)$. So by Automorphism for $\mathfrak{A}$, $(U, es(B)) \in MV^{\mathfrak{A}}$, and hence $(U, es(B)) \in MV^{\mathfrak{C}}$, as required.

Combining lemmas 36 and 37 gives:

THEOREM 40: $\mathfrak{A}$ satisfies P-Cond(F-Cond) iff it is of the form $\mathfrak{C}^p$(resp. $\mathfrak{C}^f$) for some $\mathfrak{C}$ satisfying PF-Cond.

It is almost trivial that the PF-structure $\mathfrak{C}$ satisfies PF-Cond. iff it is of the form $\mathscr{D}^{p,f}$ for some PFC-structure $\mathscr{D}$ satisfying PFC-Cond. Therefore the above theorems may readily be extended to PFC-structures.

Similar results might be established for structures subject to more or to fewer conditions. One possibility is to drop the coincidence condition on P-structures. An examination of the proof of lemma 31 then quickly reveals that when $\mathfrak{A}$ satisfies the remaining conditions of P-Cond, $\mathfrak{A}^f$ satisfies the conditions of F-Cond. other than Cardinality; and so it might be of interest to determine if a converse result holds or, if not, what further consequences the conditions on the P-structures have for the F-structures. In the other direction, one might add such conditions as Determinacy from §8 of [15] to P-Cond. and consider their equivalents for F-structures.

Given these results, it is natural to ask to what extent information is lost in the passage from propositions to facts. More exactly, we may ask to what extent the $\mathfrak{A}$ satisfying P-Cond. for which $\mathfrak{A}^f = \mathfrak{B}$, are, for a fixed $\mathfrak{B}$, unique up to isomorphism.

There are two main ways in which the $\mathfrak{A}$'s may fail to be unique. First, the cardinalities of $\{e \in P : mv^{\mathfrak{A}}(e) = \langle U, V\rangle\}$ may vary when $U \cap V = \phi$. But also $MV^{\mathfrak{A}}$, even $TS^{\mathfrak{A}}$, can vary. For consider the ω P-structure $\mathfrak{A} = (W, \bar{P}, t)$ (each $\{f : mv(f) = e\}$ countably infinite) for which:

(i) $W = \{w_1, w_2, w_3, w_4\}$, for w_1, w_2, w_3 and w_4 all distinct;

(ii) $MV^{\mathfrak{A}} = \{\langle W, W\rangle, \langle \phi, W\rangle\} \cap \{\langle U, \{w\}\rangle : U \subseteq W \text{ and } w \in W\}$.

Then $\mathfrak{A}$ satisfies P-Cond. However, the associated F-structure $\mathfrak{A}^f$ merely contains the existence-sets $\{w_i\}$, $i = 1, 2, 3, 4$, and W and so $(\mathfrak{A}^f)^p$ will merely contain the truth-sets ϕ, $\{w_i\}$, $W - \{w_i\}$, and W.

Although not provable, the requirement that the truth-sets of $\mathfrak{A}$ be recoverable from $\mathfrak{A}^f$ is a reasonable one. For it amounts to the actualist-type claim that the facts should determine all of the possibilities, that the appeal to the truth of non-existent propositions should be redundant. It is therefore natural to consider under what additional conditions on $\mathfrak{A}$ the requirement might be met.

Let us adopt the Determinacy Assumption from §8 of [15]. When stated in semantical terms, it becomes:

Determinacy: Suppose that $w, v, u \in W$, $\bar{P}_w \cap \bar{P}_v = \bar{P}_w \cap \bar{P}_u$ and that, for some $e \in \bar{P}_w$, $ts(e) \cap \{v, u\} = \{V\}$. Then for some $f \in \bar{P}_w \cap \bar{P}_v$, $ts(f) \cap \{v, u\} = \{v\}$.

The plausibility of this condition was argued for in [15].

As an almost immediate consequence, we have that worlds distinguished by an actual proposition are distinguished by an actual fact:

LEMMA 41: Suppose, for a P-structure $\mathfrak{A}$ satisfying Automorphism and Determinacy, that $w, v, u \in W$ and that, for some $e \in \bar{P}_w$, $ts(e) \cap \{v, u\}$ is singleton. Then for some $f \in \bar{P}$, both $w \in its(f)$ and $its(f) \cap \{v, u\}$ is singleton.

Proof. We distinguish two cases. (1) $\bar{P}_w \cap \bar{P}_v \neq \bar{P}_w \cap \bar{P}_u$, say $g \in \bar{P}w \cap \bar{P}_v$ but $g \notin \bar{P}_w \cap \bar{P}_v$. By Automorphism, there is an $f \in P$ such that $ts(f) = es(f) = es(g)$. It is then clear that $w \in its(f)$ and $its\ (f) \cap \{v, u\} = \{v\}$. (2) $\bar{P}_w \cap \bar{P}_v = \bar{P}_w \cap \bar{P}_u$. Suppose, without loss of generality, that $ts(e) \cap \{v, u\} = \{v\}$. By Determinacy, there is a $g \in \bar{P}_w \cap \bar{P}_v$ for which $ts(g) \cap \{v, u\} = \{v\}$. If $w \in ts(g)$, let $f = g$. Otherwise, choose f, by Automorphism, to have modal value $\langle W - ts(e), es(e)\rangle$.

We may now obtain:

THEOREM 42: Suppose that the P-structure $\mathfrak{A}$ satisfies World Actualism, Automorphism, and Determinacy. Then $\bar{F}_w$ determines U in $\mathfrak{B} = \mathfrak{A}^f$ whenever $U \in TS_w$.

Proof. Suppose that $\mathfrak{A}$ satisfies the conditions and that $U \in TS^{\mathfrak{A}}_w$, with $e_o \in \bar{P}_w$ and $ts(e_o) = U$. By World Actualism and Automorphism, there is, for any $u \in U$ and $v \in W - U$, an e, call it $e_{u,v}$, for which $ts(e) \cap \{u, v\}$ is singleton. So by lemma 41, there is an $f_{u,v} \in \bar{P}_w$ such that $w \in its(w)$ and $its(f_{u,v}) \cap \{u, v\}$ is singleton. Since $its(f_{u,v})$ is non-empty, each $f_{u,v} \in F$ and $es^{\mathfrak{B}}(f_{u,v}) = its^{\mathcal{A}}(f_{u,v})$.

It now follows that $\bar{F}_w$ determines U in $\mathfrak{B}$. It suffices to show that, for each α fixed on $\bar{F}_w$, $\alpha(U) \subseteq U$; for if α is fixed on $\bar{F}_w$, so is α^{-1}; and $\alpha(U) \subseteq U$ and $\alpha^{-1}(U) \subseteq U$ imply $\alpha(U) = U$. Suppose then, for an α fixed on $\bar{F}_w$, that $\alpha(U) \nsubseteq U$; say $u \in U$ and $v = \alpha(u) \notin U$. Now $f_{u,v} \in \bar{F}_w$ and $es^{\mathfrak{B}}(f_{u,v}) \cap \{u, v\}$ is singleton. But then $es^{\mathfrak{B}}(f_{u,v})$ cannot be invariant under α, and so α is not fixed on $\bar{F}_w$ after all.

Note that the proof does not call for a full use of the Automorphism condition, but merely of the truth of a few instances of the Comprehension scheme. By examining the proof, we also see that U may be expressed by a proposition in disjunctive normal form from atomic propositions to the effect that a given fact $f_{u,v}$ exists:

The above result is related to the result on the definability of outer truth T in terms of inner truth T^+, presented in section 8 of [15]. Both make use of the Determinacy condition; and the inner truth of a proposition corresponds, in the natural way, to the existence of a fact. However, the present result cannot be derived from the earlier one by the simple device of replacing reference to propositions and their inner truth in the definition of T by reference to facts and their existence; for the definition also makes use of the actualist notion $\Rightarrow^{+}$ of strict implication, which is not eliminable in this way.

Although the requirement that the truth-sets of propositions be recoverable from the facts is a reasonable one and can be proved under suitable assumptions, the stronger requirement that all modal values be recoverable from the facts is not reasonable and cannot be proved, even with the addition of Determinacy. For since it may be the P-structure $\mathfrak{A} = \mathfrak{B}^p$ for which $\mathfrak{A}^f = \mathfrak{B}$, what will be required is that, for each modal value $mv(e) = (U, V)$, there should be a set of facts G such that G determines U and $es(G) = V$. But although some facts will determine U, there will be no guarantee that the existence-conditions for those facts can be made to coincide with V, because of peculiarities in the way the existence-conditions of the facts are obtained. To be specific, let $\mathfrak{A} = (W, \bar{P}, t)$ be the ω P-structure for which:

(i) $W = \{w_1, w_2, w_3, w_4\}$, for w_1, w_2, w_3, w_4 all distinct;
(ii) $MV^{\mathfrak{A}} = \{\langle W, W\rangle, \langle\phi, W\rangle\} \cup \{\langle U, \{w_i, w_j\}\rangle :$
$U \subseteq W, 1 \leq i \leq j \leq 4\}$.

Then $\mathfrak{A}$ satisfies both P-Cond. and Determinacy. But the existence-sets of $\mathfrak{A}^f$ consist of W, $\{w_i\}$ and $\{w_i, w_j\}$ for $i \neq j$. So $\langle\{w_3\}, \{w_1, w_2\}\rangle$ is not a modal value of $(\mathfrak{A}^f)^p$, even though it is of $\mathfrak{A}$.

8. OBJECTUALIST THEORIES

We shall consider the various objectualist theories, both in a semantical and in a syntactical light. In each case, we shall investigate their status, as decidable or axiomatizable, and their relationship to the other theories.

Let P-Th*, F-Th*, PF-Th*, PFC-Th* be the theories determined by the respective conditions P-Cond, F-Cond, PF-Cond, and PFC-Cond. For ϕ from $\mathcal{L}^f$, let ϕ^0 be the result of relativizing the variables to the predicate F. Then from the previous equivalence theorems, the following conservative extension results may readily be proved:

THEOREM 43:
(i) For each sentence ϕ of $\mathcal{L}^f$, F-Th* $\vdash \phi$ iff PF-Th* $\vdash \phi^0$;
(ii) For each sentence ϕ of $\mathcal{L}^p$, P-Th* $\vdash \phi$ iff PF-Th* $\vdash \phi^0$;
(iii) For each sentence ϕ of $\mathcal{L}^{p,f}$, PF-Th* $\vdash \phi$ iff PFC-Th* $\vdash \phi$.

Some translations between the various theories may also be set up. Between $\mathscr{L}^f$ and $\mathscr{L}^p$, we may use the translation ϕ^p from §5:

THEOREM 44: For each sentence ϕ of $\mathscr{L}^f$, F-Th*$\vdash \phi$ iff P-Th*$\vdash \phi^p$.

Proof. From lemma 14 and theorem 34.

However, in this case, no reverse reduction, from $\mathscr{L}^p$ to $\mathscr{L}^f$ would appear to be possible.

Between PF-Th* and P-Th*, the translations ϕ^0 and ϕ^p may again be used:

THEOREM 45: For each sentence ψ of $\mathscr{L}^p$ and ϕ of $\mathscr{L}^{p,f}$

(i) P-TH*$\vdash \psi \Rightarrow PF$-Th*$\vdash \psi^o$;

(ii) PF-Th*$\vdash \phi \Rightarrow$ P-Th*$\vdash \phi^p$;

(iii) P-Th*$\vdash \psi \equiv (\psi^o)^p$ and PF-Th*$\vdash \phi \equiv (\phi^p)^o$.

Proof.

(i) Trivial.

(ii) From theorem 34.

(iii) One part is trivial and the other follows from lemma 21.

In the case of Th*-PF and Th*-PFC, we can no longer use the reduction in which $Px \wedge Fy \wedge \Box(T^+x \equiv Ey)$ replaced xCy, since the empirical criterion of identity is no longer available. However, the translation ϕ^p may be extended even further. For given a formula ϕ of $\mathscr{L}^{p,f,c}$, ϕ^p may be obtained by making the previous replacements in ϕ^+ and, in addition, by replacing sub-occurrences of $\rho_i Cf_j$ with $\rho_i = \sigma_j$ and sub-occurrences of $\rho_i C\rho_i$, $f_j Cf_j$ or $f_j C\rho_i$ with $\perp$. In analogy to lemma 21, it may now be shown that:

LEMMA 46: For $\phi' = \phi'(\rho_1, \ldots, \rho_m, f_1, \ldots, f_n)$ a formula of $\mathscr{L}^{p,f,c}$ with free variables among $\rho_1, \ldots, \rho_m$, $f_1, \ldots, f_n$, the sentence $(\Box)\left[\bigwedge_{j=1}^{n} \sigma_j Cf_j \supset \phi'(\rho_1, \ldots, \rho_m, f_1, \ldots, f_n) \equiv \phi^{p,o}(\rho_1, \ldots, \rho_m, \sigma_1, \ldots, \sigma_n)\right]$ is a logical consequence of Distinctness, Non-truth and the Correspondence axioms.

From this it then follows that:

THEOREM 47: For each sentence ϕ of $\mathscr{L}^{p,f,c}$:
(i) PFC-Th* $\vdash \phi$ iff PF-Th* $\vdash \phi^{p,o}$;
(ii) PFC-Th* $\vdash \phi \equiv \phi^{p,o}$.

Proof. (ii) follows from the lemma. (i) then follows from (ii) and theorem 43.

The various theories P-Th*, F-Th*, PF-Th* and PFC-Th* are all non-axiomatizable. The non-axiomatizability result for the theory P in §6 of [15] is of no help in this regard, since there the Coincidence condition was not adopted. However, a new method may be used to establish the non-axiomatizability of F-Th* and hence of the other theories.

THEOREM 48: The theory F-Th* is not axiomatizable.

Proof. Let T^+ be the second-order theory of two equivalence relations E_1 and E_2 (the quantification is over arbitrary sets and all valid sentences are to be theorems). By methods deriving from Rogers [34], it may be shown that T^+ is non-axiomatizable and, indeed, is equivalent in undecidability to full second-order logic. For any sentence ϕ of T^+, we produce a translate ϕ^* in $\mathscr{L}^f$ as follows. First, divide the fact variables into two distinct groups: $a_1, a_2, b_1, b_2, \ldots$ and $f_1, f_2, \ldots$ respectively. Second, replace each atomic formula $x_i E_\pi x_j$, for $\pi = 1, 2$, by $\Pi f[\Box(Ea_\pi \supset Ef) \supset (\Box(Eb_i \supset Ef) \equiv \Box(Eb_j \supset Ef))]$, and each atomic formula $x_i \in X_j$ by $\Box(Eb_i \supset Ef_j)$. Finally, replace each subformula $\forall x_i \psi$ by $\Pi b_i(\Diamond Wb_i \wedge \Diamond(Eb_i \wedge -Ea_1 \wedge -Ea_2)) \supset \psi)$ and each subformula $\forall X_i \psi$ by $\Pi f_i \psi$. Then it may be shown that:

> the sentence ϕ is a theorem of T^+ iff $\Pi a_1 \Pi a_2[(\Diamond Wa_1 \wedge \Diamond Wa_2) \supset \phi^*]$ is a theorem of Th*-F.

The proof of the right-to-left direction is relatively straightforward. For the proof of the other direction, suppose that the sentence ϕ is not a theorem of T^+. Then ϕ is false in some classical structure $\mathscr{C} = \langle X, \epsilon_1, \epsilon_2 \rangle$ with two equivalence relations ϵ_1 and ϵ_2 on X. Without loss of generality, we may suppose that X is infinite, say of cardinality κ. Now define a structure $\mathfrak{B} = (W, \bar{F})$ by:

(i) $W = X \cup \{w_1, w_2\}$, for w_1, w_2 distinct elements foreign to X,

(ii) $F = \{\langle V, \xi\rangle : V \subseteq W,\ \xi \leq 2^{\kappa}$ and V is closed under ϵ_1 if it contains w_1 and is closed under ϵ_2 if it contains $w_2\}$;

(iii) $\bar{F}_w = \{\langle V, \xi\rangle \in F : w \in V\}$.

Then it may be shown that $\mathfrak{B}$ satisfies F-Cond. and that $\mathfrak{B}$ is also counter-model for $\Pi a_1 \Pi a_2[(\Diamond Wa_1 \wedge \Diamond Wa_2) \supset \phi^*]$.

Theorem 48 also provides a new proof of the non-axiomatizability of the theory P from [15]. For let ϕ_0 be the sentence $(\Box)[T^+\rho \supset \exists\sigma(\Box(T\sigma \equiv E\sigma) \wedge \Box(T^+\sigma \equiv T^+\rho))]$. Then ϕ_0 expresses exactly the content of the Coincidence Condition. Therefore the sentence ϕ will be a theorem of P-Th* iff $\phi_0 \supset \phi$ is a theorem of P; and so the non-axiomatizability of P-Th* will carry over to P.

Two further features of this result are worth noting. One is that it is not essential to the proof that the language $\mathcal{L}^f$ of F-Th* contain identity. Thus we have another example of a pure existence theory that is of intuitive significance and yet is non-axiomatizable. Secondly, the theory F-Th* (without identity) can be expressed within the classical language of second-order logic, with quantification over sets corresponding to quantification over facts. Now as is well-known, the classical logic is decidable. But the present theory, in its translation, only differs from that logic in the intended range of the set variables; for the logic it is arbitrary, while for the theory it is subject to the automorphism condition. It is therefore of interest that such a small, but well-motivated, change, should turn a decidable system into one that is not axiomatizable.

Let us now attempt a partial axiomatization of the various semantically determined theories, in analogy to the treatment in §5 of [15].

First, we present a theory F-Th for facts alone. The objectualist abstraction axiom for facts takes the form:

Abstraction: $\Box\forall f_1 \ldots \forall f_n[(\phi \wedge Ef_1 \wedge \ldots \wedge Ef_n) \supset \exists g\Box(Eg \equiv (\phi \wedge Ef_1 \wedge \ldots \wedge Ef_n))]$, where $f_1, \ldots, f_n$ are exactly the free variables of ϕ and g itself does not occur free in ϕ.

The single propositional requirement on the existence of facts has been replaced by a composite propositional-cum-objectual requirement. Note that this is a scheme whose form is sensitive to the underlying language. Thus if there is reference to individuals or to other constituents, then their existence must be included in the existence conditions.

This scheme differs in subtle ways from its naive counterpart.

Define the negation, conjunction and disjunction of facts by:

g Neg f for $\Box(Eg \equiv -Ef)$;
h Conj g, f for $\Box(Eh \equiv (Eg \wedge Ef))$;
h Disj g, f for $\Box(Eh \equiv (Eg \vee Ef))$.

We then have, as with the naive scheme:

LEMMA 49: From F-Abstraction may be derived

$$\Box\forall f\forall g\exists h(h \text{ Conj } f, g).$$

However, the analogues of this result for Disj and Neg cannot be derived. Certain difficulties over $\Box\forall f\exists g(g \text{ Neg } f)$ have nothing to do with objectual considerations; for the fact f may be necessary and, in any case, the fact g will not co-exist with f. But the amended principle for Neg., $(\Box)[-Ef \supset \exists g(g \text{ Neg } f)]$, and the standard principle for Disj., $\Box\forall f, g\exists h(h \text{ Disj}, f, g)$, will still fail. For let $\mathfrak{B}$ be the ω F-structure in which $W = \{w_1, w_2, w_3, w_4\}$ and $ES^{\mathfrak{B}} = \{\{w_i\}: i = 1, \ldots, 4\} \cup \{\{w_i, w_j\}: 1 \leq i < j \leq 4\} \cup \{W\}$. Then $\mathfrak{B}$ satisfies F-Cond. But $\{w_2, w_3, w_4\} \notin ES^{\mathfrak{B}}$, which, given $\{w_1\} \in ES^{\mathfrak{B}}$, invalidates the principle, for Neg, and which, given $\{w_1, w_2\}$, $\{w_1, w_3\} \in ES^{\mathfrak{B}}$, invalidates the principle for Disj.

The axioms of the theory F-Th are now F-Abstraction, F-Diversity, World-Fact, and Conjunctive Closure for facts. The last three axioms have already been stated in §5 and so need not be restated here.

It will be recalled that a covering axiom was required for the theory of determinate facts in §2 and for the objectualist theory of propositions in [15]. It is therefore remarkable that Covering is redundant in the present context, though the proof is by no means trivial.

LEMMA 50: Covering is a theorem of F-Th.

Proof. Let us proceed semantically. Suppose $(\mathfrak{A}, w) \models -Ef$, for $f \in F$, and, using World-Fact, choose $e_0 \in F$ for which $w \models We_0$. By Abstraction, there is a $g \in F$ such that $\mathfrak{A} \models \Box(Eg \equiv (Ee_0 \wedge -\exists h\Diamond(Ee_0 \wedge -Eh)))$. Then $\mathfrak{A} \models \Box(Eg \supset -Ef)$. For otherwise, $(\mathfrak{A}, v) \models Eg \wedge Ef$ for some $v \in W$. But then $(\mathfrak{A}, v) \models \exists h\Diamond(Ee_0 \wedge -Eh)$, contradicting $\mathfrak{A} \models \Box(Eg \equiv (Ee_0 \wedge -\exists h\Diamond(Ee_0 \wedge -Eh)))$.

It may be shown that the theory Th-F is sound for its intended semantics.

THEOREM 51: The theorems of Th-F are true in each F-structure satisfying F-Cond.

Proof. The only problematic case is Abstraction, and that may be proved by techniques similar to those used in the proof of validity for P-Abstraction in theorem 31 of [15].

By following the proof of theorem 33 in [15], it may also be shown that:

THEOREM 52: The four axioms of F-Th are independent.

Proof. The modifications to the earlier proof are, for the most part, clear. In the case of conjunctive closure, one may let $\mathfrak{B}$ be the ω F-structure $(W, \bar{F})$ for which $W = \{0, 1, 2, \ldots\}$ and $ES = \{V \subseteq W : V$ is co-finite if $0, 1 \in V$ and V is finite if $0 \in V$ but $1 \notin V\}$; and one may let the formula ψ, in Conjunctive Closure, be $\Box(Eh \supset Ef)$. Then, as before, it may be shown that $\mathfrak{B}$ is a model for the other axioms, but that $(\mathfrak{B}, 0) \not\models \exists g(g \ \mathrm{Conj}_f \Box(Eh \supset Ef)$, for $es(h) = \{1\}$.

As in the case of the objectualist propositional systems, what is most striking about this result is the independence of Conjunctive Closure from Abstraction.

Comparable systems may be set up for the other semantical theories. For the theory P-Th., we require the axioms of the theory DV of [15] – viz. Abstraction, Covering, World-Proposition, Conjunctive Closure and Diversity, plus:

Coincidence: $\Box\forall\rho[T\rho \supset \exists\sigma(\Box(T\sigma \equiv E\sigma) \wedge \Box(T\sigma \equiv T^{+}\rho))]$

For the theory PF-Th., we require the axioms or schemes of P-Th. and F-Th., plus the axioms:

Non-truth: $(\Box)(-Tf)$; and

Distinctness: $(\Box)(\rho \neq f)$.

The theory PFC-Th requires, in addition, the following six axioms for C:

Correspondence:

(i) $(\Box)[xCy \supset Px \wedge Fy]$;

(ii) $(\Box)[(\rho Cf \wedge \sigma Cg) \supset (\rho = \sigma \equiv f = g)]$;

(iii) $(\Box)(\rho Cf \supset \Box\rho Cf]$;

(iv) $\Box\forall\rho(T\rho \supset \exists f(\rho Cf))$;

(v) $\Box\forall f\exists\rho(\rho Cf)$;

(vi) $(\Box)[\rho Cf \supset \Box(T^{+}\rho \equiv Ef)]$.

Axioms (ii) is a principle of internal identity, (iii) of external identity,

(iv) and (v) of internal existence, and (vi) of external existence. Axioms (iv) and (vi) combine to form the correspondence thesis $\Box\forall\rho(T\rho \supset \exists f\Box(Ef \equiv T^{+}\rho))$.

Great simplifications may be made to the mixed systems *Pf*-Th and *PFC*-Th. The axiom of Coincidence is redundant in both. For arguing intuitively, suppose we are given a proposition ρ for which $T^{+}\rho$. Then there exists a fact f for which $\Box(T^{+}\rho \equiv Ef)$ by *F*-Abstraction. So there exists a proposition σ for which $\Box(T\sigma \equiv Ef)$ and $\Box(E\sigma \equiv Ef)$, by *P*-Abstraction. But then $\Box(T\sigma \equiv E\sigma)$ and $\Box(T\sigma \equiv T^{+}\rho)$, as required.

The axiom of Conjunctive Closure for facts is also redundant in both systems. For let ϕ be the formula $\exists f(\psi \wedge \Box(T\rho \equiv Ef) \wedge \Box(E\rho \equiv Ef))$. From Conjunctive Closure for propositions, $\exists\sigma(\sigma \text{ Conj } \phi)$ may be derived. But by letting "g" be the fact that "σ" is true, we may then derive $\exists g(g \text{ Conj}_f\, \psi)$.

As with the anti-objectualist systems, *F*-Abstraction may be replaced with *PF*-Correspondence; $\Box\forall\rho(T\rho \supset \exists f\Box(Ef \equiv T^{+}\rho))$, and World-Fact may be dropped. Also, *P*-Abstraction may be limited to the case in which ϕ is a sentence of $\mathscr{L}^{p}$. But then *FP*-Correspondence must be added and Coincidence retained as axioms.

In Th-*PFC*, the existence for facts can be defined in terms of correspondence and propositional truth via the equivalence $(\Box)$ $[Ef \equiv \exists\rho(\rho Cf \wedge T\rho)]$. Hence all of the axioms for facts alone can be derived from this definition and the related theorems for propositions. Thus the theory *PFC*-Th is seen to provide a natural setting for an account of facts, with the principles concerning facts alone having no independent status, but merely being consequences of the propositional theory and of the general principles governing the relation of correspondence.

Two further simplifications may be made to *PFC*-Th. First, *F*- and hence *P*-Rigidity may be dropped, since they are derivable from the correspondence axioms (v), (iii), and (i). Secondly, given lemma 46, *P*-Abstraction may be restricted to formulas ϕ of $\mathscr{L}^{p}$.

Some of the results on conservative extensions and translations for the semantical systems may be extended to the syntactical ones. In analogy to parts (ii) and (iii) of theorem 43, it may be shown that:

THEOREM 53:

(i) For each sentence ϕ of $\mathscr{L}^{p}$, $P\text{-Th} \vdash \phi$ iff $PF\text{-Th} \vdash \phi^{0}$;

(ii) For each sentence ϕ of $\mathscr{L}^{p,f}$, $PF\text{-Th} \vdash \phi$ iff $PFC\text{-Th} \vdash \phi$.

Proof. (i) $\Rightarrow$ Trivial

$\Leftarrow$ Suppose $\mathfrak{A}$ is a model for P-Th and $\mathfrak{A} \not\models \phi$. Then show that $\mathfrak{A}^{p,f}$ is a model for PF-Th and that $\mathfrak{A}^{p,f} \not\models \phi^0$. To this end, it is best to use the formulation of PF-Th in which P-Abstraction is restricted to formulas ϕ of $\mathscr{L}^p$ and PF- and FP-Correspondence are axioms.

(ii) $\Rightarrow$ Trivial

$\Leftarrow$ Suppose that, for some model $\mathcal{N} = (\mathfrak{C}, w)$ for PF-Th, $\mathcal{N} \not\models \phi$. By the Skolem–Löwenheim theorem, we may suppose that W and $A = P \cup F$ are countable. But it can then be easily shown that there is a one-one correspondence γ between $\{e \in P : its(e) \neq \phi\}$ and F. Adding γ to $\mathcal{N}$ gives a PFC-model $\mathcal{N}^+$. It may now be shown that $\mathcal{N}^+$ makes ϕ false and, given that C need not appear in the instances of P-Abstraction, that $\mathcal{N}^+$ is a model for PFC-Th.

Turning to reductions, we have in analogy to the substantial parts of theorems 45 and 47:

THEOREM 54: For each sentence ϕ of $\mathscr{L}^{p,f}$:

(i) $PF\text{-Th} \vdash \phi$ iff $P\text{-Th} \vdash \phi^p$;

(ii) $PF\text{-Th} \vdash \phi \equiv (\phi^p)^o$.

THEOREM 55: For each sentence ϕ of $\mathscr{L}^{p,f,c}$:

(i) $PFC\text{-Th} \vdash \phi$ iff $P\text{-Th} \vdash \phi^p$;

(ii) $PFC\text{-Th} \vdash \phi \equiv \phi^p$.

Proof. Given theorem 53, part (i) of each theorem will follow from part (ii). Part (ii) then follows from lemmas 21 and 46.

This is as far as I have been able to take such results. I have not been able to show that for ϕ a sentence of $\mathscr{L}^f$, $PF\text{-Th} \vdash \phi^o$ implies $F\text{-Th} \vdash \phi$ or, equivalently, given the other results, that $P\text{-Th} \vdash \phi^p$ implies $F\text{-Th} \vdash \phi$; though the result seems plausible.

In analogy to §7 of [15], many different extensions or modifications to our basic system may be considered. Rather than give a survey, let us merely give a few examples along with their more interesting features. First of all, the syntactic consequences of imposing certain metaphysical conditions on a structure may be considered. Some of these, such as Determinacy, will be exactly expressible within the language, but others will not.

Secondly, one might experiment with the criteria of identity. In one direction, one might adopt an empirical criterion, either for facts alone or for facts and propositions together. In this case, the previous difficulties over cardinality on the semantical approach would disappear, although the negative results on axiomatizability would still obtain, since they did not depend upon the presence of identity in the language. In the other direction, one might explore some of the more refined consequences of adopting a structural criterion of identity. For example, one might follow §2 of [15] by assuming $\text{card}\{e \in A : mv(e) = \langle U, V\rangle\} = \text{card}\{e \in A : mv(e) = \langle W - U, V\rangle\}$ and generalisations of that sort. Such assumptions would then have corresponding consequences for the identity of facts.

Finally, one might add a notation for fact abstracts, introduce quantification over sets of facts, or allow for the use of pseudo-classes in the formulation of the axioms. If fact abstracts are introduced, with $\urcorner\phi$ denoting the fact that ϕ, then F-Abstraction will take the form:

$$(\Box)[(\phi \wedge Et_1 \wedge \ldots \wedge Et_n) \supset \Box(E\urcorner\phi \equiv (\phi \wedge Et_1 \wedge \ldots \wedge Et_n))],$$

where $t_1, \ldots, t_n$ are all of the 'individual' terms to occur in ϕ. If, in addition, propositional abstraction (§) and correspondence (C) are introduced, then the scheme:

$$\exists f(f = \urcorner\phi) \supset \S\phi C \urcorner\phi$$

should be added as an axiom. For the most part, though, the treatment will follow that in [15].

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NOTES

[1] Proponents of sentential reference include Russell [34], Prior [27] and Rundle [33]. Belnap and others [4] have tried to avoid singular reference to propositions in their pro-sentential theory of truth, and they might wish to apply similar considerations to facts. Those on the other side include Baylis [3], Slote [37] and Williamson [44].

[2] Here, as elsewhere, I have been sloppy over use-mention; but my meaning should be clear, even when my language is not. The above account is, of course, an elucidation, not a definition, of correspondence, since the sentence ϕ has a schematic role. It might be thought that correspondence could be defined as f being identical to the fact that p is true; but this would not be correct on a highly structuralist conception of facts.

[3] To give the general form of the argument, let us use the quantifier $\forall x$ to range over the actuals of each world and the quantifier Πx to range over a more inclusive domain. Use $F^\Diamond$ as a predicate for the possible $F = ers$ and E as a predicate for existence. Then from $\Box\forall x(Fx \supset -Gx)$ and the auxiliary premisses $\Pi x(F^\Diamond x \supset \Diamond(Ex \wedge Fx))$ and $\Pi x(Gx \supset \Box(Ex \supset \Box(Ex \supset Gx))$, it logically follows that $\Pi x(F^\Diamond x \supset -Gx)$. Thus given the

categorial nature of the objects that satisfy G, the actual F's and the possible F's cannot be distinguished in regard to whether they are G's.

4 This is a point that is rarely appreciated and which I myself overlooked in the reduction of possibilist discourse in [10]. I there proposed a separate reduction for possible individuals and possible worlds. But there is, in fact, a common reduction, taking quantification over the possibles, be they individuals or worlds, into quantification over the actuals. What confuses the issue is that (a) the worlds may also be used to secure back reference to the given world in the common reduction, though other devices may be used in their place and (b) the quantification over actual worlds may be further reduced, to world-propositions, perhaps, or world facts. If the aspects (a) or (b) are combined with the world-reduction, then the common element with the individual-reduction is easily overlooked.

5 Note that the argument does not require that the reduced theory contain the identity predicate. Identity provides the simplest case, but any predicate that enables one to discriminate some entities $y_1, y_2, \ldots, y_m$ of $Y - X$ from all entities $x_1, x_2, \ldots, x_m$ of X will do.

6 Indeed, it is doubtful that Quine's condition of proxyhood (see [31] and [32]) or any other formal conditions are both necessary and sufficient for a legitimate reduction.

7 The need for a distinction among forms of being is briefly defended in §E1 of [14].

8 The general topic of ontological genesis is discussed more thoroughly in a book under preparation, *Objects Under a Description.* I can only be brief here.

9 The concretizer C should be distinguished both from the operation of *glossing*, in *Objects Under a Description*, that takes an individual x and property P into the new individual x qua P, and from the much mooted operation of *exemplication*, that takes x and P into the state of x's being P. But these are differences that will not concern us in the paper.

10 There is the question as to whether the distinction between determinate and non-determinate facts can sensibly be made out. I think it can; but I shall not discuss the issue here.

11 Again, the distinction between the two relations may be traced back to Moore ([24], p. 145) in his contrast between direct and indirect verification. The distinction between determinate and non-determinate facts and its consequences for verification are also very clearly presented in Clark [7].

12 Shorter [37] has also suggested that there is an indeterminacy in our ordinary usage, but it is drawn between facts of the world and true propositions, not between facts and truths. He also discusses the bearing of such a distinction on the questions of negative facts and reducibility; but his conclusions are slightly different from my own.

13 See [16] for further discussion of these distinctions.

14 I have assumed that there are necessary facts. Without this assumption, contingency may be defined as $\exists f(f = \urcorner \phi \vee f = \urcorner - \phi)$ and then the other modal notions defined in terms of contingency.

15 In [13], I was led through different considerations altogether to distinguish between propositional and constitutive content.

16 A similar point has been made in regard to semantical theories of truth. My point here is that the mere reference to facts does not help us to get a realist position.

17 The relevance of questions of identity to modal reduction was discussed, all too briefly, in §6 of [10].

REFERENCES

[1] Ackerman, W. (1954): *Solvable Cases of the Decision Problem*, North-Holland: Amsterdam

[2] Austin, J. L. (1961): *Philosophical Papers* (ed. J. O. Urmson and G. J. Warnock), Oxford: Clarendon Press.

[3] Baylis, C. A. (1948): 'Facts, Propositions, Exemplification and Truth', *Mind* **57**, 459–479.

[4] Belnap, N. D., Camp, J. L. and Grover, D. L. (1975): 'A Pro-sentential Theory of Truth', *Philosophical Studies* **27**, 73–125.

[5] Brownstein, D. (1976): 'Denoting Corresponding and Facts', *Theoria* **42**, 115–138.

[6] Carnap, R. (1947): *Meaning and Necessity*, Chicago: The University of Chicago Press.

[7] Clark, R. L. (1975): 'Facts, Fact-correlates, and Fact-surrogates', in *Fact, Value and Perception: Essays in Honor of Charles A. Baylis* (ed. P. Welsh), Durham, N.C.: Duke University Press.

[8] Clark, R. W. (1976): 'What Facts Are', *Southern Journal of Philosophy* **14**, 257–268.

[9] Ducasse, C. J. (1940): 'Propositions, Opinions, Sentences and Facts', *Journal of Philosophy* **37**, 707–711.

[10] Fine, K. (1977): 'Postscript' to *Worlds, Times and Selves* (with A. N. Prior), London: Duckworth.

[11] Fine, K. (1978): 'Model Theory for Modal Logic I – The De Re/De Dicto Distinction', *Journal of Philosophical Logic* **7**, 125–156.

[12] Fine, K. (1979): 'Failures of the Interpolation Lemma in Quantified Modal Logic', *Journal of Symbolic Logic* **44**, 201–206.

[13] Fine, K. (1979): 'Analytic Implication' in *Papers on Language and Logic* (ed. J. Dancy), England: Keele University Library.

[14] Fine, K. (1982): 'The Problem of Non-existents I – Internalism', to appear in *Topoi.*

[15] Fine, K. (1979): 'First-Order Modal Theories II – Propositions', *Studia Logica* **39**, 159–202.

[16] Fine, K. (1981): 'First-Order Modal Theories I – Sets', *Noûs* **15**, 117–206.

[18] Kripke, S. A. (1972): 'Naming and Necessity', in *Semantics of Natural Language* Theory', *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik* **8**, 113–116.

[18] Kripke, S. A. (1972): 'Naming and Necessity', in *Semantics of Natural Language* (ed. Harman and Davidson), Holland: Reidel, reprinted and revised in book form, Harvard University Press (1980).

[19] Löwenheim, L. (1915): 'Über Möglichkeiten im Relativkalkül, *Math. Annalen* **76**, 137–148.

[20] Malcolm, N. (1940): 'The Nature of Entailment', *Mind* **49**, 333–347.

[21] Martin J. (1975): 'Facts and the Semantics of Gerunds', *Journal of Philosophical Logic* **4**, 439–454.

[22] Moore, G. E. (1953): *Some Main Problems in Philosophy*, London: Unwin.

[23] Moore, G. E. (1959): *Philosophical Papers*, London: Unwin.

[24] Moore, G. E. (1966): *Lectures on Philosophy* (ed. C. Lewy), London: Unwin.

[25] Prior, A. N. (1948): 'Facts, Propositions and Entailment', *Mind* **57**, 62–68.
[26] Prior, A. N. (1957): *Time and Modality*, Oxford: Clarendon Press.
[27] Prior, A. N. (1967): 'Correspondence Theory of Truth', in *Encyclopedia of Philosophy* (ed. P. Edwards), New York: Macmillan.
[28] Prior, A. N. (1968): 'Tense Logic and the Logic of Earlier and Later' in *Papers on Time and Tense*, England: Oxford University Press.
[29] Prior, A. N. (1971): *Objects of Thought*, England: Oxford University Press.
[30] Prior, A. N. (1977): *Worlds, Times and Selves* (with K. Fine), London: Duckworth.
[31] Quine, W. V. (1966): 'Ontological Reduction and the World of Numbers', *Journal of Philosophy* **61**, 209–216 (1964); reprinted in *The Ways of Paradox*, New York: Random House.
[32] Quine, W. V. (1969): *Ontological Relativity*, New York: Columbia University Press.
[33] Ramsey, F. P. (1931): 'Facts and Propositions', *Aristotelian Society, Suppl. Vol.* 7, 153–170 (1927); reprinted in *The Foundations of Mathematics* (ed. R. B. Braithwaite), London: Routledge and Kegan Paul.
[34] Rogers, H. (1956): 'Certain Logical Reductions and Decision Problems', *Annals of Mathematics* **64**, 264–284.
[35] Rundle, B. (1979): *Grammar in Philosophy*, Oxford: Clarendon Press.
[36] Russell, B. (1956): 'The Philosophy of Logical Atomism', in *Logic and Knowledge* (ed. R. C. Marsh), London: Unwin.
[37] Shorter, J. M. (1962): 'Facts, Logical Atomism and Reduction', *Australasian Journal of Philosophy* **40**, 283–302.
[38] Slomson, A. B. (1969): 'An Undecidable Two Sorted Predicate Calculus', *Journal of Symbolic Logic*, **4**, 21–23.
[39] Slote, M. A. (1974): *Metaphysics and Essence*, Oxford: Basil Blackwell.
[40] Sprigge, T. L. S. (1970): *Facts, Words and Beliefs*, New York: Humanities Press.
[41] Strawson, P. (1971): *Logico-Linguistic Papers*, London: Methuen.
[42] Tarski, A. (1949): 'Undecidability of the Theories of Lattices and Projective Geometrics, *Journal of Symbolic Logic* **14**, 77–78.
[43] Taylor, B. (1976): 'States of Affairs' in *Truth and Meaning* (ed. G. Evans and J. McDowell), Oxford: Clarendon Press.
[44] Toms, E. (1940): *Fact and Entailment*, *Mind* **49**, 451–454.
[45] van Fraassen, B. C. (1969): 'Facts and Tautological Entailments', *The Journal of Philosophy* **66**, 477–487.
[46] Wells, R. S. (1949): 'The Existence of Facts', *Review of Metaphysics* **3**, 1–20.
[47] Williamson, J. (1976): 'Facts and Truth', *Philosophical Quarterly* **26**, 203–216.
[48] Wisdom, J. (1969): *Logical Constructions* (ed. J. J. Thomson), New York: Random House.