

SOME RELATIONS BETWEEN POTENTIAL THEORY AND
THE WAVE EQUATION

by
D. A. Darling

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ABSTRACT

Solutions to the Dirichlet and Neumann problems for the region exterior to the intersection of two regions whose individual electrostatic Green's functions are known are developed. The method is applied specifically to obtain solutions for the exterior of a solid finite circular cone with a spherical cap. The solutions to the vector and scalar wave equations for long wavelengths can be expressed in terms of these Dirichlet and Neumann solutions. This will be the subject of a forthcoming report.

SCOPE OF REPORT AND SUMMARY OF RESULTS

Our ultimate purpose is to obtain solutions to the scalar wave equation

$$\nabla^2 \phi + k^2 \phi = 0$$

over the exterior of a compact set Ω in 3-space, subject to vanishing normal derivatives on the boundary of Ω , and in the presence of a source ψ — i. e. essentially the Green's solution. We intend to get explicit solutions, moreover, in terms of classical functions, and indeed in terms of the Green's functions for regions simpler than Ω . In this paper we study only the case $k = 0$ (potential problem), and we will subsequently use these results to obtain the so-called Rayleigh solutions for $k \neq 0$.

This program is accomplished under the following two limitations:

- (1) The region Ω is the intersection of two regions Ω_1 , and Ω_2 for each of which the exterior electrostatic Green's function is known.
- (2) The region Ω is, moreover, axially symmetric as is the function ψ , and these axes of symmetry coincide.

Using the present method it does not appear possible to lighten these restrictions essentially (it will be seen under (1) that we could have an arbitrary number of regions instead of just two). These two restrictions,

which correspond roughly to the two major parts of this work, are imposed only as they are needed to "make the method work", since a greater generality at the intermediate steps adds nothing to the complexity of the treatment and is conceivably of independent interest.

In Part I, in which we assume only the restrictions under (1), we solve the general Dirichlet problem for a region Ω which is the intersection of two regions Ω_1 and Ω_2 — i. e. $\Omega_1 \cap \Omega_2$ — for which individually the electrostatic Green's functions G_1 and G_2 are known. The method is probabilistic in nature, using Brownian motion theory. It is not indispensable to use this theory to achieve our results, but it was the way it was discovered and it has the advantage of being relatively simple, and in addition gives simple bounds on the iterative method it leads to.

In Part II we assume conditions (1) and (2) and using the result of Part I we show how to solve the Neumann problem (vanishing normal derivatives on the boundary of Ω) with a suitable singularity (the axially integrated Green's solution). This gives, as a special case, of course, the dipole source at infinity directed along the axis of symmetry analogous to the "radiation condition" at infinity). The method employed is due essentially to Bassett and is very old, but seems not to have been noticed by later writers. The present modification, in particular, seems new.

In Part III some of these results are explicitly carried through for a specific example — i. e. a solid finite cone capped by a spherical segment, the center of the sphere coinciding with the apex of the cone. We give calculations for the solution of the Dirichlet problem with constant boundary values, and those boundary values which in turn yield the solution the Neumann problem corresponding to a dipole source at infinity. These calculations are rather involved but can be, as appears to be generally the case, expressed neatly in terms of explicit matrix calculations. The main numerical problem, indeed, is that of inverting a certain matrix.

In Part IV we discuss several aspects of the method of Bassett; namely its uniqueness and its relation to the "method of generalized electrostatics" developed recently by Weinstein and Payne.

I

PROBABILISTIC METHODS AND THE DIRICHLET PROBLEM

1.1 Notation and Terminology

The points of 3-space E^3 are denoted by vectors p, q, x, y, \dots which will be co-ordinatized when necessary. We consider only scalar-valued functions f, g, U, V, \dots . The volume differential at p is denoted by $d v(p)$.

We consider certain regions $\Omega, \Omega_1, \Omega_2, \dots$ in E^3 which, if bounded, will be compact. The complement of Ω is denoted by $\bar{\Omega}$ and the boundary of Ω by $B(\Omega)$. We denote by $\frac{\partial}{\partial n_p}$ a derivative in the direction of the outward drawn normal to $B(\Omega)$ at the point $p \in B(\Omega)$. The restrictions placed on Ω here and in later developments will be clear from the context. For $p \in B(\Omega)$ we denote by $d \sigma(p)$ the surface area differential.

By the electrostatic Green's function associated with Ω we mean the function $G(x, y)$, $x, y \in \bar{\Omega}$ which as a function of y is harmonic in $\bar{\Omega}$ except at $y = x$, which vanishes on $B(\Omega)$, and is such that $G(x, y) \sim \frac{1}{4\pi |x - y|}$, $x \rightarrow y$. If $B(\Omega)$ is considered as a (possibly unbounded) grounded conductor, $G(x, y)$ is the potential induced at x by a unit charge at y .

1.2 Probability Considerations

We let $x(t)$, $0 \leq t < \infty$ be the three dimensional Wiener-Einstein stochastic process; i. e. $x(t) = (x_1(t), x_2(t), x_3(t))$ where $x_i(t)$ are three independent one-dimensional processes with $x(0) = 0$. Let a compact Ω be given and a function $f(p)$, $p \in B(\Omega)$, be defined over $B(\Omega)$. The following probabilistic fact is the cornerstone of the method employed here.

Let $x_0 \in \bar{\Omega}$, and let $T = T(x_0)$ be the time at which $x(t) + x_0$ first intersects Ω , if it ever does. If $x(t) + x_0$ never intersects Ω , set $T = \infty$. Thus

$$T(x_0) = \sup_{t > 0} \left\{ t \mid x(\tau) + x_0 \in \bar{\Omega}, \quad 0 \leq \tau < t \right\}.$$

From the continuity of $x(t)$ we conclude $x(T) + x_0 \in B(\Omega)$ and we define a random variable $Y = Y(x_0)$ by

$$Y = \begin{cases} f(x(T) + x_0), & T < \infty \\ 0, & T = \infty \end{cases}$$

Then it is true that $U(x_0) = E(Y(x_0))$, where E is the expectation, is the solution to the exterior Dirichlet problem. That is, $U(x)$ is harmonic for $x \in \bar{\Omega}$, assumes the value $f(x)$ for $x \in B(\Omega)$, and $U(x)$ vanishes like $\frac{1}{|x|}$, for $|x| \rightarrow \infty$.

These facts are relatively well known — cf [1, 2], [3] — and may be deduced from the more elementary fact that if $G_{\Omega}(x, y)$ is the electrostatic Green's function for Ω , then $\frac{\partial G_{\Omega}(x_0, p)}{\partial n_p} d\sigma(p)$ is the (infinitesimal) probability that $x(t) + x_0$ first enters Ω via the surface element $d\sigma(p)$ at p . We then obtain by integration

$$E(Y(x_0)) = \int_{B(\Omega)} f(p) \frac{\partial G_{\Omega}(x_0, p)}{\partial n_p} d\sigma(p),$$

which is the classical solution.

1.3 The Dirichlet Problem for Two Intersecting Regions

Let Ω be the intersection of two regions Ω_1 and Ω_2 ; $\Omega = \Omega_1 \cap \Omega_2$ and suppose that the electrostatic Green's functions G_1 and G_2 are known for Ω_1 and Ω_2 . Exploiting the probabilistic interpretation of the Dirichlet problem of Section 1.2, we show how to solve the Dirichlet problem for Ω .

The method will also yield the electrostatic Green's function for Ω (which we need in Parts II and III) by the simple expedient of solving the Dirichlet problem for those boundary values induced on $B(\Omega)$ by the potential

$$\frac{1}{4\pi |x - x_0|}.$$

For convenience we assume at least one of the two regions

Ω_1 and Ω_2 is bounded, so that the same is true of $\Omega = \Omega_1 \cap \Omega_2$. The

method is applicable if Ω is the intersection of $\Omega_1, \Omega_2, \dots, \Omega_n$ where

the individual Green's functions are known, but the calculational complexity increases rapidly.

Let $B(\Omega_1) = S_1 \cup S_1'$ and $B(\Omega_2) = S_2 \cup S_2'$ where $S_1 = B(\Omega_1) \cap \Omega_2$,
 $S_1' = B(\Omega_1) \cap \bar{\Omega}_2$ and similarly for S_2, S_2' ; $S_2 = B(\Omega_2) \cap \Omega_1$,
 $S_2' = B(\Omega_2) \cap \bar{\Omega}_1$. To make the problem non-trivial we assume these four
 components non-void.

In connection with this notation we refer to Figure 1 below in which
 Ω_1 is an infinite cone and Ω_2 is a sphere, an example treated in detail in
 part III.

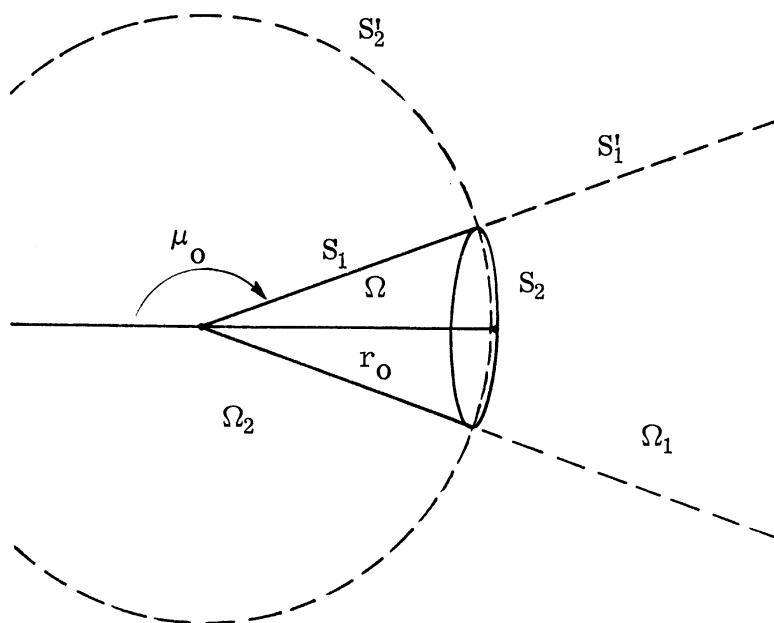


Figure 1

Here the radius of the sphere is r_0 and the semi vertex angle of the cone is $\pi - \cos^{-1} \mu_0$.

Let $f(p)$ be the desired boundary value defined over $B(\Omega) = S_1 \cup S_2$ and consider a point $p \in \overline{\Omega}_1 \cap \overline{\Omega}_2$. Let us denote by $E_1(p)$, $p \in S_2^l$ the required expectation, which is the potential sought, where the Wiener process starts at a point $p \in S_2^l$ and similarly $E_2(p)$, $p \in S_1^l$ for a starting position $p \in S_1^l$. For the particle starting at $p \in S_2^l$, it may intersect S_1 before intersecting S_1^l , or S_1^l before S_1 , or may never intersect either. If we denote by $G_1(p, q)$ and $G_2(p, q)$ the respective electrostatic Green's functions for $\overline{\Omega}_1$ and $\overline{\Omega}_2$ the sum of the expectations for these three cases is

$$E_1(p) = F_1(p) + \int_{S_1^l} \frac{\partial G_1(p, x)}{\partial n_x} E_2(x) d\sigma(x), \quad (2)$$

$p \in S_2^l,$

where

$$F_1(p) = \int_{S_1} \frac{\partial G_1(p, x)}{\partial n_x} f(x) d\sigma(x), \quad (3)$$

$p \in S_2^l.$

In an exactly similar way we find the corresponding equations for a point $p \in S_1^l$

$$E_2(p) = F_2(p) + \int_{S_2^!} \frac{\partial G_2(p, x)}{\partial n_x} E_1(x) d\sigma(x) \quad (4)$$

$p \in S_1^!$

where

$$F_2(p) = \int_{S_2^!} \frac{\partial G_2(p, x)}{\partial n_x} f(x) d\sigma(x) \quad (5)$$

$p \in S_1^!$

Equations (2) and (4) constitute a pair of integral equations for E_1 and E_2 which we propose to solve. We remark that if we have done so, and F_1 and F_2 are defined for an arbitrary point $p \in \bar{\Omega}_1 \cap \bar{\Omega}_2$ by (3) and (5), then the right hand sides of either (2) or (4) gives the potential sought, and they must agree in this region. If, however, as in Figure 1, the region Ω_2 is bounded and Ω_1 is not then for $p \in \Omega_1 \cap \bar{\Omega}_2$ only (4), and not (2), gives the correct result. Similarly, for $p \in \bar{\Omega}_1 \cap \Omega_2$, (2) only gives the correct result. Indeed it is easily seen that the above enumeration of cases is valid for both (2) and (4) only if $p \in \bar{\Omega}_1 \cap \bar{\Omega}_2$.

Let us define

$$H_1(p, q) = \frac{\partial G_1(p, q)}{\partial n_q}, \quad q \in S_1^! \quad (6)$$

$$H_2(p, q) = \frac{\partial G_2(p, q)}{\partial n_q}, \quad q \in S_2^! \quad (7)$$

$$\mathcal{J}_1(p) = F_1(p) + \int_{S_1^I} H_1(p, x) F_2(x) d\sigma(x) \quad (8)$$

$p \in S_2^I$

$$\mathcal{J}_2(p) = F_2(p) + \int_{S_2^I} H_2(p, x) F_1(x) d\sigma(x) \quad (9)$$

$p \in S_1^I$

$$K_1(p, q) = \int_{S_1^I} H_1(p, x) H_2(x, q) d\sigma(x) \quad (10)$$

$p, q \in S_2^I$

$$K_2(p, q) = \int_{S_2^I} H_2(p, x) H_1(x, q) d\sigma(x) \quad (11)$$

$p, q \in S_1^I$

where F_1 and F_2 are defined in (3) and (5). Then (2) and (4) yield the following Fredholm equations for E_1 and E_2 ,

$$E_1(p) = \mathcal{J}_1(p) + \int_{S_2^I} K_1(p, x) E_1(x) d\sigma(x), \quad (12)$$

$p \in S_2^I$

$$E_2(p) = \mathcal{J}_2(p) + \int_{S_1^I} K_2(p, x) E_2(x) d\sigma(x) \quad (13)$$

$p \in S_1^I$

These are two ordinary Fredholm equations which determined E_1 and E_2 uniquely and, as mentioned above, they completely determine the solution.

1.4 The Iterative Solutions for E_1 and E_2

In these integral equations the classical iteration procedure can be carried out profitably (in contrast to the double-layer integral equation for which the iteration procedure diverges).

Considering E_1 for example, we let $K_1^{(n)}(p, q)$ be the n^{th} iterate of the kernel K_1

$$K_1^{(1)}(p, q) = K_1(p, q)$$

$$K_1^{(n+1)}(p, q) = \int_{S_2'} K_1^{(n)}(p, x) K_1(x, q) d\sigma(x)$$

Then we have the classical Neumann expansion

$$E_1(p) = F_1(p) + \sum_{n=1}^{\infty} \int_{S_2'} K_1^{(n)}(p, x) F_1(x) d\sigma(x)$$

and this series converges exponentially fast.

To verify this last remark, we notice that $K_1^{(n)}(p, q) d\sigma(q)$ is the probability that the Brownian motion particle has made n double

transfers between S_2' and S_1' , terminating at an area element $d\sigma(q)$ on S_2' without ever having entered $S_1 \cap S_2 = B(\Omega)$. If Ω_1 , say, is bounded it is clear, at least intuitively, that the event of n such double transfers, for large n , has a probability $O(a^n)$ for some $0 < a < 1$.

More precisely, put

$$a = \sup_{x \in S_2'} \int_{S_2'} K(x, q) d\sigma(q)$$

then $a < 1$ and it follows easily

$$\int_{S_2'} K_1^{(n)}(x, y) d\sigma(y) \leq a^n, \quad n = 1, 2, \dots$$

and also if we put

$$b = \max_{x \in S_2'} |F_1(x)|$$

we shall have

$$E_1(p) = F_1(x) + \sum_{j=1}^n \int_{S_2'} K_1^{(j)}(p, q) F_1(q) d\sigma(q) + J$$

where

$$|J| \leq \sum_{j=n+1}^{\infty} b a^{j-1} = \frac{b a^n}{1-a}$$

giving a simple bound on the error made by truncating the series at n terms.

It is easily seen that if we know $G_2(p, q)$ for a region Ω_2 which "differs" but little from Ω , then a is very small, and as a matter of fact, pursuing this remark it is possible to get an exact perturbation formula for a region of the form $\Omega = \Omega_2 \cup T_\epsilon$ where T_ϵ is a region whose capacity goes to zero with ϵ .

In the practical calculations we have encountered thus far all the iterates can be explicitly calculated in terms of matrix calculations, and the resolvent kernel for these integral equations is expressible as the inverse of an explicit matrix, cf Part III.

II

SOLUTION OF THE NEUMANN PROBLEM IN TERMS
OF THE DIRICHLET PROBLEM2.1 Assumptions and Notations

From this point on we assume, as stated in the introduction, that $\Omega = \Omega_1 \cap \Omega_2$ is axially symmetric, or is a body of rotation. The sets Ω_1 and Ω_2 need not be, however. We also adopt, from this point on, the spherical co-ordinate system (ρ, ϕ, θ) to describe the vectors p, q, \dots . In this notation ρ is the distance from the origin $\rho = |p|$, ϕ is the co-latitude $0 \leq \phi \leq \pi$ and θ is the longitude $0 \leq \theta \leq 2\pi$. We adopt the transformation $\mu = \cos \phi$, so that $1 \geq \mu \geq -1$, and consider, when convenient, the system (ρ, μ, θ) .

We choose the axis of symmetry of Ω to be the polar axis $\phi = 0$. Given a region Ω we say that a function $U(x)$, $x \in \bar{\Omega}$, is a solution of the Neumann problem if $U(x)$ is harmonic, $x \in \bar{\Omega}$, and $\partial/\partial n_x U(x) = 0$, $x \in B(\Omega)$. In order to avoid yielding the trivial result $U = \text{const.}$ the condition of harmonicity in $\bar{\Omega}$ must be relaxed and U must possess certain singularities either in the finite part of E^3 or "at infinity". The precise nature of these is discussed later.

2.2 Harmonic Functions and Stream Functions

We denote by ∇_1^2 and E^2 the operators

$$\nabla_1^2 \psi = \frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial \psi}{\partial \rho} \right) + \frac{\partial}{\partial \mu} \left((1 - \mu^2) \frac{\partial \psi}{\partial \mu} \right) + \frac{1}{1 - \mu^2} \frac{\partial^2 \psi}{\partial \theta^2} \quad (14)$$

$$E^2 g = \rho^2 \frac{\partial^2 g}{\partial \rho^2} + (1 - \mu^2) \frac{\partial^2 g}{\partial \mu^2} \quad (15)$$

We apply E^2 only to axially symmetric functions, so the term in θ does not appear. We say that a pair of functions, axially symmetric, are conjugate, if they satisfy the Beltrami-Stokes equations. That is, ϕ and h are conjugate if and only if

$$(1 - \mu^2) \frac{\partial \phi}{\partial \mu} = \frac{\partial h}{\partial \rho} \quad (16)$$

$$\rho^2 \frac{\partial \phi}{\partial \rho} = - \frac{\partial h}{\partial \mu} \quad (17)$$

It follows easily that if ϕ and h are conjugate, ϕ must satisfy $\nabla_1^2 \phi = 0$ and h must satisfy $E^2 h = 0$. Moreover, if ϕ and h are conjugate they are orthogonal - $\nabla \phi \cdot \nabla h = 0$. The pair of equations (16) and (17) correspond to the ordinary Cauchy-Riemann equations, and given any ψ satisfying $\nabla_1^2 \psi = 0$ there corresponds, by means of them, a function g satisfying $E^2 g = 0$, and conversely; given g satisfying $E^2 g = 0$ there exists

a ϕ satisfying $\nabla_1^2 \phi = 0$. The correspondence is unique up to an additive constant which may be determined by prescribing the value at one point, or at infinity. We give the exact reciprocity relations later.

The goal, which we described in the introduction, of solving only Dirichlet boundary value problems could be achieved if, given Ω , we could solve the problem $E^2 h = 0$, $p \in \overline{\Omega}$, $h(\sigma) = \text{const}$, $\sigma \in B(\Omega)$, for then the function ϕ conjugate to h would be harmonic with the Neumann boundary conditions. It turns out to be possible to do this by solving an accessory Dirichlet problem.

2.3 The Method of Bassett

Bassett noticed that the following relation is true, for any function $u = u(\rho, \mu)$ with requisite derivatives,

$$\cos \theta E^2 \left(\rho \sqrt{1-\mu^2} u(\rho, \mu) \right) = \rho \sqrt{1-\mu^2} \nabla_1^2 (\cos \theta u(\rho, \mu)) \quad (18)$$

This correspondence between E^2 and ∇_1^2 , which is easily proved by substituting in (14) and (15), is the link by which we connect the Dirichlet and Neumann problems. Bassett's results are credited to him by Hobson in [4]. A related procedure, due to Weinstein and Payne, is sketched briefly in Part V.

From (18) it is clear that if we want to solve the problem $E^2 g = 0$ with boundary values $b(p)$, $p \in B(\Omega)$ we can do so by solving $\nabla_1^2 \psi = 0$ with boundary values $c(p) = \frac{\cos \theta}{\sqrt{1-\mu^2}} b(p)$, and in fact the function g will then be given by

$$g = \frac{\rho \sqrt{1-\mu^2}}{\cos\theta} \psi \quad (19)$$

If we now find a function ϕ conjugate to g , ϕ will be harmonic and its equipotential surfaces will be orthogonal to the level surfaces of g .

In particular, suppose $v(x)$ is harmonic in $\bar{\Omega}$, except for prescribed singularities. In order to solve the Neumann problem we consider the function \bar{v} conjugate to v and choose $b(p) = \bar{v}(p)$, $p \in B(\Omega)$. Then the function $\bar{v} - g$, with g as above has zero boundary values and its conjugate function $v - \psi$ is harmonic and has vanishing normal derivatives on $B(\Omega)$, and thus furnishes the solution with the prescribed singularities. This procedure is in fact the way the further development proceeds.

We remark that the restriction to axially symmetric bodies and boundary conditions is indispensable for the use of (18), and within this framework cannot be extended.

2.4 Solution of Reciprocity Equations

We consider the equations (16) and (17), showing how to express ϕ in terms of h and vice versa.

Suppose ϕ is an axially symmetric harmonic function defined on $\bar{\Omega}$. Then there exist measures ν_1 and ν_2 over the non-negative reals such that for sufficiently small ρ

$$\phi(\rho, \mu) = \int \rho^\lambda P_\lambda(\mu) d\nu_1(\lambda) \quad (20)$$

and for ρ sufficiently large

$$\phi(\rho, \mu) = \int \frac{1}{\rho^{\lambda+1}} P_\lambda(\mu) d\nu_2(\lambda) \quad (21)$$

These are the "normal solutions" and generally ν_1 and ν_2 will be discrete so that the above integrals will be sums - that is, ν_1 , for example, will assign a weight a_n at the value $\lambda_n \geq 0$, $n = 0, 1, \dots$.

When ϕ is expressed in this fashion its conjugate $\bar{\phi} = h$ has a particularly simple form. We obtain, namely, by solving (16) and (17), that for ρ in the range where (20) is valid

$$\bar{\phi} = h(\rho, \mu) = -\rho \sqrt{1-\mu^2} \int \frac{\rho^\lambda}{\lambda+1} P_\lambda^{(1)}(\mu) d\nu_1(\lambda) \quad (22)$$

and for ρ in the range of (21)

$$\bar{\phi} = h(\rho, \mu) = \rho \sqrt{1-\mu^2} \int \frac{1}{\lambda \rho^{\lambda+1}} P_\lambda^{(1)}(\mu) d\nu_2(\lambda) \quad (23)$$

These formulas determine h except for an additive constant. We omit the derivation which is straightforward.

Conversely, any axially symmetric stream function h can be written in the form of (22) or (23) with suitable ν_1 and ν_2 , and the harmonic function which is conjugate to it is given by (20), (21) with these same ν_1, ν_2 .

2.5 Solution of the Neumann Problem

Let $G(\rho, \mu)$ be an axially symmetric function harmonic in Ω except for prescribed singularities. Using the preceding results we are now in a position to solve the Neumann problem for $\bar{\Omega}$, that is finding a function $U(\rho, \mu)$ harmonic in $\bar{\Omega}$ with vanishing normal derivatives on Ω and with the same singularities as G .

We write

$$U(p) = G(p) - V(p) \quad (24)$$

so that $V(p)$ is harmonic everywhere in Ω , and we require it to vanish for $\rho \rightarrow \infty$, thus requiring U and G have the same limiting behavior.

Write G as

$$G = \int \rho^\lambda P_\lambda(\mu) d\nu(\lambda) \quad \rho < \rho_0 \quad (25)$$

on supposing the singularities of G are exterior to the sphere $\rho = \rho_0$.

The function conjugate to G is then, by the preceding section,

$$\bar{G} = -\rho \sqrt{1-\mu^2} \int \frac{\rho^\lambda}{\lambda+1} P_\lambda^{(1)}(\mu) d\nu(\lambda)$$

and we write for a point $p \in B(\Omega)$

$$b(p) = - \int \frac{\rho^\lambda}{\lambda+1} P_\lambda^{(1)}(\mu) d\nu(\lambda) \cos\theta \quad (26)$$

Let $V_1(p) \cos \theta$ be the solution to the Dirichlet problem for $\bar{\Omega}$ with boundary values $b(p)$ - it is simple to prove that the solution is of the form $V_1(\rho, \mu) \cos \theta$. Then $\rho \sqrt{1-\mu^2} V_1(\rho, \mu)$ is a stream function and V_1 can thus be written in the form of (22) or (23)

$$\begin{aligned} V_1(\rho, \mu) &= \int \rho^\lambda P_\lambda^{(1)}(\mu) d\tau_1(\lambda) \\ V_1(\rho, \mu) &= \int \frac{1}{\rho^{\lambda+1}} P_\lambda^{(1)}(\mu) d\tau_2(\lambda) \end{aligned} \quad (27)$$

We now set V as the function conjugate to $\rho \sqrt{1-\mu^2} V_1$, so that we have from the reciprocity formulas of Section 2.4

$$\begin{aligned} V &= - \int (\lambda + 1) \rho^\lambda P_\lambda(\mu) d\tau_1(\lambda) \\ V &= \int \frac{\lambda}{\rho^{\lambda+1}} P_\lambda(\mu) d\tau_2(\lambda) \end{aligned} \quad (28)$$

for small and large ρ respectively. The requirement that $V \rightarrow 0$, $\rho \rightarrow \infty$ then shows the additive undetermined constant must be, in fact, zero.

Since $\bar{G} - \rho \sqrt{1-\mu^2} V_1$ is a stream function vanishing on $B(\Omega)$ its conjugate has normal derivatives zero on $B(\Omega)$, as required.

To recapitulate, the normal solution to the Neumann problem, given Ω and G , is

$$U = G - \overline{\rho \sqrt{1-\mu^2} V_1}$$

(the bar representing a conjugate) where $V_1 \cos\theta$ is the solution to the Dirichlet problem for Ω with boundary values b given by (26), ν being determined by (25). The function $V = \frac{1}{\rho \sqrt{1-\mu^2}} V_1$ is given by (28) where τ_1 and τ_2 are determined by (27).

2.6 The Green's Solution

We now specialize the solution of the preceding section to the case where the singularity is that generated by "charged ring". This will give the Green's solution, or fundamental solution.

Thus in equation (24) we take for the function G

$$G(\rho_o, \mu_o; \rho, \mu) = \frac{1}{4\pi} \int_0^{2\pi} \frac{d\theta}{|p - p_o|} \quad (29)$$

and this can be expanded

$$\begin{aligned} G(\rho_o, \mu_o; \rho, \mu) &= \sum_0^{\infty} \frac{\rho^n}{\rho_o^{n+1}} P_n(\mu_o) P_n(\mu) \quad , \quad \rho < \rho_o \\ &= \sum_0^{\infty} \frac{\rho_o^n}{\rho^{n+1}} P_n(\mu_o) P_n(\mu) \quad , \quad \rho > \rho_o \end{aligned} \quad (30)$$

and this is the Green's function for the operator ∇_1^2 defined by (14) with respect to the differential $d\rho d\mu$. That is

$$\nabla_1^2 \int_0^\infty \int_{-1}^1 G(\rho_o, \mu_o; \rho, \mu) \xi(\rho, \mu) d\mu d\rho = \xi(\rho_o, \mu_o) \quad (31)$$

for continuous functions ξ .

By (22) and (23) the function conjugate to this G is

$$\bar{G}(\rho_o, \mu_o; \rho, \mu) = -\rho \sqrt{1-\mu^2} \sum_1^\infty \frac{\rho^n}{(n+1)\rho^{n+1}} P_n^{(1)}(\mu) P_n(\mu_o) \quad (32)$$

$$\rho < \rho_o$$

$$\bar{G}(\rho_o, \mu_o; \rho, \mu) = \rho \sqrt{1-\mu^2} \sum_1^\infty \frac{\rho_o^n}{n\rho^{n+1}} P_n^{(1)}(\mu) P_n(\mu_o) \quad (33)$$

$$\rho > \rho_o$$

The term $n = 0$ does not appear since $P_o^{(1)}(\mu) = 0$.

By (26) we consider, for $(\rho, \mu) \in B(\Omega)$ and for $\rho < \rho_o$ when

$(\rho, \mu) \in B(\Omega)$, the boundary value

$$b(\rho, \mu, \theta) = - \sum_1^\infty \frac{\rho^n}{(n+1)\rho_o^{n+1}} P_n^{(1)}(\mu) P_n(\mu_o) \cos\theta \quad (34)$$

and we let $V_1(\rho, \mu)\cos\theta$ be the solution to the ordinary Dirichlet problem

with these boundary values on $B(\Omega)$. The function $V_1(\rho, \mu)$ can then be

expressed, by (27), in the normal form

$$V_1(\rho, \mu) = \sum_{n=0}^{\infty} a_n \rho^{\lambda_n} P_{\lambda_n}^{(1)}(\mu) \quad (35)$$

$$V_1(\rho, \mu) = \sum_{n=0}^{\infty} \frac{a_n}{\rho^{\lambda_n+1}} P_{\lambda_n}^{(1)}(\mu)$$

for small and large ρ respectively, the constants a_n depending on ρ_o, μ_o .

For the function conjugate to V_1 we have by (28)

$$V(\rho, \mu) = \sum_{n=0}^{\infty} a_n (\lambda_n + 1) \rho^{\lambda_n} P_{\lambda_n}(\mu) \quad (36)$$

$$V(\rho, \mu) = \sum_{n=0}^{\infty} a_n \frac{\lambda_n}{\rho^{\lambda_n+1}} P_{\lambda_n}(\mu)$$

and consequently the required solution to the Neumann problem is

$$U(\rho_o, \nu_o; \rho, \mu) = G(\rho_o, \mu_o; \rho, \mu) - V(\rho, \mu) \quad (37)$$

where G is given by (32) and V by (36).

To recapitulate, given Ω we solve the Dirichlet problem with boundary values $b(\rho, \mu, \theta)$ given by (34). The solution can be expressed as $V_1 \cos \theta$ which determines the constants $a_n = a_n(\rho_o, \mu_o)$ in (33). These determine $V(\rho, \mu)$ via (36), and then U is given by (37).

III

AN APPLICATION TO A SOLID FINITE CONE.

3.1 The Problem Considered

Using spherical coordinates let Ω be the intersection of

$$\Omega_1: \mu \leq \mu_0$$

$$\Omega_2: \rho \leq r_0$$

The relevant regions and surfaces are as in Fig. 1.

In this part we are going to carry through some of the calculations for Ω that we developed in parts I and II. The calculations, though elementary, become somewhat involved; but fortunately the use of matrix notation permits a significant compression.

We solve here the Neumann problem for Ω , corresponding to a singularity at infinity of the form $\rho\mu$, $\rho \rightarrow \infty$. This is done in reasonable detail and the answer, given by (45), is expressed in terms of the quite simple matrix $\begin{smallmatrix} \diagup \\ \diagdown \end{smallmatrix}$ given by (43).

For the Dirichlet problem for the exterior of Ω , that is the solution with boundary values one on $B(\Omega)$, is similar, except that it is not necessary to find the conjugate functions, and is given by (46).

3.2 The Neumann Problem

As mentioned in the preceding section we do not use the singularity generated by (30), but only its limiting form when $\rho_0 \rightarrow \infty$ and $\mu_0 = 1$.

This will be the analogue of the "radiation condition" and is that induced by an indefinitely large charge on the remote axis of symmetry - i. e. The potential behaves like $\rho\mu$, $\rho \rightarrow \infty$ as can be seen by considering

$$\lim_{\rho_0 \rightarrow \infty} \rho_0^2 \left(G - \frac{1}{\rho_0} \right)$$

in equation (30).

By considering (32) this induces the boundary values

$$\begin{aligned} \lim_{\rho_0 \rightarrow \infty} \rho_0^2 b(\rho, \mu, \theta) &= - \frac{1}{2} \rho P_1^{(1)}(\mu) \cos \theta \\ &= \frac{1}{2} \rho \sqrt{1 - \mu^2} \cos \theta \end{aligned}$$

when we set $\mu_0 = 1$ in (34) and take the limit. Thus in (26) we have just one term and the boundary value for the associated Dirichlet problem is

$$b(\rho, \mu, \theta) = \frac{1}{2} \rho \sqrt{1 - \mu^2} \cos \theta. \quad (38)$$

3.3 Solution of the Associated Dirichlet Problem

Let $a_j(m)$ be the positive zeros of $P_x^{(m)}(\mu_0)$ considered as a function of x , $j = 1, 2, \dots$, $m = 0, 1, \dots$ and put

$$\begin{aligned} \alpha_j(m) &= \left. \frac{\partial P^{(m)}(x)}{\partial x} \right|_{x=\mu_0} \\ \beta_j(m) &= \left. \frac{\partial P_x^{(m)}(\mu_0)}{\partial x} \right|_{x=a_j(m)}. \end{aligned}$$

Then the Greens function for Ω_1 is

$$G_1(p, q) = \sum_{m=0}^{\infty} \sum_{j=1}^{\infty} \frac{-2(2 - \delta_{m,0})}{(1 - \mu_o^2)^{4\pi}} \frac{\rho_p^{a_j(m)}}{\rho_q^{a_j(m)+1}} \frac{P_{a_j(m)}^{(m)}(\mu_p) P_{a_j(m)}^{(m)}(\mu_q)}{\alpha_j(m) \beta_j(m)} \cos m(\theta_p - \theta_q) \quad (39)$$

if $\rho_q \geq \rho_p$; if $\rho_q \leq \rho_p$ we interchange p and q in (39).

For Ω_2 , the sphere, we have

$$G_2(p, q) = \frac{1}{4\pi} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(1 - \frac{r_o^{2n+1}}{\rho_p^{2n+1}}\right) \frac{\rho_p^n}{\rho^{2n+1}} P_n^{(m)}(\mu_p) P_n^{(m)}(\mu_q) \cos m(\theta_p - \theta_q). \quad (40)$$

for $\rho_p \leq \rho_q$. We make a similar interchange of p and q if $\rho_p \geq \rho_q$.

For $B(\Omega_1)$ and $B(\Omega_2)$ we have

$$\left. \begin{aligned} d\sigma_1(p) &= \rho \sqrt{1 - \mu_o^2} d\rho d\theta \\ \frac{\partial}{\partial n_p} &= \frac{1}{\rho} \sqrt{1 - \mu_o^2} \frac{\partial}{\partial \mu} \Big|_{\mu = \mu_o} \end{aligned} \right\} p \in B(\Omega_1)$$

$$\left. \begin{aligned} d\sigma_2(p) &= r_o^2 d\theta d\mu \\ \frac{\partial}{\partial n_p} &= \frac{\partial}{\partial \rho} \Big|_{\rho = r_o} \end{aligned} \right\} p \in B(\Omega_2)$$

In calculating H_1 , H_2 , \mathcal{J}_1 , \mathcal{J}_2 , K_1 , K_2 of (6), (7), (8), (9), (10) and (11) we need only retain the terms corresponding to $m = 1$ in (39) and (40) because of the boundary condition (38). cf. also the remark following (26).

We let a_j , α_j , β_j be respectively $a_j(1)$, $\alpha_j(1)$, $\beta_j(1)$, and in the sequel we replace the cosines by complex exponentials, it being understood that in the ensuing calculations we are to take real parts and subsequent to any θ integrations we are to supply a factor π , since an easy calculation

shows

$$\int_0^{2\pi} \operatorname{Re} \left(e^{i\theta_p - i\theta_x} \right) \operatorname{Re} \left(e^{i\theta_x - i\theta_q} \right) d\theta_x$$

$$\pi \operatorname{Re} \left(e^{i\theta_p - i\theta_q} \right)$$

This notational device enables us to write H_1 and H_2 , as well as the other relevant functions, as rather compact bilinear forms.

Thus retaining only the terms $m = 1$ in (39), (40) and defining the following column vectors, with $p = (\rho, \mu, \theta)$,

$$U_1(p) = \begin{pmatrix} a_j^{(1)} \\ \rho \mathbf{P}_{a_j}(\mu) e^{i\theta} \end{pmatrix}$$

$$V_1(p) = \begin{pmatrix} \frac{1}{\pi \sqrt{1 - \mu_o^2}} & \frac{e^{-i\theta}}{\rho a_j + 2\beta_j} \end{pmatrix}$$

$$U_2(p) = \begin{pmatrix} \mathbf{P}_{a_j}^{(1)}(\mu) \\ \frac{a_j}{\rho a_j + 1} e^{i\theta} \end{pmatrix}$$

$$V_2(p) = \begin{pmatrix} \frac{\rho_j^{a_j} - 1}{\pi \sqrt{1 - \mu_o^2}} & \frac{e^{-i\theta}}{\beta_j} \end{pmatrix}$$

we have for H_1 , defined by (6)

$$H_1(p, q) = U_1^t(p) V_1(q), \quad \rho_p < \rho_q$$

$$= U_2^t(p) V_2(q), \quad \rho_p > \rho_q$$

$$q \in B(\Omega_1),$$

where the accent denotes the transpose and where, without specifically indicating it, real parts are to be taken.

Similarly defining

$$Y(p) = \begin{pmatrix} \frac{P_j^{(1)}(\mu)}{\rho_j^{j+1}} e^{i\theta} \end{pmatrix}$$

$$Z(p) = \begin{pmatrix} \frac{2j+1}{4\pi} r_o^{j-1} P_j^{(1)}(\mu) e^{-i\theta} \end{pmatrix}$$

we obtain from (40) and (7)

$$H_2(p, q) = Y^t(p) Z(q)$$

$$q \in B(\Omega_2)$$

The follow two vectors A , B and three matrices M , N_1 , N_2 , whose components are constants, are needed. In the designation of a matrix as c_{jk} , the index j refers to the row index and k the column index. In denoting

by d_j a column vector, d_j refers to the j^{th} component of it.

$$A = \int_{S_1} V_2(x) b(x) d\sigma_1(x) \\ = \frac{\sqrt{1 - \mu_o^2} r_o^{a_j + 2}}{2\beta_j (a_j + 2)} \quad (41)$$

$$B = \int_{S_2} Z(x) b(x) d\sigma_2(x) \\ = \frac{(1 - \mu_o^2) (2j + 1) r_o^{j + 2}}{8(j - 1)(j + 2)} \left\{ \frac{\mu_o}{\sqrt{1 - \mu_o^2}} P_j^{(1)}(\mu_o) - \sqrt{1 - \mu_o^2} P_j^{(1)'}(\mu_o) \right\}, j > 1$$

$$\frac{r_o^3}{8} (3\mu_o - \mu_o^3 - 2), j = 1$$

$$M = \int_{S'_1} V_1(x) Y'(x) d\sigma_1(x) \\ = \frac{P_k^{(1)}(\mu_o)}{\beta_j (a_j + k + 1) r_o^{a_j + k + 1}}$$

$$j = 1, 2, \dots$$

$$k = 0, 1, \dots$$

$$N_1 = \int_{S'_2} Z(x) U_1'(x) d\sigma_2(x)$$

$$= (1 - \mu_o^2) \frac{2j+1}{4} \frac{r_o^{a_k+j+1} P_j^{(1)}(\mu_o) \alpha_k}{(a_k - j)(a_k + j + 1)}$$

$$j = 0, 1, \dots$$

$$k = 1, 2, \dots$$

$$N_2 = \int_{S_2^1} Z(x) U_2^1(x) d\sigma_2(x) \\ = (1 - \mu_o^2) \frac{2j+1}{4} \frac{P_j^{(1)}(\mu_o) \alpha_k}{(a_k - j)(a_k + j + 1) r_o^{a_k - j}}$$

$$j = 0, 1, \dots$$

$$k = 1, 2, \dots$$

In these integrations the function $b(p)$ is given by (38) and a factor π has been supplied in lieu of the θ integrations, as explained above.

In terms of these expressions it is simple to calculate the quantities in (3) and (5), and (8) thru (11). They are, namely, the real parts of the left hand sides of the following:

$$F_1(p) = U_2^1(p) A, \quad p \in \bar{\Omega}_1 \cap \bar{\Omega}_2$$

$$F_2(p) = Y'(p) B, \quad \rho_p \geq r_o$$

$$\mathcal{F}_1(p) = U_2^1(p) A + U_1^1(p) M B, \quad p \in S_2^1$$

$$\mathcal{F}_2(p) = Y'(p) (B + N_2 A), \quad p \in S_1^1$$

$$K_1(p, q) = U_1'(p) M Z(q), \quad p, q \in S_2'$$

$$K_2(p, q) = Y'(p) N_1 V_1(q), \quad p, q \in S_1'$$

In terms of these expressions we can easily solve (12) and (13).

From (13) we obtain for E_2 , namely,

$$\begin{aligned} E_2(p) = & Y'(p) B + Y'(p) N_2 A \\ & + Y'(p) N_1 \int_{S_1'} V_1(x) E_2(x) d\sigma_1(x) \end{aligned} \quad (42)$$

$p \in S_1'$

Denote by X the vector

$$X = \int_{S_1'} V_1(x) E_2(x) d\sigma_1(x),$$

then multiply (42) by $V_1(p)$ and integrate over S_1' , obtaining

$$X = MB + MN_2 A + MN_1 X$$

giving

$$X = (I - MN_1)^{-1} (MB + MN_2 A)$$

Substituting this in (42) we obtain

$$\begin{aligned} E_2(p) = & Y'(p) \left[B + N_2 A + N_1 (I - MN_1)^{-1} M (B + N_2 A) \right] \\ = & Y'(p) \left[I + N_1 (I - MN_1)^{-1} M \right] (B + N_2 A) \\ = & Y'(p) (I - N_1 M)^{-1} (B + N_2 A) \end{aligned}$$

Denoting the vector Λ by

$$\Lambda = (\lambda_j) = (I - N_1 M)^{-1} (B + N_2 A) \quad (43)$$

we have thus

$$E_2(p) = \sum_j \lambda_j \frac{P_j^{(1)}(\mu)}{\rho^{j+1}} \cos \theta,$$

which, though initially defined only for $\mu = \mu_0$ ($p \in S_1^I$) is now seen to be valid everywhere in $\rho \geq r_0$ ($p \in \bar{\Omega}_2$) by continuation, this being a consequence of the remark following (5). This remark is also verified after we calculate E_1 .

To calculate E_1 , we obtain from (12)

$$\begin{aligned} E_1(p) &= U_2^I(p) A + U_1^I(p) M B \\ &\quad + U_1^I(p) M \int_{S_2^I} Z(x) E_1(x) d\sigma(x) \end{aligned}$$

This is solved exactly as for E_2 , and we obtain

$$\begin{aligned} E_1(p) &= U_2^I(p) A + U_1^I(p) M B \\ &\quad + U_1^I(p) M (I - N_1 M)^{-1} (N_2 A + N_1 M B), \end{aligned}$$

$$p \in S_2^I.$$

On using (4) we obtain for the potential $W(p)$, $p \in \bar{\Omega}_2$, the expression

$$\begin{aligned}
 W(p) &= Y'(p) B + Y'(p) \left[N_2 A + N_1 M B \right. \\
 &\quad \left. + N_1 M (I - N_1 M)^{-1} (N_2 A + N_1 M B) \right] \\
 &= Y'(p) \left[B + (I - N_1 M)^{-1} (N_2 A + N_1 M B) \right] \\
 &= Y'(p) (I - N_1 M)^{-1} (N_2 A + B) \\
 &= Y'(p) \bigwedge
 \end{aligned} \tag{44}$$

in agreement with (43)

A similar argument with (2) yields

$$\begin{aligned}
 W(p) &= F_1(p) + \int_{S_1^I} H_1(p, x) E_2(x) d\sigma(x) \\
 &= U_2^I(p) A + \left\{ U_2^I(p) \int_{\rho_x < \rho_p} V_2(x) Y'(x) d\sigma(x) \right. \\
 &\quad \left. + U_1^I(p) \int_{\rho_x > \rho_p} V_1(x) Y'(x) d\sigma(x) \right\} [I - N_1 M]^{-1} (B + N_2 A).
 \end{aligned}$$

For the k^{th} component of the vector in brackets we have

$$e^{i\theta} \frac{P_k^{(1)}(\mu_o)}{\rho^{k+1}} \sum_{j=1}^{\infty} \frac{P_{a_j}^{(1)}(\mu)}{\beta_j} \left\{ \frac{1}{a_j - k} + \frac{1}{a_j + k + 1} \right\}$$

$$- \sum_{j=1}^{\infty} \frac{P_{a_j}^{(1)}(\mu)}{\rho^{a_j-1}} e^{i\theta} \frac{P_k^{(1)}(\mu_o) r_o^{a_j-k}}{\beta_j (a_j - k)}$$

in which we have set $p = (\rho, \mu, \theta)$.

In the first term the summation on j can be performed and the result is simply $P_k^{(1)}(\mu) / P_k^{(1)}(\mu_o)$. This follows directly in expanding $P_k^{(1)}(\mu)$ as a Fourier series in the complete orthogonal set $\{P_{a_j}^{(1)}(\mu)\}$ over the interval $1 \geq \mu \geq \mu_o$, the sum of the series equaling this function except perhaps at the end points of this interval.

Thus, defining the matrix Q by

$$Q = \frac{P_k^{(1)}(\mu_o) r_o^{a_j-k}}{\beta_j (a_j - k)}, \quad \begin{matrix} j = 1, 2, \dots \\ k = 0, 1, \dots \end{matrix}$$

we obtain for $p \in \Omega_1$

$$\begin{aligned} W(p) &= U_2'(p) A + Y'(p) \left[I - N_1 M \right]^{-1} (B + N_2 A) \\ &\quad - U_2'(p) Q \left[I - N_1 M \right]^{-1} (B + N_2 A). \end{aligned}$$

These two expressions for W , namely this latter and (44) must agree over their common domain of definition, i. e. $\overline{\Omega}_1 \cap \overline{\Omega}_2$.

The potential sought is then

$$W(p) = \sum \lambda_j \frac{P_j^{(1)}(\mu)}{\rho^{j+1}} \cos \theta, \quad \rho > r_o.$$

where the λ_j are the components of \bigwedge given by (43).

The stream function for the body Ω of Fig. 1 is then, by (23),

$$\rho \sqrt{1 - \mu^2} \sum_{j=0}^{\infty} \lambda_j \frac{P_j^{(1)}(\mu)}{\rho^{j+1}}$$

and the solution to the Neumann problem for Ω is by (24) and (28)

$$N(p) = \rho \mu - \sum_{n=1}^{\infty} n \lambda_n \frac{P_n(\mu)}{\rho^{n+1}} \quad (45)$$

To recapitulate, let λ_n be the n^{th} component of the vector Λ , given by (43), and A by (41). Then the solution to the Neumann problem for Ω , which behaves as $\rho\mu$ for $\rho \rightarrow \infty$, is given by (45).

We remark that in the matrix Q , if μ_o is such that $a_j = k$ for some integers j and k , that element in Q becomes indeterminate, and is to be replaced by its limiting value when $k \rightarrow a_j$, regarding k as a continuous variable. This limiting value is readily seen to be unity.

Also, as explained in section 1.4, the inverse in (43) can be approximated by

$$(I - N_1 M)^{-1} = \sum_{k=0}^n (N_1 M)^k + J_n$$

where the bound for J_n is known, as in section 1.4.

3.4 Solution of the Conductor Potential Problem

We solve here the Dirichlet problem for Ω , in which the function $b(p) = 1$ instead of (38), and seek the solutions which vanish as ρ^{-1} for $\rho \rightarrow \infty$. The calculations are formally nearly identical to those in the preceding section, except that because of the simple constant $b(p)$ the θ integrations can be avoided.

Define

$$a_j = j^{\text{th}} \text{ positive zero of } P_x(\mu_o)$$

$$\alpha_j = \left. \frac{\partial P_{a_j}(x)}{\partial x} \right|_{x=\mu_o}$$

$$\beta_j = \left. \frac{\partial P_x(\mu_o)}{\partial x} \right|_{x=a_j}$$

and the following vectors A, B , and matrices M, N_1, N_2

$$A = \frac{r_o^{a_j+1}}{\beta_j(a_j+1)}$$

$$B = \frac{2j+1}{2} r_o^{j+1} \frac{(\mu_o^2 - 1) P_j'(\mu_o)}{j(j+1)}, \quad j > 0$$

$$\frac{1}{2} r_o(\mu_o + 1), \quad j = 0$$

$$M = \frac{P_k(\mu_o)}{\beta_j(a_j + k + 1) r_o^{a_j + k + 1}}$$

$$N_1 = (1 - \mu_o^2) \frac{2j+1}{2} \frac{r_o^{a_k+j+1} P_j(\mu_o) \alpha_k}{(a_k - j)(a_k + j + 1)}$$

$$N_2 = (1 - \mu_o^2) \frac{2j+1}{2} \frac{P_j(\mu_o) \alpha_k}{(a_k - j)(a_k + j + 1) r_o^{a_k - j}}$$

These correspond to the previous designated quantities with the same letters, and we have for the solution of the Dirichlet problem

$$D(p) = \sum_{n=0}^{\infty} \lambda_n \frac{P_n(\mu)}{\rho^{n+1}} \quad (46)$$

where λ_n is the n^{th} component of Λ given by

$$\Lambda = (I - N_1 M)^{-1} (B + N_2 A),$$

as before.

IV

THE METHOD OF BASSETT

4.1 Preliminary Remarks

The success of the present method rests very intimately on Bassett's identity (18) . This identity enabled us to revert the solution of the Neumann problem to that of a Dirichlet problem. There is thus a certain interest in attempting to extend and simplify it, if possible.

In this part of the report we give a sketch of the method of Weinstein and Payne which is related to Bassett's method, and show that Bassett's identity is essentially the only one of its kind.

4.2 The Uniqueness of Bassett's Method

The crux of the method of Bassett consists in finding a universal boundary function, given in (26), such that the solution to the Dirichlet problem corresponding to it could be, under a suitable transformation, changed to the Neumann solution by considering the conjugate function to the transformed function. For the method to be universally applicable to every Ω the boundary function and transformation employed must be independent of Ω .

From this last remark it is simple to deduce that the boundary function must be a harmonic function. We only have to consider a sequence of boundaries Ω_n converging to Ω and employ Harnack's theorem. Also the transformation

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must be linear in the sense that if D is the solution to the Dirichlet problem the transform of D must be $aD + b$ where a and b are certain point functions.

If we let $aD + b = F$ we obtain $D = \alpha F + \beta$ for some functions α and β , and if we want $F = \text{const.}$ for a point on $B(\Omega)$ for every Ω , then $\alpha = B$ is the universal bounding function sought and $\beta = \text{const.}$ and we may as well assume $\beta = 0$. We now consider only axially symmetric Ω - if uniqueness can be established for this subset of all Ω it will follow over the larger class.

The condition required now is that $\nabla^2 BF = 0$ implies $E^2 F = 0$ and conversely. From the fact that the classes of harmonic functions and stream functions form linear manifolds we have then $\nabla^2 BF = CE^2 F$ for every F , and we seek functions B and C such that this is an identity in F .

The calculation becomes simpler in cylindrical co-ordinates, and (14), (15) become with co-ordinates (z, r, θ)

$$\nabla^2 = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

$$E^2 = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r}$$

so that

$$E^2 = \nabla^2 - \frac{2}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

If we are then to have the identity $\nabla^2 BF = CE^2 F$, we obtain for F axially symmetric

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$$\nabla^2_{BF} = F \nabla^2 B + 2 \nabla F \cdot \nabla B + B \nabla^2 F$$

$$CE^2_F = C \nabla^2 F - \frac{2C}{r} \frac{\partial F}{\partial r}$$

If these expressions are to be identical in F then

$$\nabla^2 B = 0$$

$$B = C$$

which verifies the previous remark that the bounding function is harmonic.

The equation in F then becomes

$$2 \nabla F \cdot \nabla B = - \frac{2B}{r} \frac{\partial F}{\partial r}$$

which shows B cannot depend on z. Since in that case $\nabla F \cdot \nabla B = \frac{\partial F}{\partial r} \frac{\partial B}{\partial r}$

we obtain that B must satisfy

$$\frac{\partial B}{\partial r} = - \frac{B}{r}$$

and $B = \frac{C}{r}$ where C is some function of θ . Finally if B is to be harmonic

we must have $C''(\theta) + C'(\theta) = 0$ and $C = A \cos \theta + A' \sin \theta$.

Thus we obtain $B = \frac{1}{r} (A \cos \theta + A' \sin \theta)$ or expressed in spherical co-ordinates

$$B = \frac{A \cos \theta + A' \sin \theta}{\rho \sqrt{1 - \mu^2}}$$

for some constants A, A'. The choice used in (19) was $A = 1, A' = 0$.

4.3 The Method of Weinstein-Payne

Using what they term the "method of generalized electrostatics", Weinstein and Payne [7] have developed a procedure similar to the above to solve the stream function equation in terms of the solution of a Dirichlet problem. Though their method is not relevant to the development presented here it seems worthwhile to point out the similarity.

Their method depends on the fact there exists an identity similar to Bassett's in the case when the dimensionality of the problem is stepped up by two. Using the cylindrical co-ordinates again we let ∇_n^2 and E_n^2 be the Laplacian and stream operators in n dimensions with axial symmetry. Thus for $n > 2$

$$\nabla_n^2 = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{n-2}{r} \frac{\partial}{\partial r}$$

$$E_n^2 = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} - \frac{n-2}{r} \frac{\partial}{\partial r} .$$

The identity which corresponds to (18) is then

$$\nabla_{n+2}^2 \left(\frac{1}{r^{n-1}} \psi \right) = \frac{1}{r^{n-1}} E_n^2 \psi$$

and the reciprocity relations (16), (17) are

$$\frac{\partial \phi}{\partial z} = \frac{1}{r^{n-2}} \frac{\partial \psi}{\partial r} \quad ; \quad \frac{\partial \phi}{\partial r} = \frac{1}{r^{n-2}} \frac{\partial \psi}{\partial z}$$

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Choosing $n = 3$ it is thus seen that by solving the conductor potential problem (the electrostatic problem) in 5 dimensions we can find the stream function in 3 dimensions and thus the Neumann problem in 3 dimensions. This method has the advantage that the Dirichlet problem considered is one with constant boundary values, but it has the serious disadvantage of requiring 5 dimensional potentials. In our work it has seemed preferable to use only 3 dimensional potentials with non-constant boundary values.

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