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Geometric and Analytic Views in Existence Theorems for Optimal Control. II. Distributed and Boundary Controls[1]

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Abstract. Existence theorems are proved for Lagrange problems of optimization in a given domain G with possibly unbounded distributed controls in G and on the boundary of G, and with functional relations on G and on the boundary represented by closed operators, not necessarily linear. The case where the functional relations are partial differential equations is emphasized. Recent work concerning the reduction or elimination of seminormality requirements is taken into account. Many examples are given.

Key Words. Existence theorems, orientor fields, Lagrange problems, distributed and boundary controls, closed graph and convergence properties of operators, analytic criteria for seminormality.

1. Introduction

We are interested in existence theorems for nonlinear problems of control with possibly unbounded strategies. We present here existence theorems for problems with distributed and boundary controls of the Lagrange type.

We shall make use here of recent results which have been obtained in different directions.

First, it has been shown that drastic reductions, or complete elimination, of seminormality properties can be achieved by requesting simple properties—geometric in character—of the relevant sets, properties which are already commonly required in controllability theory (Cesari, Refs. 1–3).

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Secondly, extensive contributions have been made in the use of seminormality conditions, as properties (K), (Q), and intermediate (Q_ρ), in connection with various closure and convergence properties of the operators in Banach space and modes of convergence (Cesari, Ref. 4; Cesari and Cowles, Ref. 5; Cesari and Kaiser, Ref. 6).

Finally, it has been shown that suitable analytic properties of the relevant functions [Lipschitz-type and growth-type conditions (F), (G), (H)] have relevant consequences on the character of the corresponding Nemitsky's operators, and again lead to closure and lower closure theorems with no seminormality conditions (Cesari and Suryanayarana, Refs. 7 and 8).

Concerning the continuity and convergence properties of the operators under consideration, we shall need here definitions which represent a finer analysis of these properties than in our previous paper (Ref. 3): definitions of closure on a given set, closure of the graph property, and convergence properties in connection with various topologies.

Also we need to emphasize in this paper the interplay of the properties of the operators, the properties of the relevant sets $\tilde{\tilde{Q}}(t, y)$, $\tilde{\tilde{R}}(t, y)$, and the properties of the representative functions f_0, $f = (f_1, ..., f_r)$, g_0, $g = (g_1, ..., g_{r'})$. The stronger the topological properties of the operators, and of the modes of convergence under consideration, the weaker are the needed properties on the sets $\tilde{\tilde{Q}}$, $\tilde{\tilde{R}}$ and on the functions f_0, f, g_0, g.

We state here existence theorems for strong solutions. Analogous theorems for weak solutions will be presented in a forthcoming paper in this same Journal. Many examples are given to illustrate our statements.

2. Lagrange-Type Problems with Distributed and Boundary Controls

Problems of optimal control with distributed and boundary controls often concern the minimum of a given integral expression over an open set G of E^ν, or over a part Γ of the boundary ∂G of G, or of a sum of two integrals, one over G and one over Γ. The latter case is of course more general if we do not exclude that one or the other be zero. Each integral may involve arbitrary measurable functions on G (distributed controls), and on Γ (boundary controls), whose values are assumed to belong to given fixed or variable sets (control spaces). Finally, partial differential equations must be satisfied on G (state equations on G); boundary data or more partial differential equations must be satisfied on Γ (state equations on Γ). Often, Γ is divided into parts on each of which different boundary data or partial differential equations are assigned. Important examples of such problems can be seen, for instance, in Butkovsky's book (Ref. 9). However, it is convenient to formulate the general problem in a slightly more general form.

Indeed, it would be enough to say that the functional to be minimized is the finite sum

$$I = \sum_{j=1}^{N} \int_{G_j} F_j d\mu_j$$

of integrals, each over a measure space $(G_j, \mathscr{A}_j, \mu_j)$, $j = 1,..., N$, on each of which we have a measure μ_j, a collection $\mathscr{A}_j$ of measurable subsets, and certain functional relations $L_j x = N_j[x, u_j]$ to satisfy, each involving certain μ_j-measurable control functions u_j on G_j, while the unknown x is an element of some abstract topological space X.

The present paper, however, is meant to have practical significance and thus we prefer to be more specific. We simply assume that one of the spaces is a domain G of E^ν of points t, with Lebesgue measure dt, and a functional relation $Lx = N[x, u]$ to satisfy, involving a measurable control function u with values in E^m, and that there is only another space, namely, a measure space $(\Gamma, \mathscr{A}, \mu)$ of points τ, with measure μ, and a functional relation $Jx = N'[x, v]$ to satisfy, involving a μ-measurable control function v with values in $E^{m'}$. Actually, in all applications and examples G is a Morrey's type domain, Γ is a part of the boundary of G of some dimension $1 \leqslant \sigma \leqslant \nu - 1$, and μ is simply the σ-measure (area, length, or in general σ-area subsumed by G on Γ). The specific interest arises when X is a Sobolev space $W_p{}^l(G)$ on G, Γ is a part of the boundary of G for which Sobolev's imbedding theorems hold, and then the relative dimensions ν of G, σ of Γ, and the indices l and p may have relevant implications on the properties of the operators L on G and J on Γ. This formulation seems to be rather easy to apply, as we shall see in the examples. Any extension to more than two integrals and spaces would be rather trivial. Instead, we shall give emphasis to the fact that L and J may well represent systems of partial differential equations, or functional relations, or data, and that the operators in the single equations may have far different topological properties. We shall see in this paper that, by a suitable and subtle analysis of their properties, the needed conditions of convexity and seminormality to guarantee the existence of an absolute minimum can be drastically reduced.

Let G be any measurable subset of E^ν with finite Lebesgue measure. Let (Γ, d) be a metric space, and let $(\Gamma, \mathscr{A}, \mu)$ be a finite, complete measure space such that the σ-algebra $\mathscr{A}$ contains the Borel sets of

(Γ, d). Let $(X, \mathfrak{T})$ be any given topological space with S a given subset of X. Let $T = (\mathfrak{m}(G))^m$ denote the collection of all measurable functions $u(t) = (u^1, \ldots, u^m)$, $t \in G$, and let $\mathring{T} = (\mathfrak{m}(\Gamma))^{m'}$ denote the collection of all μ-measurable functions $v(\tau) = (v^1, \ldots, v^{m'})$, $\tau \in \Gamma$. Let L, M, J, K be operators defined on S, not necessarily linear, whose values are L-integrable vector functions on G, and μ-integrable vector functions on Γ, namely $L\colon S \to (L_1(G))^r$, $M\colon S \to (L_1(G))^s$, $J\colon S \to (L_1(\Gamma))^{r'}$, $K\colon S \to (L_1(\Gamma))^{s'}$, where $r, s, r', s' \geqslant 0$ are (nonnegative) integers. Actually, in Theorem 8.1 below, we shall allow some (or even all) of the components of L, M, J, K to have their values in the space $\mathfrak{m}(G)$, or $\mathfrak{m}(\Gamma)$, of all measurable functions on G, or μ-measurable functions on Γ.

As usual a subset $A(t)$ of E^s is assigned for every $t \in G$, and a subset $U(t, y)$ of E^m is assigned for every $(t, y) \in A = [(t, y) \mid t \in G, y \in A(t)]$. We denote by M the set of all (t, y, u) with $(t, y) \in A$, $u \in U(t, y)$, and by $f_0(t, y, u), f(t, y, u) = (f_1, \ldots, f_r)$ given functions defined on $M \subset E^{\nu+r+m}$. Analogously, a subset $B(\tau)$ of $E^{s'}$ is assigned for every $\tau \in \Gamma$, and a subset $V(\tau, \mathring{y})$ of $E^{m'}$ is assigned for every $(\tau, \mathring{y}) \in B = [(\tau, \mathring{y}) \mid \tau \in \Gamma, \mathring{y} \in B(\tau)]$. We denote by $\mathring{M}$ the set of all $(\tau, \mathring{y}, v)$ with $(\tau, \mathring{y}) \in B$, $v \in V(\tau, \mathring{y})$, and by $g_0(\tau, \mathring{y}, v)$, $g(\tau, \mathring{y}, v) = (g_1, \ldots, g_{r'})$ given functions defined on $\mathring{M} \subset \Gamma \times E^{r'+m'}$.

As usual, functions $y(t) = (y^1, \ldots, y^s)$, $t \in G$, $y \in (L_1(G))^s$, $y(t) \in A(t)$, are said to be state functions on G; functions $\mathring{y}(\tau) = (y^1, \ldots, y^{s'})$, $\tau \in \Gamma$, $\mathring{y} \in (L_1(\Gamma))^{s'}$, $\mathring{y}(\tau) \in B(\tau)$, are said to be state functions on Γ. Also, functions $u(t) = (u^1, \ldots, u^m)$, $t \in G$, $u \in T$, $u(t) \in U(t, y(t))$, are said to be control functions on G, and functions $v(\tau) = (v^1, \ldots, v^{m'})$, $\tau \in \Gamma$, $v \in \mathring{T}$, $v(\tau) \in V(\tau, \mathring{y}(\tau))$ are said to be control functions on Γ.

We now consider the problem of finding elements $x \in S$, $u \in T$, $v \in \mathring{T}$ which minimize the functional

$$I[x, u, v] = \int_G f_0(t, Mx(t), u(t))\, dt + \int_\Gamma g_0(\tau, Kx(\tau), v(\tau))\, d\mu, \tag{1}$$

subject to the state equations

$$Lx(t) = f(t, Mx(t), u(t)), \qquad t \in G, \text{ (a.e.)}, \tag{2}$$

$$Jx(\tau) = g(\tau, Kx(\tau), v(\tau)), \qquad \tau \in \Gamma, \text{ (μ-a.e.)}, \tag{3}$$

and the constraints

$$Mx(t) \in A(t), \qquad u(t) \in U(t, Mx(t)), \qquad t \in G, \text{ (a.e.)}, \tag{4}$$

$$Kx(\tau) \in B(\tau), \qquad v(\tau) \in V(\tau, Kx(\tau)), \qquad \tau \in \Gamma, \text{ (μ-a.e.)}. \tag{5}$$

Here, $f(t, y, u) = (f_1, \ldots, f_r)$, $(t, y, u) \in M$, $g(\tau, \mathring{y}, v) = (g_1, \ldots, g_{r'})$, $(\tau, \mathring{y}, v) \in \mathring{M}$, and correspondingly we may write $L = (L_1, \ldots, L_r)$, $J = (J_1, \ldots, J_{r'})$, with $r \geqslant 0$, $r' \geqslant 0$.

In applications, the functions f_0, f, g_0, g are continuous and the sets A, M, B, $\mathring{M}$ are closed. Actually, much less is needed. We say that the sets A, M and the functions f_0, f satisfy a Carathéodory-type condition (C) on G if for every $\epsilon > 0$ there is a compact subset K of G such that $| G - K | \leqslant \epsilon$, the sets $A_K = [(t, y) \in A \mid t \in K]$ and $M_K = [(t, y, u) \in M \mid t \in K]$ are closed, and the functions $f_0(t, y, u)$, $f(t, y, u)$ restricted to K are continuous. An analogous condition (C) on Γ may be defined for B, $\mathring{M}$, g_0, g. This condition (C) on A, M, B, $\mathring{M}$, f_0, f, g_0, g will be assumed throughout this paper.

We seek optimal solutions for problems (1–5), and these will be denoted as usual strong optimal solutions. For weak optimal solutions, state equations (2)–(3) will be written in the corresponding weak form.

3. Topological View: Property (Q) and Variants

As in previous papers (Refs. 4–6, 10), we shall need properties of the sets

$$\tilde{Q}(t, y) = [(z^0, z) \mid z^0 \geqslant f_0(t, y, u),\ z = f(t, y, u),\ u \in U(t, y)] \subset E^{r+1},$$

$$\tilde{R}(\tau, \mathring{y}) = [(z^0, z) \mid z^0 \geqslant g_0(\tau, \mathring{y}, v),\ z = g(\tau, \mathring{y}, v),\ v \in V(\tau, \mathring{y})] \subset E^{r'+1}.$$

The projections of these sets on the z-spaces E^r, or $E^{r'}$, are the sets $Q(t, y) = f(t, y, U(t, y)) \subset E^r$, $R(\tau, \mathring{y}) = g(\tau, \mathring{y}, V(\tau, \mathring{y})) \subset E^{r'}$. We state the definitions below in terms of the sets $\tilde{Q}$. Analogous statements hold for the sets $\tilde{R}$.

The sets $\tilde{Q}(t, y)$ are said to satisfy property (K) at (t_0, y_0) with respect to y provided

$$\tilde{Q}(t_0, y_0) = \bigcap_{\epsilon>0} \operatorname{cl} \bigcup_{y \in N_\epsilon(y_0)} \tilde{Q}(t_0, y),$$

where $N_\epsilon(y_0) = [y \in A(t_0),\ | y - y_0 | \leqslant \epsilon]$. The same sets $\tilde{Q}(t, y)$ are said to satisfy property (Q) at (t_0, y_0) with respect to y provided

$$\tilde{Q}(t_0, y_0) = \bigcap_{\epsilon>0} \operatorname{cl} \operatorname{co} \bigcup_{y \in N_\epsilon(y_0)} \tilde{Q}(t_0, y).$$

Sets satisfying property (K) are closed; sets satisfying property (Q) are closed and convex. Property (Q) implies property (K). [Properties (K) and (Q) were introduced by Kuratovsky (Ref. 11) and Cesari (Ref. 12). See Refs. 12–13 for a discussion of these properties and simple criteria for property (Q).]

There are intermediate properties (Q_ρ), $0 \leqslant \rho \leqslant r+1$, between properties (K) and (Q) above. To present them, let us decompose the $\tilde{z}$-space E^{r+1}, $\tilde{z} = (z^0, z) = (z^0, z^1, ..., z^r)$ into the product $E^\rho \times E^{r+1-\rho}$ of the $z^{(1)}$-space E^ρ, or $E^{(1)}$, $z^{(1)} = (z^0, z^1, ..., z^{\rho-1})$, and the $z^{(2)}$-space $E^{r+1-\rho}$, or $E^{(2)}$, $z^{(2)} = (z^\rho, ..., z^r)$. We say that the sets $\tilde{\tilde{Q}}(t, y)$ satisfy property (Q_ρ) at (t_0, y_0) with respect to y provided, for every $z_0^{(2)} = (z_0^\rho, ..., z_0^r) \in E^{r+1-\rho}$, we have

$$\tilde{\tilde{Q}}(t_0, y_0) \cap [E^\rho \times \{z_0^2\}]$$
$$= \bigcap_{\epsilon>0} \bigcap_{\delta>0} \operatorname{cl} \operatorname{co} \left\{ \bigcup_{y \in N(y_0)} \tilde{\tilde{Q}}(t_0, y) \cap [E^\rho \times \{z^2\} \mid |z^2 - z_0^2| \leqslant \delta \right\}.$$

For $\rho = r+1$, we understand that the sets in brackets in the first and second members of this relation coincide with E^{r+1}, and property (Q_{r+1}) coincides with property (Q). For $\rho = 0$, property (Q_0) coincides with property (K). Owing to the special structure of the sets $\tilde{\tilde{Q}}(t, y)$ we consider here, if the sets $\tilde{\tilde{Q}}(t, y)$ have property (K) or (Q_0), they also have property (Q_1). Finally, for any integer ρ, $0 \leqslant \rho \leqslant r$, property $(Q_{\rho+1})$ implies property (Q_ρ). (See Refs. 14–15 for proofs and details.)

This intermediate property (Q_ρ), $0 \leqslant \rho \leqslant r+1$, can be actually expressed as a property (Q) of suitably defined auxiliary sets (see Ref. 15). Indeed, if we take

$$Q^*(t, y, z^{(2)}) = \tilde{\tilde{Q}}(t, y) \cap (E^\rho \times \{z^2\}),$$

then the sets $\tilde{\tilde{Q}}(t, y)$ have property (Q_ρ) with respect to y at (t_0, y_0) iff the sets $Q^*(t, y, z^{(2)})$ have property (Q) with respect to $(y, z^{(2)})$ at $(y_0, z_0^{(2)})$ for every $z_0^{(2)} \in E^{r+1-\rho}$.

To simplify the presentation, we have decomposed here E^{r+1} into E^ρ and $E^{r+1-\rho}$ by using the coordinates $z^0, z^1, ..., z^{\rho-1}$ in E^ρ and the coordinates $z^\rho, ..., z^r$ in $E^{r+1-\rho}$. Obviously, we could have used any two complementary systems of ρ and $r+1-\rho$ of the $R+1$ coordinates $z^0, z^1, ..., z^{r+1}$. We shall denote them as the first and second system of ρ and $r+1-\rho$ coordinates $z^1, ..., z^r$, respectively.

Below, we shall need properties (K), or (Q), or (Q_ρ) to hold globally; by this we mean that there is some subset T_0 of G of measure zero (possibly empty) such that the sets $\tilde{\tilde{Q}}(t, y)$ have property (K), or (Q), or (Q_ρ) with respect to y at every $(t_0, y_0) \in A$, $t_0 \in G - T_0$.

4. Geometric View: Property (P) and Variants

For the sake of simplicity, we present property (P) for the sets $\tilde{Q}(t, y)$ as a global property (Ref. 1). For every $N \geqslant 0$, we shall denote by $V(0, N)$ the closed ball of center the origin in E^r (or $E^{r'}$) and radius N. For every $(t, y) \in A$ and $z \in Q(t, y)$, we denote by $T(z; t, y)$ the scalar function $T(z; t, y) = \operatorname{Inf}[z^0 \mid (z^0, z) \in \tilde{Q}(t, y)]$. We say that the sets $\tilde{Q}(t, y)$ satisfy property (P) with respect to y provided: (P1) there is a measurable bounded function $p(t)$, $t \in G$, p: $G \to E^r$, say $|p(t)| \leqslant \sigma$, and a constant $c \geqslant 0$ such that $p(t) \in Q(t, y)$ for all $(t, y) \in A$, $t \in G - T_0$; $|T(z; t, y)| \leqslant c$ for all $(t, y) \in A$, $t \in G - T_0$, $z \in Q(t, y) \cap V(0, 2\sigma)$; (P2) for every $N > 0$, the sets $\tilde{Q}(t, y) \cap (E^1 \times V(0, N))$ have property (Q) with respect to y at every $(t_0, y_0) \in A$, $t_0 \in G - T_0$ [see Cesari, Ref. 1, for details and proofs on this property (P)].

First, a remark is relevant here. One is that (P2) implies that the sets $\tilde{Q}(t, y) \cap [E^1 \times V(0, N)]$, $N \geqslant \sigma$, are closed and convex and hence the sets $\tilde{Q}(t, y)$ themselves are closed and convex. Conversely, if we assume that the sets $\tilde{Q}(t, y)$ are closed and convex, then the sets $\tilde{Q}(t, y) \cap [E^1 \times V(0, N)]$, $N \geqslant \sigma$, are also closed and convex, and because of their special structure, property (K) of the sets $\tilde{Q}(t, y) \cap [E^1 \times V(0, N)]$ implies property (Q) of the same sets. (This was proved in Ref. 15.) Thus, if we assume that the sets $\tilde{Q}(t, y)$ are closed and convex, it suffices to require property (K) in (P2) above.

Note that property $p(t) \in Q(t, y)$ is certainly satisfied if all sets $Q(t, y)$ contain a fixed point $\bar{z} \in E^r$, say $\bar{z} = 0$, the latter case being rather common in applications. Note that the condition $|T(z; t, y)| \leqslant c$ is certainly satisfied if $f_0(t, y, u) \geqslant -c$ for all $(t, y, u) \in M$, and $f_0(t, y, u) \leqslant c$ for all $(t, y, u) \in M$ with $|f(t, y, u)| \leqslant 2\sigma$. This condition is also very mild and usually satisfied in application.

Finally, if we know that

(*) for every t_0 fixed, $t_0 \in G - T_0$, we have $|f(t_0, y, u)| \to +\infty$ as $|u| \to \infty$, $u \in U(t, y)$, uniformly on every compact subset of $A(t_0)$,

then the sets $\tilde{Q}(t, y) \cap [E^1 \times V(0, N)]$ certainly have property (K) with respect to y at every $(t_0, y_0) \in A$, $t_0 \in G - T_0$. This is a mere consequence of condition (C) as it has been proved in Ref. 1. Thus, all we have to verify for (P2) is that the sets $\tilde{Q}(t, y)$ are closed and convex, because they will have property (K) [as a consequence of property (C)], and property (Q) (as a consequence of their special structure), as stated above.

We see that property (P) as explained above, is a very mild one, similar to the one required in Filippov's existence theorem for equibounded controls (Ref. 16).

We come now to variants of property (P). We say that the sets $\tilde{Q}(t, y)$ satisfy property (P_0) with respect to y if (P1) holds as above and (P2) is replaced by the analogous condition (P_02) with property (K) instead of property (Q). Thus, if (P_0) holds, all sets $\tilde{Q}(t, y) \cap [E^1 \times V(0, N)]$, $N \geqslant \sigma$, are closed, and the sets $\tilde{Q}(t, y)$ themselves are closed, but not necessarily convex $((t, y) \in A,\ t \in G - T_0)$. Conversely, as above, under conditions (C) and (*), all we have to verify for (P_02) is that the sets $\tilde{Q}(t, y)$ are closed (or the sets $\tilde{Q}(t, y) \cap [E^1 \times V(0, N)]$, $N \geqslant \sigma$, are closed), $((t, y) \in A,\ t \in G - T_0)$.

Finally, we say that the sets $\tilde{Q}(t, y)$ in E^{r+1} satisfy the intermediate property (P_ρ), $0 \leqslant \rho \leqslant r + 1$, provided (P1) holds as above and (P2) is replaced by the analogous condition $(P_\rho 2)$, with property (Q_ρ) instead of property (Q). Note that property (P_{r+1}) is equivalent to property (P), that property (P_0) corresponds to property (K), and that, for every ρ, $0 \leqslant \rho \leqslant r$, property $(P_{\rho+1})$ implies property (P_ρ).

Note that, if (P_ρ) holds, then all sets $\tilde{Q}(t, y) \cap [E^1 \times V(0, N)]$, $N \geqslant \sigma$, are closed, and for every $z_0^{(2)} = (z_1^\rho, \ldots, z_0^r)$ the sets $\tilde{Q}(t, y) \cap [E^1 \times V(0, N)] \cap [E^\rho \times \{z^{(2)}\}]$ are convex, and consequently the sets $\tilde{Q}(t, y)$ themselves are closed, and for every $z_0^{(2)}$ the sets $\tilde{Q}(t, y) \cap [E^\rho \times \{z_0^{(2)}\}]$ are convex $((t, y) \in A,\ t \in G - T_0)$. Conversely, as above, under conditions (C) and (*), all we have to verify for $(P_\rho 2)$ is that the sets $\tilde{Q}(t, y)$ are closed and that, for every $z_0^{(2)}$, the sets $\tilde{Q}(t, y) \cap [E^\rho \times \{z_0^{(2)}\}]$ are convex (or the sets $\tilde{Q}(t, y) \cap [E^1 \times V(0, N)]$, $N \geqslant \sigma$, are closed and the sets $\tilde{Q}(t, y) \cap [E^1 \times V(0, N)] \times [E^\rho \times \{z_0^{(2)}\}]$ are convex), $((t, y) \in A$, $t \in G - T_0)$.

The following new variants of properties (P) and (P_ρ) (Ref. 17) are useful. The variants concern property (P1) and will be expressed in terms of sequences of functions $y_k(t)$, $t \in G$, $k = 1, 2, \ldots$ (In applications, this will be any minimizing sequence of state functions.)

We say that the sets $\tilde{Q}(t, y)$ satisfy property (P′1) provided, for any sequence $y_k(t)$, $t \in G$, $k = 1, 2, \ldots$, of measurable functions (or at least for any minimizing sequence of state functions), $y_k(t) \in A(t)$, $y_k(t) \to y(t)$ in measure in G as $k \to \infty$, there are functions $\mu(t)$, $\mu_k(t)$, $p(t)$, $p_k(t)$, $t \in G$, $k = 1, 2, \ldots$, μ, $\mu_k \in L_1(G)$, p, $p_k \in (L_1(G))^r$, such that

$$(\mu_k(t), p_k(t)) \in \tilde{Q}(t, y_k(t)), \qquad t \in G, \text{ (a.e.)}, \qquad k = 1, 2, \ldots,$$

$$\mu_k \to \mu \qquad \text{weakly in } L_1(G),$$

$$p_k \to p \qquad \text{strongly in } (L_1(G))^r \text{ as } k \to \infty.$$

Actually, it is enough that this occurs for sequences y_k with $y_k \to y$, say in $L_1(G)$, or in $L_p(G)$, as for minimizing sequences.

We shall say that the sets $\tilde{Q}(t, y)$ satisfy condition (P′) [or (P_ρ′)], provided the same sets satisfy condition (P′1) and (P2) [or (P′1) and (P_ρ2)], with N replaced by $\max(N, \bar{p}(t))$, where $\bar{p} \in L_1(G)$ and $| p_k(t)| \leqslant \bar{p}(t)$ for all k.

We conclude this section with the remark that the natural geometrical properties required under (P1) allow a drastic reduction of the *seminormality* requirements (Q) or (Q_ρ). Under conditions (C) and (*), no seminormality requirement is needed: only closedness and suitable convexity properties.

5. Analytic View: Properties (F), (G), (H)

In Refs. 7–8, Cesari and Suryanarayana took into consideration a great many general analytic conditions on the functions f_0, f, g_0, g, which are easy to verify and which dispense with properties (Q) or (P) or their variants. [Actually, we proved in Ref. 8 that these analytic conditions do imply weak forms of property (Q).]

We describe these properties in terms of the function f, but the same statements hold for f_0, g_0, g. In any case, we assume that f_0, f, g_0, g satisfy Carathéodory condition (C), and that $U(t)$, $V(\tau)$ depend on t, τ only (not on y, or $\mathring{y}$). Here are some of the properties of interest. Below, y_k, u_k denote state and control functions on G.

(F_p) For $1 \leqslant p < \infty$, and $y_k, y \in (L_p(G))^s$, $\| y_k - y \|_p \to 0$ as $k \to \infty$, $u_k \in T$, we have

$$\begin{aligned} &| f(t, y_k(t), u_k(t)) - f(t, y(t), u_k(t))| \\ &\qquad \leqslant F(t, u_k(t))\, h(| y_k(t) - y(t)|), \qquad k = 1, 2,..., \qquad t \in G, \end{aligned}$$

where $h(\zeta) \geqslant 0$, $0 \leqslant \zeta < \infty$, is a given monotone nondecreasing function with $h(0+) = 0$, $h(\zeta) \leqslant c\zeta$ for all $\zeta \geqslant \zeta_0 \geqslant 0$, and $F(t, u_k(t)) \in L_{p'}(G)$, $1/p' + 1/p = 1$, $\| F(t, u_k(t)\|_{p'} \leqslant C$, where ζ_0, c, C are constants and $F(t, u)$ a given nonnegative function on $G \times E^m$.

(F_∞) For $y, y_k \in (L_\infty(G))^s$, $\| y_k - y \|_\infty \to 0$ as $k \to \infty$, $u_k \in T$, we have

$$\begin{aligned} &| f(t, y_k(t), u_k(t)) - f(t, y(t), u_k(t))| \\ &\qquad \leqslant F_k(t)\, h(| y_k(t) - y(t)|), \qquad k = 1, 2,..., \qquad t \in G, \end{aligned}$$

where $h(\zeta) \geqslant 0$, $0 \leqslant \zeta < \infty$, is a given monotone nondecreasing function with $h(0+) = 0$, and $F_k(t) \geqslant 0$, $t \in G$, $F_k \in L_1(G)$, are given functions with $\| F_k \|_1 \leqslant C$, a given constant.

(G_{pq}) For $1 \leqslant p$, $q < \infty$, $y_k, y \in (L_q(G))^s$, $u_k \in (L_q(G))^m$, $\| y \|_p$, $\| y_k \|_p \leqslant L_0$, $\| u_k \|_q \leqslant L, L_0$, L given constants, $y_k(t) \to y(t)$

pointwise a.e. in G as $k \to \infty$, and there are constants c, c', α, β, $0 < \alpha \leqslant p$, $0 < \beta \leqslant q$, and a function $\psi(t) \geqslant 0$, $t \in G$, $\psi \in L_1(G)$, such that, for all (t, y_1, u), $(t, y_2, u) \in M$, we have

$$|f(t, y_1, u) - f(t, y_2, u)| \leqslant \psi(t) + c(|y_1|^{p-\alpha} + |y_2|^{p-\alpha}) + c' |u|^{q-\beta}.$$

($\mathrm{G}_{\infty q}$) For $1 \leqslant q < \infty$, y, $y_k \in (L_\infty(G))^s$, $u_k \in (L_q(G))^m$, $\|y\|_\infty$, $\|y_k\|_\infty \leqslant L_0$, $\|u_k\|_q \leqslant L$, L_0, L given constants, $y_k(t) \to y(t)$ pointwise a.e. in G as $k \to \infty$, and there are constants c', β, $0 < \beta \leqslant q$, a function $\psi(t) \geqslant 0$, $t \in G$, $\psi \in L_1(G)$, and a monotone nondecreasing function $\sigma(\zeta) \geqslant 0$, $0 \leqslant \zeta < \infty$, such that for all (t, y_1, u), $(t, y_2, u) \in M$, we have

$$|f(t, y_1, u) - f(t, y_2, u)| \leqslant \psi(t)\,(\sigma(|y_1|) + \sigma(|y_2|)) + c' |u|^{q-\beta}.$$

(H_q) For $1 \leqslant q < \infty$, y, y_k measurable, $y_k(t) \to y(t)$ in measure in G as $k \to \infty$, $u_k \in (L_q(G))^m$, $\|u_k\|_q \leqslant L$, a constant, there are constants c', β, $0 < \beta \leqslant q$, and a function $\psi(t) \geqslant 0$, $t \in G$, $\psi \in L_1(G)$, such that for all (t, y_1, u), $(t, y_2, u) \in M$, we have

$$|f(t, y_1, u) - f(t, y_2, u)| \leqslant \psi(t) + c' |u|^{q-\beta}.$$

(H_∞) For y, y_k measurable, $y_k(t) \to y(t)$ in measure in G as $k \to \infty$, $u_k \in (L_\infty(G))^m$, $\|u_k\|_\infty \leqslant L$, there are a function $\psi(t) \geqslant 0$, $t \in G$, $\psi \in L_1(G)$, and a monotone nondecreasing function $\sigma(\zeta) \geqslant 0$, $0 \leqslant \zeta < \infty$, such that for all $(t, y, u) \in M$, we have

$$|f(t, y, u)| \leqslant \psi(t)\,\sigma(|u|).$$

Whenever u_k is bounded in norm, as for instance under hypotheses (G) and (H) above, the requirements on f_0 may be relaxed (see Ref. 7).

6. Growth Conditions on f_0 and g_0

The following growth conditions on $f_0(t, y, u)$, or $g_0(\tau, \mathring{y}, v)$, are relevant in our existence theorems below. The growth conditions are stated here in terms of $f_0(t, y, u)$ and G only; analogous statements hold for $g_0(\tau, \mathring{y}, v)$ and Γ.

We say that f_0 satisfies condition (α) on G provided there is some function $\psi(t) \geqslant 0$, $t \in G$, $\psi \in L_1(G)$, so that $f_0(t, y, u) \geqslant -\psi(t)$ for all $(t, y, u) \in M$.

We say that $f_0(t, y, u)$ and another function $h(t, y, u)$ satisfy condition (ψ_ϵ) on G if, for any given $\epsilon > 0$ there is some function $\psi_\epsilon(t) \geqslant 0$, $t \in G$,

$\psi_\epsilon \in L_1(G)$, so that $|h(t, y, u)| \leqslant \psi_\epsilon(t) + \epsilon f_0(t, y, u)$ for all $(t, y, u) \in M$.

If f_0, h satisfy (ψ_ϵ), then, by taking $\epsilon = 1$, we see that f_0 also satisfies (α) with $\psi(t) = \psi_1(t)$.

We say that $f_0(t, y, u)$ and a function $h(t, y, u)$ satisfy growth condition (β) on G if there is some function $\psi(t) \geqslant 0$, $t \in G$, $\psi \in L_1(G)$, and a constant $\gamma \geqslant 0$ so that $f_0(t, y, u) \geqslant -\psi(t) - \gamma |h(t, y, u)|$ for all $(t, y, u) \in M$.

We say that $f_0(t, y, u)$ and a function $h(t, y, u)$ satisfy condition (γ) on G if there is some function $\psi(t) \geqslant 0$, $t \in G$, $\psi \in L_1(G)$, and constants $\gamma \geqslant 0$, $\gamma' \geqslant 0$, $p \geqslant 1$, so that $f_0(t, y, u) \geqslant -\psi(t) - \gamma |h(t, y, u)| - \gamma' |y|^p$ for all $(t, y, u) \in M$.

7. Properties of Operators

Let $\mathscr{A}$ be an operator, not necessarily linear, from a subset S of a topological space $(X, \mathscr{T})$ into a topological space $Y = (Y, \mathscr{Z})$. To denote that Y is not the empty space we say that $\mathscr{A}$ is not vacuous.

We say that the operator $\mathscr{A}: S \to (Y, \mathscr{Z})$, $S \subset (X, \mathscr{T})$ is *closed* on S provided $x_k \in S$, $k = 1, 2,\ldots$, $x \in S$, $x_k \to x$ in $(X, \mathscr{T})$, $\mathscr{A}x_k \to y$ in $(Y, \mathscr{Z})$, implies $y = \mathscr{A}x$. We say that $\mathscr{A}: S \to (Y, \mathscr{Z})$, $S \subset (X, \mathscr{T})$ has the *closed graph* property provided $x_k \in S$, $k = 1, 2,\ldots$, $x \in X$, $x_k \to x$ in $(X, \mathscr{T})$, $\mathscr{A}x_k \to y$ in $(Y, \mathscr{Z})$ implies $x \in S$ and $y = \mathscr{A}x$.

We say that the operator $\mathscr{A}: S \to (Y, \mathscr{Z})$, $S \subset (X, \mathscr{T})$, has the *convergence* property [with respect to S, $(X, \mathscr{T})$, $(Y, \mathscr{Z})$] provided, if $x_k \in S$, $k = 1, 2,\ldots$, $x \in X$, $x_k \to x$ in $(X, \mathscr{T})$, then the sequence $\mathscr{A}x_k$, $k = 1, 2,\ldots$, has a convergent subsequence in $(Y, \mathscr{Z})$; that is, there is some $y \in Y$ and a $[k_s]$ such that $\mathscr{A}x_{k_s} \to y$ as $s \to \infty$ in $(Y, \mathscr{Z})$.

Below, whenever Y is a normed space, we denote by strong (weak) closure in S, closedness of the graph, and convergence properties, the properties above relative to S, the space $(X, \mathscr{T})$, and the strong (weak) topology in Y. Whenever Y is a space of measurable functions we shall understand the properties with respect to convergence in measure in Y.

If $S = X$, that is, if $\mathscr{A}$ is defined on all of X, then *closure in* S and *closed graph property* coincide. If $(X, \mathscr{T})$ is a topological vector space, if S is a linear subspace of X, if $\mathscr{A}$ is linear, and $(Y, \mathscr{Z})$ is a normed space, then closure graph properties with respect to the strong and the weak topologies in Y are identical, by force of Banach–Saks–Mazur theorem (see Ref. 18, p. 120).

In most applications $(X, \mathscr{T})$ is a Sobolev space $W_p{}^m(G)$, $m \geqslant 1$, $p \geqslant 1$, with the weak topology $\mathscr{T}$, while Y is a product of spaces $L_p(G)$,

$L_p(\Gamma)$, $p \geqslant 1$, each with either the weak or the strong topology, and of spaces $\mathfrak{m}(G)$, $\mathfrak{m}(\Gamma)$ of measurable functions, a.e. finite, with the *convergence in measure* topology. In most cases $S = X$, that is, the operators are defined on all of X, but S may well be a proper subset of X. We shall see examples of all these cases.

For general properties of operators, we refer here to Dunford and Schwartz (Ref. 19), and also we mention the short presentation in Section 2 of Burns' paper (Ref. 20).

8. Existence Theorems for Lagrange Problems with State Equations in the Strong Form

Existence theorems concerning only G have been stated in Part 1, Section 3.2 (Ref. 3), and the same theorems hold for Γ alone as well. Because of Sobolev's embedding theorems, which relate properties and behavior at the boundary of G to the properties and behavior in G of (Sobolev) functions in G, it is of practical interest to formulate the existence theorems below involving both G and Γ, as the examples will show. These theorems differ from those in Ref. 5 because of the extremely more general conditions on the relevant operators, the functions f and g, and the relevant sets.

We shall now make full use of the notations of Section 2.

As usual, we say that a triple (x, u, v), $x \in S \subset (X, \mathfrak{T})$, $u \in T$, $v \in \mathring{T}$ is admissible for problem (1–5), if relations (2–5) are satisfied, and $f_0(\cdot, Mx(\cdot), u(\cdot)) \in L_1(G)$, $g_0(\cdot, Kx(\cdot), u(\cdot)) \in L_1(\Gamma)$. For the concept of a closed class Ω of admissible triples (x, u, v) we refer to Ref. 5. We mention here that, given any class Ω of such triples, then $\{x\}_\Omega$ denotes the set of all $x \in S$ such that $(x, u, v) \in \Omega$ for some $u \in T$, $v \in \mathring{T}$.

We shall consider below a nonempty class Ω of admissible triples (x, u, v), and we shall denote by Ω_0 the nonempty subset of only those $(x, u, v) \in \Omega$ with $I[x, u, v] \leqslant M_0$ for some constant M_0. We may denote by A_0, A, $\mathring{A}$ the sets

$$A_0 = \{x_0\}_{\Omega_0} = \{x \in X \mid (x, u, v) \in \Omega_0\},$$

$$A = \{(x, u)\}_{\Omega_0} = \{(x, u) \in X \times T \mid (x, u, v) \in \Omega_0\},$$

$$\mathring{A} = \{(x, v)\}_{\Omega_0} = \{(x, v) \in X \times \mathring{T} \mid (x, u, v) \in \Omega_0\}.$$

We denote here by S a given subset of the topological space $(X, \mathfrak{T})$, and we consider operators $L\colon S \to (L_1(G))^r$, $J\colon S \to (L_1(\Gamma))^{r'}$ with the weak topology in the range spaces, and $M\colon S \to (m(G))^s$, $K\colon S \to (m(\Gamma))^{s'}$ with the convergence in measure topology in the range spaces.

The vector functions $f(t, y, u)$, $g(\tau, \mathring{y}, v)$ actually define Nemitsky's operators

$$\begin{aligned} F[x, u](t) &= f(t, Mx(t), u(t)), && t \in G, \\ G[x, v](\tau) &= g(\tau, Kx(\tau), v(\tau)), && \tau \in \Gamma. \end{aligned}$$

Theorem 8.1. *An Existence Theorem for Optimal Strong Solutions.* Let us assume that A, M, f_0, f satisfy condition (C) on G, and that B, $\mathring{M}$, g_0, g satisfy condition (C) on Γ. Let us assume that f_0, f satisfy property (β) on G, and g_0, g satisfy (β) on Γ. Let us assume that the sets $\tilde{Q}(t, y)$ in E^{r+1} satisfy property (Q) with respect to y only at all $(t, y) \in A$, $t \in G - T_0$, $| T_0 | = 0$, and that the sets $\tilde{R}(t, \mathring{y})$ in $E^{r'+1}$ satisfy property (Q) with respect to $\mathring{y}$ only at all $(\tau, \mathring{y}) \in B$, $\tau \in \Gamma - \mathring{T}_0$, $\mu(\mathring{T}_0) = 0$. Let us assume that all operators L, J, M, K have the closure property on S, and that at least one is not vacuous and has the closed graph property on S. Also, let us assume (w_1) that L and J have the weak convergence property, and that (w_2) M and K have the convergence in measure property. Let Ω and Ω_0 as above, and let us assume that $\Lambda_0 = \{x\}_{\Omega_0}$ is relatively sequentially compact as a subset of $(X, \mathscr{C})$. Then the functional $I[x, u, v]$ has an absolute minimum in Ω.

Property (Q) above can be replaced by either property (P) or (P').

If we know that the images $F(\Lambda)$ of Λ and $G(\mathring{\Lambda})$ of $\mathring{\Lambda}$ are relatively sequentially compact, then requirement (w_1) can be omitted.

This statement is a particular case of the following one (Theorem 8.2) below.

To avoid repetitions, we assume that the r components of L and corresponding components of f in (2) are associated into four groups of, say $r_1, r_2, r_3, r_4 \geqslant 0$ components, $r_1 + r_2 + r_3 + r_4 = r$, so that $L = (L^{(1)}, L^{(2)}, L^{(3)}, L^{(4)}, f(t, y, u) = (f^{(1)}, f^{(2)}, f^{(3)}, f^{(4)}), L^{(i)}: S \to (L_1(G))^{r_i}$, $r = 1, 2, 3, 4$. Of course, any of the r_i may be zero, and the corresponding group of equations is missing. Analogously, we assume that the r' components of J and corresponding components of g in (3) are associated into four groups of, say $r_1', r_2', r_3', r_4' \geqslant 0$ components, $r_1' + r_2' + r_3' + r_4' = r'$, so that $J = (J^{(1)}, J^{(2)}, J^{(3)}, J^{(4)})$, $g(\tau, \mathring{y}, v) = (g^{(1)}, g^{(2)}, g^{(3)}, g^{(4)})$, $J_i : S \to (L_1(\Gamma))^{r_i'}$, $i = 1, 2, 3, 4$. As above, we shall denote, for short, by T_0 any subset of G of measure zero (or any subset of Γ of μ-measure zero), which may be empty.

The vector functions $f^{(i)}(t, y, u)$, $g^{(i)}(\tau, \mathring{y}, v)$, $i = 1, 2, 3, 4$, actually define Nemitsky operators $F^{(i)}$, $G^{(i)}$,

$$\begin{aligned} F^{(i)}[x, u](t) &= f^{(i)}(t, Mx(t), u(t)), && t \in G, \\ G^{(i)}[x, v](\tau) &= g^{(i)}(\tau, Kx(\tau), v(\tau)), && \tau \in \Gamma, \end{aligned}$$
$$F^{(i)}: \Lambda \to (\mathfrak{m}(G))^{r_i}, \qquad G^{(i)}: \mathring{\Lambda} \to (\mathfrak{m}(\Gamma))^{r_i'}, \qquad i = 1, 2, 3, 4.$$

Theorem 8.2. *An Existence Theorem for Optimal Strong Solutions.* Let us assume that A, M, f_0, f satisfy condition (C) on G, and that B, $\mathring{M}$, g_0, g satisfy condition (C) on Γ. Let us write for the operators L, J not necessarily linear and defined on some subset S of $(X, \mathfrak{T})$ the general decomposition $L = (L^{(1)}, L^{(2)}, L^{(3)}, L^{(4)})$, $J = (J^{(1)}, J^{(2)}, J^{(3)}, J^{(4)})$ above, $r_1 + r_2 + r_3 + r_4 = r \geqslant 0$, $r_1' + r_2' + r_3' + r_4' = r' \geqslant 0$, with the convention that any of the numbers r_i, r, r_i', r' may be zero. Let us assume that the functions f_0, $(f^{(1)}, f^{(2)})$ satisfy property (β) on G, and that the functions g_0, $(g^{(1)}, g^{(2)})$ satisfy (β) on Γ. Let us assume that the sets $\tilde{Q}(t, y)$ in E^{r+1} satisfy property $Q_{r_1+r_{2+1}}$ with respect to y only at all $(t, y) \in H$, $t \in G - T_0$, $|T_0| = 0$; and that the sets $\tilde{R}(\tau, \mathring{y})$ in $E^{r'+1}$ satisfy property $Q_{r_1'+r_{2+1}'}$ with respect to $\mathring{y}$ only at all $(\tau, \mathring{y}) \in B$, $\tau \in \Gamma - \mathring{T}_0$, $\mu(\mathring{T}_0) = 0$. Let us assume that $f^{(3)}(t, y)$, $g^{(3)}(\tau, \mathring{y})$ are independent of the controls u, v respectively. Let S be a given subset of the topological space $(X, \mathfrak{T})$. We consider operators $L^{(i)}$: $S \to (L_1(G))^{r_i}$, $J^{(i)}$: $S \to (L_1(G))^{r_i'}$, $i = 1, 2$, with the weak topology on the range spaces, operators $L^{(i)}$: $S \to (\mathfrak{m}(G))^{r^i}$, $J^{(i)}$: $S \to (\mathfrak{m}(\Gamma))^{r_i'}$, $i = 3, 4$, M: $S \to (\mathfrak{m}(G))^s$, K: $S \to (\mathfrak{m}(\Gamma))^{s'}$, with the convergence in measure topology on the range spaces. We assume that all these operators $L^{(1)}, L^{(2)}, L^{(3)}, L^{(4)}, J^{(1)}, J^{(2)}, J^{(3)}, J^{(4)}, M, K$ have the closure in S property, that at least one of them is nonvacuous and has the closed graph property, and that $L^{(2)}$, $J^{(2)}$, $L^{(4)}$, $J^{(4)}$, M, K have the convergence property. Let Ω be a nonempty closed class of admissible triples (x, u, v), let Ω_0 be the nonempty subset of only those $(x, u, v) \in \Omega$ with $I[x, u, v] \leqslant M_0$ for some constant M_0, and assume that A_0 is relatively sequentially compact as a subset of $(X, \mathfrak{T})$, and that the images $F^{(1)}(A)$ of A and $G^{(1)}(\mathring{A})$ of $\mathring{A}$ are relatively sequentially compact subsets of $(L_1(G))^{r_1}$ and $(L_1(\Gamma))^{r_1'}$, respectively. Then the functional $I[x, u, v]$ has an absolute minimum in Ω.

Alternate Assumptions. Property $(Q_{r_1+r_2+1})$ above can be replaced by either property $(P_{r_1+r_2+1})$, or $(P_{r_1'+r_2+1})$. The same holds for property $(Q_{r_1'+r_2'+1})$.

If the sets $U(t)$ depend on t only, and one of the conditions (F), or (G), or (H) holds for f_0, f, then conditions (Q_ρ), or (P_ρ) above need not be verified for the sets $\tilde{Q}(t, y)$.

If f_0 and only some of the components f_i, say for $i \in \{j\}$, a subset of $[1,..., r]$, satisfy conditions (F), or (G), or (H) (with requirements relative to the whole vectors y, y_k satisfied), then conditions (Q_ρ), (P_ρ) above can be reduced by requiring only the corresponding properties $(Q_{\rho'})$, $(P_{\rho'})$, $\rho' \leqslant \rho$, obtained by including the components z^i, $i \in \{j\}$, in the second class.

The same remarks above hold for the sets $V(\tau)$ and functions g_0, g, or g_0 and some of the components of g.

Proof. Let i be the infimum of $I[x, u, v]$ in the class Ω, and hence also in the class Ω_0, $-\infty \leqslant i \leqslant M_0 < +\infty$. We write

$$I[x, u, v] = I_1 + I_2, \qquad I_1 = \int_G f_0\, dt, \qquad I_2 = \int_\Gamma g_0\, d\mu.$$

Let (x_k, u_k, v_k), $k = 1, 2,\ldots$, be a sequence of elements in Ω_0 with $I_k = I[x_k, u_k, v_k] \to i$ as $k \to \infty$. Let I_{1k}, I_{2k} be the values of I_1, I_2 computed on the elements (x_k, u_k, v_k). Since $x_k \in \Lambda_0 \subset S$, $k = 1, 2,\ldots$, and Λ_0 is relatively sequentially compact [as a subset of $(X, \mathfrak{T})$], there is a subsequence, say still $[k]$, and an element $x \in X$, such that $x_k \to x$ in $(X, \mathfrak{T})$. Let z_{ik}, $\mathring{z}_{ik}$ denote the functions

$$\begin{aligned} z_{ik}(t) &= f^{(i)}(t, y_k(t), u_k(t)) = L^{(i)}x_k(t), \qquad t \in G, \text{ (a.e.)},\\ \mathring{z}_{ik}(\tau) &= g^{(i)}(\tau, \mathring{y}_k(\tau), v_k(\tau)) = J^{(i)}x_k(\tau), \qquad \tau \in \Gamma, \text{ (μ–a.e.)},\\ &i = 1, 2, 3, 4, \qquad k = 1, 2,\ldots, \end{aligned}$$

where $y_k(t) = Mx_k(t)$, $t \in G$, and $\mathring{y}_k(\tau) = Kx_k(\tau)$, $\tau \in \Gamma$.

By hypothesis, the functions $z_{1k}(t)$, $t \in G$, $k = 1, 2,\ldots$, are in $(L_1(G))^{r_1}$, this sequence has a weakly convergent subsequence, say still $[k]$, and thus there is some $z_1 \in (L_1(G))^{r_1}$ such that $z_{1k} \to z_1$ weakly in $(L_1(G))^{r_1}$. Analogously, there is a further subsequence, say still $[k]$, and some $\mathring{z}_1 \in (L_1(\Gamma))^{r_1'}$ such that $\mathring{z}_{1k} \to \mathring{z}_1$ weakly in $(L_1(\Gamma)^{r_1'}$. Thus, $L^{(1)}x_k = z_{1k} \to z_1$, $J^{(1)}x_k = \mathring{z}_{1k} \to \mathring{z}_1$ as $k \to \infty$.

Since $L^{(2)}$, $J^{(2)}$ have the weak convergence property, there is a further subsequence, say still $[k]$, such that $z_{2k} \to z_2$, weakly in $(L_1(G))^{r_2}$, and $\mathring{z}_{2k} \to \mathring{z}_2$ weakly in $(L_1(\Gamma)^{r_2'}$. By property (β) there are constants γ, $\gamma' \geqslant 0$ such that

$$\begin{aligned} \eta_k(t) &= f_0(t, y_k(t), u_k(t)) \geqslant -\psi(t) - \gamma(|z_{1k}(t)| + |z_{2k}(t)|), \qquad t \in G,\\ \mathring{\eta}_k(\tau) &= g_0(\tau, \mathring{y}_k(\tau), v_k(\tau)) \geqslant -\mathring{\psi}(\tau) - \gamma'(|\mathring{z}_{1k}(\tau)| + |\mathring{z}_{2k}(\tau)|), \qquad \tau \in \Gamma. \end{aligned}$$

The second members have bounded norms. Thus, both I_{1k} and I_{2k}, $k = 1, 2,\ldots$, are bounded below, hence $I_k = I_{1k} + I_{2k}$ is bounded below, and $i = \lim_k I_k$ is finite, $-\infty < i \leqslant I_k \leqslant M_0 < \infty$, with $I_k \to i$ as $k \to \infty$.

Since M and K have the convergence property, there is some subsequence, say still $[k]$, such that $y_k = Mx_k \to y$ in measure in $(\mathfrak{m}(G))^s$, and $\mathring{y}_k = Kx_k \to \mathring{y}$ in measure in $(\mathfrak{m}(\Gamma))^{s'}$, for some $y \in (\mathfrak{m}(G))^s$ and $\mathring{y} \in (\mathfrak{m}(\Gamma))^{s'}$.

Since $f_3(t, y)$, $g_3(\tau, \mathring{y})$ depend on (t, y), $(\tau, \mathring{y})$ only, by property (C) for almost all $t \in G$, f_3 is a continuous function of y, and for μ-almost all $\tau \in \Gamma$, g_3 is a continuous function of $\mathring{y}$. Then $L^{(3)}\, x_3(t) = z_{3k}(t) = f_3(t, y_k(t))$ converges in measure in G to $z_3(t) = f_3(t, y(t))$, and analogously $J^{(3)} x_k(\tau) = \mathring{z}_{3k}(\tau) = g_3(\tau, \mathring{y}_k(\tau))$ converges in measure in Γ to $\mathring{z}_3(\tau) = g_3(\tau, \mathring{y}(\tau))$.

Since $L^{(4)}$ and $J^{(4)}$ have the convergence property, there is a subsequence, say still $[k]$, such that $L^{(4)} x_4(t) = z_{4k}(t)$ converges in measure to some $z_4(t)$, $t \in G$, $z_4 \in (\mathfrak{m}(G))^{r_4}$, and $J^{(4)} x_4(t) = \mathring{z}_{4k}(\tau)$ converges in measure to some $\mathring{z}_4(\tau)$, $\tau \in \Gamma$, $\mathring{z}_4 \in (\mathfrak{m}(\Gamma))^{r_4'}$.

At least one of the ten operators $L^{(1)}, \ldots, J^{(4)}$, M, K has the closed graph property. This guarantees that $x \in S$. Since the ten operators have the closure property in S, we conclude that $L^{(i)} x_k \to L^{(i)} x = z_i$, $J^{(i)} x_k \to J^{(i)} x = \mathring{z}_i$, $M x_k \to M x = y$, $K x_k \to K x = \mathring{y}$ as $k \to \infty$, $i = 1, 2, 3, 4$.

By lower closure theorem (Ref. 15), we conclude that there are measurable functions $u(t)$, $t \in G$, and $v(\tau)$, $\tau \in \Gamma$, such that

$$
\begin{aligned}
y(t) &= Mx(t) \in A(t), \qquad u(t) \in U(t, Mx(t)),\\
Lx(t) &= f(t, Mx(t), u(t)) \qquad \text{a.e. in } G,\\
\mathring{y}(\tau) &= Kx(\tau) \in B(\tau), \qquad v(\tau) \in V(\tau, Kx(\tau)),\\
Jx(\tau) &= g(\tau, Kx(\tau), v(\tau)) \qquad \mu\text{–a.e. in } \Gamma,
\end{aligned}
$$

$$f_0(t, Mx(t), u(t)) \in L_1(G), \qquad g_0(\tau, Kx(\tau), v(\tau)) \in L_1(\Gamma),$$

$$I[x, u, v] = \int_G f_0(t, Mx(t), u(t))\, dt + \int_\Gamma g_0(\tau, Kx(\tau), v(\tau))\, d\mu \leqslant i.$$

Thus, the triple (x, u, v) is admissible, and since Ω is a closed class, there is some admissible triple (x_0, u_0, v_0) in Ω with $I[x_0, u_0, v_0] \leqslant i$. Thus $I[x_0, u_0, v_0] = i$, and the existence theorem is thereby proved under the main assumptions.

Under the set of alternate assumptions the proof is the same where use is made of the lower closure theorems proved in Refs. 1, 4, 5, 15, 17.

Remark 8.1. There is a natural situation where $S = X$ and all operators $L^{(i)}$, $J^{(i)}$, $i = 1, 2, 3, 4$, have convergence property and closure property. This occurs, for instance, when $(X, \mathscr{C})$ is a Sobolev space $W_p{}^l(G)$, $l \geqslant 1$, $1 \leqslant p < \infty$, Γ is a part of the boundary ∂G of G, Sobolev's imbedding theorems hold, $\| x \|_p^l \leqslant \gamma_0$ for some constant γ_0 and all $x \in A_0$, and the operators $L^{(i)}$, $i = 1, 2$, are linear combinations with bounded measurable coefficients of all derivatives $D^\alpha x$, $0 \leqslant |\alpha| \leqslant l$, for $p > 1$ [and even for $p = 1$ provided we know that the derivatives of maximal order $D^\alpha x$, $|\alpha| = l$, are equiabsolutely integrable in G (Ref. 21)]. In this situation, if the operators $L^{(i)}$, $i = 3, 4$, M and $J^{(i)}$,

$i = 3, 4$, K are also linear combinations with bounded measurable coefficients of all derivatives $D^\alpha x$, $0 \leqslant |\alpha| \leqslant l - 1$, on G and on Γ respectively, then the same operators can be thought of as being into L_p spaces, $1 \leqslant p \leqslant \infty$, on G and Γ, and the same operators then have convergence property and closure property with respect to strong convergence in L_p. This situation was essentially considered in Ref. 5.

Remark 8.2. The set $\Lambda_0 = \{x\}_{\Omega_0} \subset (X, \mathfrak{T})$ is certainly relatively weakly compact, if for instance $(X, \mathfrak{T})$ is a Sobolev space $W_p{}^l(G)$, $1 < p < \infty$, $l \geqslant 1$, with weak topology $\mathfrak{T}$, and it happens that $x \in \{x\}_\Omega$, $I[x, u, v] \leqslant M_0$ implies $\| x \|_q^l \leqslant M_1$ for some constants $1 < q \leqslant \infty$, $M_1 \geqslant 0$, and where $\| \quad \|_q^l$ denotes Sobolev norm in G.

The set $\Lambda_0 = \{x\}_{\Omega_0} \subset (X, \mathfrak{T})$ is also relatively weakly compact if, for instance, $(X, \mathfrak{T})$ is a Sobolev space $W_1{}^l(G)$, $l \geqslant 1$, with weak topology $\mathfrak{T}$, if $x \in \{x\}_\Omega$, $I[x, u, v] \leqslant M_0$ implies $\| x \|_1^l \leqslant M_1$ as before, and in addition a suitable growth condition (ψ_ϵ) holds guaranteeing that the derivatives of maximal order $\chi = [D^\alpha x(t), t \in G, |\alpha| = l]$ for the same elements $x \in \Lambda_0$ are equiabsolutely integrable on G (see Refs. 5 and 10).

Both situation will be shown in the examples below.

The image $F^{(1)}(\Lambda)$ is certainly relatively compact in $(L_1(G))^{r_1}$, if f_0, $f^{(1)}$ satisfy growth condition (ψ_ϵ) (see Ref. 5) or if $F^{(1)}(\Lambda)$ lies in a bounded part of some $L_q(G)^{r_1'}$, $1 < q \leqslant \infty$. The latter occurs, if for instance $(X, \mathfrak{T})$ is a Sobolev space $W_p{}^l(G)$, $1 < p \leqslant \infty$, $l \geqslant 1$, with weak topology $\mathfrak{T}$, and $\| F^{(1)}[u, x] \|_q \leqslant \gamma \| x \|_p^m + \gamma'$ for some constants γ, γ', $1 < q \leqslant \infty$, and all $(x, u) \in \Lambda$.

The image $G^{(1)}(\mathring{\Lambda})$ is certainly relatively weakly compact in $(L_1(\Gamma))^{r_1'}$ if g_0, $g^{(1)}$ satisfy a growth condition (ψ_ϵ), or if $G^{(1)}(\mathring{\Lambda})$ lies in a bounded part of some $(L_q(\Gamma))^{r_1}$, $1 < q \leqslant \infty$. The latter occurs, for instance, if $(X, \mathfrak{T})$ is a Sobolev space $W_p{}^l(G)$, $1 < p \leqslant \infty$, with weak topology $\mathfrak{T}$, if Γ is a part of the boundary ∂G of G for which Sobolev's imbedding theorems hold, and a relation $\| G^{(1)}[x, v] \|_q \leqslant \gamma \| x \|_p^l + \gamma'$ holds for some constants $\gamma, \gamma' \geqslant 0$, $1 < q \leqslant \infty$, and all $(x, v) \in \Lambda$.

Remark 8.3. For the case in which f_0, f (or g_0, g) are linear in the state variables y (or $\mathring{y}$), we have proved existence theorems without convexity conditions (see Refs. 22–23).

Remark 8.4. Properties (P) and (P_ρ) of the alternate set of hypotheses of the existence Theorem 8.1 could be replaced by the corresponding slightly extended properties taken into consideration by M. B. Suryanarayana (see Ref. 17 for lower closure theorems).

9. Some Nonlinear Operators with the Closed Graph Property

We begin with a simple statement in the form of a lemma.

Lemma 9.1. Let $A_k(t)$, $b_k(t)$, $A(t)$, $b(t)$, $z(t)$, $t \in G$, $k = 1, 2, \ldots$, be given functions defined on the set of finite measure G in E^ν, $|G| < \infty$, A_k, A measurable and finite a.e. in G, b_k, b, $z \in L_1(G)$, with $A_k b_k \in L_1(G)$, $k = 1, 2, \ldots$ If $A_k \to A$ in measure in G, $b_k \to b$ weakly in $L_1(G)$, and $A_k b_k \to z$ weakly in $L_1(G)$ as $k \to \infty$, then $Ab \in L_1(G)$, and $A(t)b(t) = z(t)$ a.e. in G.

Proof. Since $A_K \to A$ in measure in G, given $\epsilon > 0$ there is a subsequence, still denoted by $[k]$, and a compact subset K of G with $|G - K| \leqslant \epsilon$, so that all functions A_k, A are continuous on K and $A_k \to A$ uniformly on K as $k \to \infty$. Thus, if $\delta_k = \max\{|A_j(t) - A(t)| \mid t \in K, j \geqslant k\}$, $k = 1, 2, \ldots$, then $\delta_k \to 0$ as $k \to \infty$. Since $b_k \to b$ weakly in $L_1(G)$, $\| b_k \|_1 \leqslant M_0$ for some constant M_0 and all k, where $\| \;\; \|_1$ is the L_1 norm. Since $b_k \to b$ and $A_k b_k \to z$ weakly in $L_1(G)$, then $b_k \to b$ and $A_k b_k \to z$ weakly in $L_1(K)$ also. For every $\phi \in L_\infty(G)$ with $\| \phi \|_\infty \leqslant M_1$ the product $A\phi \in L_\infty(K)$, and so

$$\sigma_k = \left| \int_K (b_k(t) - b(t))\, A(t)\, \phi(t)\, dt \right| \to 0$$

as $k \to \infty$. Since

$$\begin{aligned} &\left| \int_K A_k(t)\, b_k(t)\, \phi(t)\, dt - \int_K A(t)\, b(t)\, \phi(t)\, dt \right| \\ &\qquad \leqslant \int_K |A_k(t) - A(t)|\, |b_k(t)|\, |\phi(t)|\, dt + \left| \int_K (b_k(t) - b(t))\, A(t)\, \phi(t)\, dt \right| \\ &\qquad \leqslant \delta_k M_0 M_1 + \sigma_k, \qquad k = 1, 2, \ldots, \end{aligned}$$

we conclude that $A_k b_k \to Ab$ weakly in $L_1(K)$. Since $A_k b_k \to z$ weakly in $L_1(K)$, we have $z(t) = A(t)b(t)$ a.e. in K, and finally $z(t) = A(t)b(t)$ a.e. in G. This proves our lemma.

Lemma 9.2. Let $A_k^s(t)$, $b_k^s(t)$, $A^s(t)$, $b^s(t)$, $z(t)$, $t \in G$, $k = 1, 2, \ldots$, $s = 1, \ldots, N$, be given functions defined on the set of finite measure G in E^ν, $|G| < \infty$, A_k^s, A^s measurable and finite a.e. in G, b_k^s, b^s, $z \in L_1(G)$, with $\sum_{s=1}^N A_k^s b_k^s \in L_1(G)$, $k = 1, 2, \ldots$ If $A_k^s \to A^s$ in measure in G, $b_k^s \to b^s$ weakly in $L_1(G)$ as $k \to \infty$, $s = 1, \ldots, N$, and $\sum_{s=1}^N A_k^s b_k^s \to z$ weakly in $L_1(G)$ as $k \to \infty$, then $\sum_{s=1}^N A^s b^s \in L_1(G)$ and $\sum_s A^s(t)\, b^s(t) = z(t)$ a.e. in G.

The proof is the same as above. The same lemmas extend to measure spaces.

In a Sobolev space $(X, \mathscr{T}) = W_p^l(G)$, $1 \leqslant p \leqslant \infty$, $l \geqslant 1$, with weak topology $\mathscr{T}$, let us consider the nonlinear operator $Lx = (D^\alpha x)(D^\beta x)$, the product of two derivatives $D^\alpha x$, $D^\beta x$ of orders $0 \leqslant |\alpha| \leqslant l-1$, $|\alpha| \leqslant |\beta| \leqslant l$. Let S denote the subset of all elements $x \in W_p^l(G)$ such that $Lx \in L_1(G)$. It may well happen that S is only a proper part of $W_p^l(G)$. However, by Lemma 9.1, the operator L: $S \to L_1(G)$ has the closed graph property on S [with respect to weak convergence on S and weak convergence on $L_1(G)$]. Indeed, if x_k, $x \in W_p^l(G)$, $x_k \to x$ weakly in $W_p^l(G)$, then $D^\alpha x_k \to D^\alpha x$ strongly in $L_1(G)$, $D^\beta x_k \to D^\beta x$ in $L_1(G)$, (strongly if $|\alpha| \leqslant \beta < l$, weakly if $|\alpha| < |\beta| = l$), and we know from the lemma that, if $(D^\alpha x_k)(D^\beta x_k) \to z$ weakly in L_1, then $z = (D^\alpha x)(D^\beta x)$ a.e. in G.

If $X = W_1^1(G)$, $G \subset E^\nu$, let us consider the operator on G,

$$Lx = x \sum_{i=1}^{\nu} (\partial x/\partial t^i) = x \operatorname{div} x,$$

as an operator L: $S \to L_1(G)$, defined on the subset S of W_1^1 of all $x \in W_1^1(G)$ such that $Lx \in L_1(G)$. By force of Lemma 9.2, L: $S \to L_1(G)$ has the closed graph property on S_1 [with respect to weak convergence on $W_1^1(G)$ and weak convergence on $L_1(G)$].

Again, for $X = W_1^2(G)$, $G \subset E^\nu$, the operator L: $S \to L_1(G)$, $S \subset W_1^2(G)$, defined by

$$Lx = \sum_{i=1}^{\nu} (\partial x/\partial t^i)^2 = |\operatorname{grad} x|^2$$

on the subset S of $W_1^2(G)$ of all $x \in W_1^2(G)$ such that $Lx \in L_1(G)$, has the closed graph property on S [with respect to weak convergence in both $W_1^2(G)$ and $L_1(G)$].

Similarly, for $X = W_1^2(G)$, $G \subset E^\nu$, the operator L: $S \to L_1(G)$, $S \subset W_1^2(G)$, defined by

$$Lx = \sum_{i=1}^{\nu} (\partial x/\partial t^i)(\partial^2 x/\partial t^{i2}) = (\operatorname{grad} x, \nabla^2 x),$$

on the subset S of $W_1^2(G)$ of all $x \in W_1^2(G)$ such that $Lx \in L_1(G)$, has the closed graph property on S.

Analogously, we may define the operator J: $S \to L_1(\Gamma)$ by taking $Jx = x(\partial x/\partial n)$ and S the subset of all $x \in W_1^2(G)$ for which the product $x(\partial x/\partial n) \in L_1(\Gamma)$. Here $\partial x/\partial n$ denotes normal derivative and we assume G

smooth and such that Sobolev's embedding theorems hold. However, S may well be a proper subset of $W_1^2(G)$. The same holds for the analogous operator Jx: $S \to L_1(\Gamma)$, $S \subset W_1^2(G)$, defined by $Jx = (\partial x/\partial n)^2$, where $S = [x \in W_1^2(G) \mid Lx \in L_1(G)]$.

Let $G = [(\xi, \eta) \mid \xi^2 + \eta^2 < 1]$ and let $a(\xi, \eta)$ be a measurable function on G, a.e. finite. Let $(X, \mathcal{T}) = W_2^2(G)$ with the weak topology, and let L: $S \to L_1(G)$, $S \subset W_2^2(G)$, be the operator defined by $Lx = a(\xi, \eta)(x_\xi + x_\eta)$, on the subset S of $W_2^2(G)$ of all $x \in W_2^2(G)$ for which $a(\xi, \eta)(x_\xi + x_\eta)\, x_{\xi\eta} \in L_1(G)$. Then L has the closed graph property on S [relatively to the weak convergences on $W_2^2(G)$ and $L_1(G)$].

Before considering the next example, we mention here the following well-known lemma.

Lemma 9.3. If $1 < p < \infty$, $b_k \in L_p(G)$, $k = 1, 2,\ldots$, $\| b_k \|_p \leqslant M_0$ for some constant M_0, and $b_k \to b$ in measure in G, then $b_k \to b$ weakly in $L_p(G)$ (Hewitt and Stromberg, Ref. 25).

Now let G be as above, $(X, \mathcal{T}) = W_p^2(G)$ with the weak topology, and let L: $S \to L_1(G)$, $S \subset W_p^2(G)$, be the operator $Lx = x_{\xi\eta}$ on the subset S of $W_p^2(G)$ of all $x \in W_p^2(G)$ for which $\| x_{\xi\eta} \|_1 = C$ for a given constant C. We claim that the operator L: $S \to L_p(G)$ has the closed graph property on S with respect to weak convergence in $W_p^2(G)$ and convergence in measure in $L_p(G)$. Indeed, if $x_k \in S$, $x \in W_p^2(G)$, $x_k \to x$ weakly in $W_p^2(G)$, then certainly $x_{k\xi\eta} \to x_{\xi\eta}$ weakly in $L_2(G)$ and $\| x_{k\xi\eta} \|_p \leqslant M_0$ for some constant M_0. If we know that $x_{k\xi\eta} \to y$ in measure in G to some measurable function y on G, then by Lemma 9.3, $y = x_{\xi\eta}$ a.e. in G, and $x_{k\xi\eta} \to x_{\xi\eta}$ in measure as well as weakly. Then $| x_{k\xi\eta} | \to | x_{\xi\eta} |$ in measure with $\| x_{k\xi\eta} \|_p \leqslant M_0$. Again, by Lemma 9.3, we conclude that $| x_{k\xi\eta} | \to | x_{\xi\eta} |$ weakly, and hence $\| x_{k\xi\eta} \|_1 \to \| x_{\xi\eta} \|_1$. Since $\| x_{k\xi\eta} \|_1 = C$, we have $\| x_{\xi\eta} \|_1 = C$, and L has the closed graph property.

We refer to Ref. 18 for general properties of operators. We mention here that some more examples of nonlinear operators with the closed graph property are exhibited in Ref. 24.

10. Examples

A number of examples have been already given in Ref. 5. Here we list some examples to which the present more general existence Theorem 8.1 applies. Given any function x on a domain G, we shall denote by γx the values of x on the boundary ∂G of G, whenever they are defined. Some of these examples have been elaborated in Ref. 26.

Example 10.1. Let G be a connected bounded open subset of the $\xi\eta$-plane E^2 of Morrey's class K (Ref. 21) so that the usual arc-length measure ds is defined on $\partial G = \Gamma$. We are concerned with the problem of the minimum of the functional

$$I[x] = \iint_G (\xi + \eta + |x|^p + |x_\xi|^p + |x_\eta|^p)\, d\xi\, d\eta + \int_\Gamma (\xi^2 + \eta^2)\, x^2(s)\, ds,$$

where $p > 2$ is a fixed number, with constraint

$$a(\xi, \eta)\, x + b(\xi, \eta)\, x_\xi + c(\xi, \eta)\, x_\eta \geqslant d(\xi, \eta),$$

where $\epsilon > 0$ is a given constant, and a, b, c, d are given measurable bounded functions on G. In other words, we seek the minimum of

$$I[x, u, v, w] = \iint_G (\xi + \eta + |x|^p + |u|^p + |v|^p)\, d\xi\, d\eta + \int_\Gamma (\xi^2 + \eta^2)\, x^2(s)\, ds,$$

with state equations

$$x_\xi = u, \qquad x_\eta = v, \qquad a(\xi, \eta)\, x = d(\xi, \eta) - b(\xi, \eta)\, u - c(\xi, \eta)\, v + w,$$

and controls u, v, w in G, $(u, v, w) \in U = [(u, v) \in E^2,\ w \geqslant 0]$.

Thus, $L_1 x = x_\xi$, $L_2 x = x_\eta$, $L_3 x = ax$, $Mx = x$, $Kx = \gamma x$, $f_1 = u$, $f_2 = v$, $f_3 = d - bu - cv + w$, or $f(\xi, \eta, x, u, v, w) = [u, v, d - bu - cv + w]$. Also we have

$$f_0(\xi, \eta, x, u, v, w) = \xi + \eta + |x|^p + |u|^p + |v|^p, \qquad g_0 = (\xi^2 + \eta^2)(\gamma x)^2.$$

Thus $f_0 \geqslant \xi + \eta$, $g_0 \geqslant 0$ and f_0, g_0 satisfy condition (α). Also, f_0, (f_1, f_2) satisfy a (ψ_ϵ) growth condition on G (Section 6).

Let $(X, \mathcal{T})$ be the Sobolev space $X = W_p{}^1(G)$ with weak topology. Then the operators L_1, L_2, L_3 are defined in $S = X$ and have the weak convergence and weak closure property, with range in $L_p(G)$. The operators M and K are also defined in $S = X$ and have the strong convergence and strong closure property with range in $L_p(G)$ and $L_p(\Gamma)$, respectively.

Note that if $\Omega = \{(x, u, v, w)\}$ is the class of all admissible systems, and Ω_0 the subclass of only those with $I[x, u, v, w] \leqslant M_0$ for some M_0 sufficiently large, then Ω_0 is not empty, and $A_0 = \{x\}_{\Omega_0}$ is made up of elements $x \in W_p{}^1(G)$ with $x, x_\xi, x_\eta \in L_p(G)$, $\| x \|_p + \| x_\xi \|_p + \| x_\eta \|_p \leqslant M_1$ and finally $\| x \|_p^1 \leqslant M_2$ for suitable constants M_1, M_2 depending only on M_0, p, and G. Because of property (ψ_ϵ), the derivatives x_ξ, x_η of the elements $x \in \Lambda_0 = \{x\}_{\Omega_0}$ are equiabsolutely integrable. Thus, the class Λ_0 is weakly sequentially relatively compact as a subset of $X = W_p{}^1(G)$.

Since $\nu = 2$, $l = 1$, $p > 2$, we have $\nu < lp$. Hence, by Sobolev's imbedding theorem, the elements $x \in \Lambda_0 = \{x\}_{\Omega_0}$ are equibounded continuous functions on cl G, say $|x(\xi, \eta)| \leqslant c_0$, where c_0 depends only on G, M_2, ϵ.

Thus, for $|z| \leqslant R$, $z = (z^1, z^2, z^3)$, that is, $|(u, v, d - bu - cv + w)| \leqslant R$, we certainly have $|u| \leqslant R$, $|v| \leqslant R$, and

$$\xi + \eta \leqslant f_0(\xi, \eta, x, u, v) \leqslant \xi + \eta + c_0{}^p + 2R^p.$$

In other words, $|T(z; \xi, \eta, x)| \leqslant c$ for some constant c and all $|z| \leqslant R$. The sets $Q(\xi, \eta, x)$ are here defined by

$$Q(\xi, \eta, x) = [z = (z^1, z^2, z^3) \mid z^1 = u, z^2 = v, z^3 = d - bu - cv + w, (u, v, w) \in U]$$

and thus $p(\xi, \eta) \in Q(\xi, \eta, x)$ for all x if we take $p(\xi, \eta) = (0, 0, d(\xi, \eta))$, a bounded measurable function $p: G \to E^3$.

The sets $\tilde{Q}(\xi, \eta, x)$ are here defined by

$$\tilde{Q}(\xi, \eta, x) = [(z^0, z^1, z^2, z^3) \mid z^0 \geqslant \xi + \eta + |x|^p + |u|^p + |v|^p,$$
$$z^1 = u, z^2 = v, z^3 = d - bu - cv + w],$$

where $(u, v, w) \in U = [(u, v) \in E^2, w \geqslant 0]$. First, the sets $\tilde{Q}(\xi, \eta, x)$ are convex since f_0 is convex in u, v and f_1, f_2, f_3 are linear in u, v, w, and obviously closed. Let us prove that $|f| \to +\infty$ as $|(u, v, w)| \to +\infty$. Indeed,

$$|f| = |u| + |v| + |d - bu - cv - w|$$

and if $D = \sup |d|$, $B = \max [\sup |b|, \sup |c|]$, then, given any $M > 0$, if we take $|u| + |v| + |w| \geqslant N = (B + 2) M + D$, then either $|u| + |v| \geqslant M$ and then $|f| \geqslant M$; or $|u| + |v| \leqslant M$, and then $|w| \geqslant (B + 1) M + D$, and $|f| \geqslant |w| - B(|u| + |v|) - D \geqslant (B + 1) M + D - BM - D = M$. We have proved that $|f| \to +\infty$ as $|u| + |v| + |w| \to +\infty$. We now apply Theorem 8.1 with $r = 3$, $r_1 = 2$, $r_2 = 1$, $f^{(1)} = (f_1, f_2)$, $f^{(2)} = f_3$, $r_3 = r_4 = 0$, $r' = 0$. The sets $\tilde{Q}(t, x)$ have here property (P) by the remarks in Section 4. The sets $\tilde{R}(\xi, \eta, \mathring{y})$ are the trivial sets $[(z^0, z) \mid z^0 \geqslant (\xi^2 + \eta^2)(\mathring{y})^2, z = 0]$, which certainly are closed, and have property (K) [even property (Q)] with respect to $\mathring{y}$, $\mathring{y} \in E^1$. The functional under consideration has an absolute minimum.

Remark 10.1. The functional of Example 10.1 has an absolute minimum even for any fixed $p > 1$. In this situation we consider any minimizing sequence $[x_k]$, and $y_k = Mx_k = x_k$. For some subsequence, say still $[k]$, we have $x_k \to x$ weakly in $W_p{}^1(G)$, $x_k \to x$ strongly in $L_p(G)$, and if we take $p_k(\xi, \eta) = p(\xi, \eta) = (0, 0, d(\xi, \eta))$, $\mu_k(\xi, \eta) = \xi + \eta + |x_k(\xi, \eta)|^p$, $\mu(\xi, \eta) = \xi + \eta + |x(\xi, \eta)|^p$, we see that $p_k \to p$ strongly in $(L_1(G))^3$, $\mu_k \to \mu$ strongly in $L_1(G)$, and that

$$(\mu_k(\xi, \eta), p_k(\xi, \eta)) \in \tilde{\tilde{Q}}(\xi, \eta, y_k(\xi, \eta))$$

for all $(\xi, \eta) \in G$, $k = 1, 2,...$ Thus, the sets $\tilde{\tilde{Q}}$ have property (P′), and Theorem 8.1 still applies.

Example 10.2. Let G be as above, $\Gamma = \partial G$, and let us consider the functional

$$I[x, u, v] = \iint_G (\xi + \eta + |x|^p + |x_\xi|^p + |x_\eta|^p + |u|)\, d\xi\, d\eta$$
$$+ \int_\Gamma [(\xi^2 + \eta^2)\, |\gamma x|^p + |\gamma x_\xi|^p + |\gamma x_\eta|^p + |v|]\, d\mu$$

for some fixed $p > 1$, with state equations

$$x_{\xi\xi} + x_{\eta\eta} = \xi^2 + \eta^2 + |x|^p + |x_\xi|^p + |x_\eta|^p + d(\xi, \eta)u \qquad \text{on} \quad G,$$
$$a(\xi, \eta)\,\gamma x + b(\xi, \eta)\,\gamma x_\xi + c(\xi, \eta)\,\gamma x_\eta = |\gamma x|^p + |\gamma x_\xi|^p + |\gamma x_\eta|^p + e(\xi, \eta)v \qquad \text{on} \quad \Gamma,$$

and constraints

$$|x_\xi| \leqslant C_1, \qquad |x_\eta| \leqslant C_2, \tag{6}$$

$$\iint_G (|x_{\xi\xi}|^p + |x_{\xi\eta}|^p + |x_{\eta\eta}|^p)\, d\xi\, d\eta \leqslant C. \tag{7}$$

The problem is immediately written in the form (1–5) with

$$\begin{aligned}
&Lx = x_{\xi\xi} + x_{\eta\eta}, \qquad Mx = (x, x_\xi, x_\eta),\\
&Jx = a\gamma x + b\gamma x_\xi + c\gamma x_\eta, \qquad Kx = (\gamma x, \gamma x_\xi, \gamma x_\eta),\\
&f_0 = \xi + \eta + |x|^p + |x_\xi|^p + |x_\eta|^p + |u|, \qquad u \in U = E^1,\\
&g_0 = (\xi^2 + \eta^2)(\gamma x)^2 + (\gamma x_\xi)^2 + (\gamma x_\eta)^2 + |v|, \qquad v \in V = E^1,\\
&\ f = \xi^2 + \eta^2 + |x|^p + |x_\xi|^p + |x_\eta|^p + d(\xi, \eta)u,\\
&\ g = (\gamma x)^2 + (\gamma x_\eta)^2 + (\gamma x_\eta)^2 + e(\xi, \eta)v,
\end{aligned}$$

where a, b, c, e are measurable bounded functions on Γ, d is a measurable bounded function on G, and p, C, C_1, C_2 positive constants. Thus

$r = r' = 1$, $s = s' = 3$, $m = m' = 1$. We take for X the Sobolev space $X = W_p{}^2(G)$. Here $f_0 \geqslant 0, g_0 \geqslant 0$, and thus f_0, g_0 satisfy condition (α). Also f_0, g_0 are convex in u and v, respectively, and f, g are linear in u, v. Thus the sets $\tilde{\tilde{Q}} \subset E^2$, $\tilde{\tilde{R}} \subset E^2$ are certainly convex, and obviously closed.

If $\Omega = \{(x, u, v)\}$ is the class of all admissible systems, and Ω_0 the subclass of only those with $I[x, u, v] \leqslant M_0$ for M_0 sufficiently large, then Ω_0 is not empty, and $A_0 = \{x\}_{\Omega_0}$ is made up of elements $x \in W_p{}^2(G)$ with $\| x \|_p^2 \leqslant M_2$ for a suitable constant M_2 which depends solely on M_0, G, p, C. Since $\nu = 2$, $l = 2$, $p > 1$, we have $\nu < lp$. Hence, the elements $x \in A_0$ are equibounded continuous functions on cl G, say $|x(\xi, \eta)| \leqslant C_0$, where C_0 depends solely on G, M_2, p. Here we assume $d(\xi, \eta) \geqslant \gamma > 0$, $e(\xi, \eta) \geqslant \gamma' > 0$ for some constants $\gamma, \gamma' > 0$. For $y = (x, x_\xi, x_\eta)$, the sets $Q(\xi, \eta, y)$ are the sets $Q(\xi, \eta, y) = [z = \xi^2 + \eta^2 + \lambda + d(\xi, \eta)u$, $u \in E^1]$, where $\lambda = | x |^p + | x_\xi |^p + | x_\eta |^p$. Hence, $Q(\xi, \eta, y) = E^1$, and for $p(\xi, \eta) = 0$ we certainly have $p(\xi, \eta) \in Q(\xi, \eta, y)$ for all y. Also, for $| z | \leqslant R$, or $| z | = | \xi^2 + \eta^2 + \lambda + d(\xi, \eta)u \leqslant R$, we certainly have $| u | \leqslant R'/\gamma$ with $R' = R + \xi^2 + \eta^2 + \lambda$, and then

$$\xi + \eta \leqslant f_0(\xi, \eta, y, u) \leqslant \xi + \eta + C_0{}^p + C_1{}^p + C_2{}^p + R'/\gamma.$$

In other words, $| T(z; \xi, \eta, y)| \leqslant c$ for some constant c and all $| z | \leqslant R$. Finally, $| f | \to \infty$ as $| u | \to \infty$. Thus, the sets $\tilde{\tilde{Q}} \subset E^2$ satisfy property (P). The same argument holds for the sets $\tilde{\tilde{R}} \subset E^2$ which, therefore, also have property (P).

As before, $A = \{x\}_\Omega$ is a subset of $X = W_p{}^2(G)$ which lies in a bounded part of X. Since $p > 1$, the set A is relatively weakly sequentially compact in $W_p{}^1(G)$. The operators L, J, M, K are defined in $S = X$ and have the required properties. The problem above has an absolute minimum.

Remark 10.2. The functional of Example 10.2 has an absolute minimum even without constraints (6). In this situation, we consider a minimizing sequence $[x_k]$, and $y_k = Mx_k = (x_k, x_{k\xi}, x_{k\eta})$. For some subsequence, say still $[k]$, we have $x_k \to x$ weakly in $W_p{}^2(G)$, $x_{k\xi} \to x_\xi$, $x_{k\eta} \to x_\eta$ strongly in $L_p(G)$, and if we take

$$\begin{aligned} u_k(\xi, \eta) &= -d^{-1}(\xi, \eta)(\xi^2 + \eta^2 + | x_k |^p + | x_{k\xi} |^p + | x_{k\eta} |^p), \\ p_k(\xi, \eta) &= p(\xi, \eta) = 0, \\ \mu_k(\xi, \eta) &= \xi + \eta + d^{-1}(\xi, \eta)(\xi^2 + \eta^2) \\ &\quad + [1 + d^{-1}(\xi, \eta)](| x_k |^p + | x_{k\xi} |^p + | x_{k\eta} |^p), \end{aligned}$$

then μ_k converges strongly in $L_1(G)$, and

$$(\mu_k(\xi, \eta), p_k(\xi, \eta)) \in \tilde{\tilde{Q}}(\xi, \eta, y_k(\xi, \eta)),$$

for all $(\xi, \eta) \in G$, $k = 1, 2,...$ Thus, the sets $\tilde{\tilde{Q}}$ have property (P′), and Theorem 8.1 still applies.

Example 10.3. Let G be as above, and let us consider the problem of the minimum of the functional

$$I = \iint_G (|x|^p + |x_\xi|^p + |x_\eta|^p + |x_{\xi\xi}|^p + |x_{\xi\eta}|^p + |x_{\eta\eta}|^p)\, d\xi\, d\eta \tag{8}$$

for some fixed $p > 1$, with the differential equation

$$x_{\xi\eta} = f(\xi, \eta, x, x_\xi, x_\eta) \tag{9}$$

and the constraint

$$\iint_G |x_{\xi\eta}(\xi, \eta)|\, d\xi\, d\eta = C_0 . \tag{10}$$

Let $(X, \mathcal{T}) = W_p^2(G)$ with the weak topology, and let

$$S = \{x \in W_p^2(G) |\, \| x_{\xi\eta} \|_1 = C_0\}.$$

The problem above is the problem of the minimum of the functional

$$I[x, u, v, w] = \iint_G (|x|^p + |x_\xi|^p + |x_\eta|^p + |u|^p + |v|^p + |w|^p)\, d\xi\, d\eta,$$

with differential equations and constraints

$$x_{\xi\xi} = u, \quad x_{\xi\eta} = v, \quad x_{\eta\eta} = w, \quad (u, v, w) \in U = E^3,$$
$$x_{\xi\eta} = f(\xi, \eta, x, x_\xi, x_\eta), \quad \| x_{\xi\eta} \|_1 = C_0 .$$

Thus, we take here $U = E^3$,

$$L^{(2)}x = (x_{\xi\xi}, x_{\xi\eta}, x_{\eta\eta}), \quad L^{(3)}x = x_{\xi\eta}, \quad Mx = (x, x_\xi, x_\eta),$$
$$r_1 = 0, \quad r_2 = 3, \quad r_3 = 1, \quad r_4 = 0, \quad r = 4, \quad s = 3, \quad m = 3,$$

and, if $Mx = (y_1, y_2, y_3)$, we actually have

$$f_0(\xi, \eta, y_1, y_2, y_3, u, v, w) = |y_1|^p + |y_2|^p + |y_3|^p + |u|^p + |v|^p + |w|^p,$$
$$f^{(2)}(\xi, \eta, y_1, y_2, y_3, u, v, w) = (u, v, w),$$
$$f^{(3)}(\xi, \eta, y_1, y_2, y_3, u, v, w) = f(\xi, \eta, y_1, y_2, y_3).$$

Here $f_0 \geqslant 0$ and thus f_0 satisfies condition (α), and $f^{(3)}$ is independent of the control variables. Here $A_0 = \{x\}_{\Omega_0}$ is weakly relatively sequentially compact in $W_p^2(G)$, since we are minimizing the norm of $W_p^2(G)$. The operator $L^{(2)}$ has the weak convergence property in S and also has the closed graph property on S, as proved in Section 9. The operator $L^{(3)}$ is certainly closed in measure on S, and M has the convergence property (with respect to convergence in measure). Also, L_3 and M have the closure property. Finally, the sets

$$\begin{aligned}\tilde{Q}(\xi, \eta, y_1, y_2, y_3) = [(z^0, z^1, z^2, z^3, z^4)|\ & z^0 \\ & \geqslant | y_1 |^p + | y_2 |^p + | y_3 |^p + | u |^p + | v |^p + | w |^p, \\ & z^1 = u,\ z^2 = v,\ z^3 = w, \\ & z^4 = f(\xi, \eta, y_1, y_2, y_3),\ (u, v, w) \in U = E^3]\end{aligned}$$

have property (Q) = (Q_4) by force of criterion (6.1) in Ref. 13. Thus, the functional under consideration has an absolute minimum.

Example 10.4. The same as before, for the minimum of the functional

$$I = \iint_G (| x |^p + | x_\xi |^p + | x_\eta |^p + | u |^p + | v |^p + | w |^p)\, d\xi\, d\eta,$$

for some $p > 2$ with the differential equations and constraint

$$\begin{aligned} & x_{\xi\xi} = u, \qquad x_{\xi\eta} = v, \qquad x_{\eta\eta} = w, \\ & x_{\xi\eta} = f(\xi, \eta, x, x_\xi, x_\eta), \qquad \| x_{\xi\eta} \|_1 = C_0, \\ & x_\xi + x_\eta = a(\xi, \eta, x, x_\xi, x_\eta)\omega. \end{aligned}$$

Here the control variables (u, v, w, ω) take their values in $U = E^4$, and $f(\xi, \eta, x, x_\xi, x_\eta)$, $a(\xi, \eta, x, x_\xi, x_\eta)$ satisfy Carathéodory condition, with $a \geqslant \gamma > 0$, γ a constant. Here $L^{(2)}$, $L^{(3)}$, M are as in Example 10.3, we have one more operator $L^{(4)} = x_\xi + x_\eta$, $r_1 = 0$, $r_2 = 3$, $r_3 = 1$, $r_4 = 1$, $r = 5$, $s = 3$, $m = 4$,

$$\begin{aligned} & f_0(\xi, \eta, y_1, y_2, y_3, u, v, w, \omega) \\ & \qquad = | y_1 |^p + | y_2 |^p + | y_3 |^p + | u |^p + | v |^p + | w |^p, \\ & f^{(4)}(\xi, \eta, y_1, y_2, y_3, u, v, w, \omega) \\ & \qquad = a(\xi, \eta, y_1, y_2, y_3)\omega, \end{aligned}$$

and $f^{(2)}, f^{(3)}$ are as in Example 10.3. The same discussion as before holds. We can think of $L^{(4)}$ as having range in $L_p(G)$, and $L^{(4)}$ has closure property and strong convergence property. Here the sets $\tilde{Q}(\xi, \eta, y) \subset E^5$,

$y = (y_1, y_2, y_3)$, are closed and convex, and have property (Q_3) [but not necessarily property $(Q) = (Q_5)$].

Example 10.5. Let $G = [(\xi, \eta) \mid \xi^2 + \eta^2 < 1]$, let $\Gamma = \partial G$, and let s denote the arc length measure on Γ. We consider the problem of the minimum of the functional

$$I[x, u_1, u_2, v] = \int_G (x^2 + x_\xi^2 + x_\eta^2 + u_1^2 + u_2^2(1 - u_2)^2)\, d\xi\, d\eta + \int_\Gamma (\gamma x - 1)^2\, ds,$$

with differential equations

$$\begin{aligned} \alpha(\xi, \eta)(x_\xi + x_\eta)x_{\xi\eta} &= u_1 && \text{in } G,\\ x_\xi x_\eta &= u_2 && \text{in } G,\\ \gamma x_\xi = \cos v, \qquad \gamma x_\eta &= \sin v && \text{on } \Gamma, \end{aligned}$$

where $(u_1, u_2) \in U = E^2$, $v \in V = E^1$, in the class Ω of all systems (x, u_1, u_2, v), $x \in S$, with $S = [x \in W_2^2(G),\ \alpha(\xi, \eta)(x_\xi + x_\eta)x_{\xi\eta} \in L_1(G)]$, u_1, u_2 measurable in G, v s-measurable on Γ, and

$$\| x_{\xi\xi} \|_2 + \| x_{\xi\eta} \|_2 + \| x_{\eta\eta} \|_2 \leqslant C.$$

Above $\alpha(\xi, \eta)$ is a given measurable function on G (not necessarily bounded or L_1-integrable). Here the constant C is assumed to be sufficiently large so that Ω is not empty. We take here $X = W_2^2(G)$ with the weak topology, and we set

$$L^{(1)}x = \alpha(\xi, \eta)(x_\xi + x_\eta)x_{\xi\eta}, \qquad L^{(4)}x = x_\xi x_\eta, \qquad Mx = (x, x_\xi, x_\eta),$$
$$J^{(4)}x = (\gamma x_\xi, \gamma x_\eta), \qquad Kx = \gamma x,$$

and thus $r_1 = 1$, $r_4 = 1$, $r_2 = r_3 = 0$, $r = 2$, $s = 3$, $m = 2$, $r_1' = r_2' = r_3' = 0$, $r_4' = 2$, $r' = 2$, $s' = 1$, $m' = 1$, and

$$f_0(\xi, \eta, y_1, y_2, y_3, u_1, u_2) = y_1^2 + y_2^2 + y_3^2 + u_1^2 + u_2^2(1 - u_2)^2,$$

$$f_1^{(1)}(\xi, \eta, y_1, y_2, y_3, u_1, u_2) = u_1, \qquad g_0(\xi, \eta, \mathring{y}, v) = (\mathring{y} - 1)^2,$$

$$f_4^{(4)}(\xi, \eta, y_1, y_2, y_3, u_1, u_2) = u_2, \qquad g^{(4)}(\xi, \eta, \mathring{y}, v) = (\cos v, \sin v).$$

The functions f_0 and $f^{(1)}$ satisfy condition (ψ_ϵ) with $\psi_\epsilon = \epsilon^{-1}$. Also, $f_0 \geqslant 0$, $g_0 \geqslant 0$, and thus f_0, g_0 satisfy condition (α). For any sequence $[x_k]$ of elements from $A_0 = \{x\}_{\Omega_0}$ there is a subsequence, say still $[k]$, such that $x_k \to x$ weakly in $W_2^2(G)$. Here, $L^{(1)}$ has the weak closed graph property on S as proved in Section 9. The operators $L^{(4)}$, M, $J^{(4)}$, K can be thought of as having range in $L_1(G)$ and $L_1(\Gamma)$ spaces, and have the strong convergence and the strong closure property on S. Finally, the sets

$$\tilde{\tilde{Q}}(\xi, \eta, y_1, y_2, y_3) = [(z^0, z^1, z^2) \mid z^0 \geqslant y_1^2 + y_2^2 + y_3^2 + u_1^2 + u_2^2(1 - u_2)^2, \\ z^1 = u_1, z^2 = u^2, (u^1, u^2) \in E^2]$$

have property (Q) = (Q_2) [but not property (Q) = (Q_3)] with respect to (y_1, y_2, y_3); the sets

$$\tilde{\tilde{R}}(\xi, \eta, \mathring{y}) = [(z^0, z^1, z^2) \mid z^0 \geqslant (\mathring{y} - 1)^2, z^1 = \cos v, z^2 = \sin v, v \in E^1]$$

have property (Q_1) [but neither property (Q_2) nor property (Q) = (Q_3)] with respect to $\mathring{y}$. Hence existence Theorem 8.1 applies, and the functional under consideration has an absolute minimum in Ω.

Example 10.6. In this example, we discuss a problem similar to those considered by Butkovsky (Ref. 9). A thin metallic billet of length $L > 0$ is moved (through a heating medium) in time $T > 0$ with given velocity $V(t) \geqslant 0$, $0 \leqslant t \leqslant T$. The state of the material is characterized by the temperature distribution function $x(t, \xi)$, $0 \leqslant t \leqslant T$, $0 \leqslant \xi \leqslant L$. The process of internal heat transfer between the stationary heating medium, characterized by the temperature distribution function $u(t, \xi)$, $0 \leqslant t \leqslant T$, $0 \leqslant \hat{\xi} \leqslant L$, and the material passing through, is given by the differential equation

$$b(t, \xi)(\partial x/\partial t) + b(t, \xi)\, V(t)(\partial x/\partial \xi) + x = u(t, \xi),$$

with boundary condition at the entrance of the heating zone $x(t, 0) = x_0(t)$. The function $b(t, \xi)$ describes the thermophysical properties of the metallic billet.

We assume that there are constants C_1, C_2, $C_3 > 0$ such that $0 \leqslant V(t) \leqslant C_1$ and $C_2 \leqslant b(t, \xi) \leqslant C_3$ on $(0, T)$ and on $G = (0, T) \times (0, L)$, respectively. We seek the minimum of the functional

$$I[x, u_1, u_2] = \int_G (|\, x - x^* \,|^p + |\, u_1 \,|^p + |\, u_2 \,|^p)\, dt\, d\xi,$$

where $p > 2$ is a constant, $x^*(t, \xi)$ a given function, $x^* \in L_p(G)$, with differential equations

$$b(t, \xi)(\partial x/\partial t) + b(t, \xi)\, V(t)(\partial x/\partial \xi) + x = u_1, \qquad \partial x/\partial \xi = u_2,$$

in the class Ω of all systems (x, u_1, u_2), u_1, u_2 measurable in G, x any element of the set $S = [x \in W_p{}^1(G) \mid x(t, 0) = x_0(t)]$ satisfying the above relation and such that the integrand function in I is L_1-integrable in G. We take here $(X, \mathcal{T}) = W_p{}^1(G)$ with weak topology, we take $Mx = x$, and

$$L^{(2)}x = [b(\partial x/\partial t) + bV(\partial x/\partial \xi) + x, \partial x/\partial \xi],$$

so that $r_1 = 0$, $r_2 = 2$, $r_3 = 0$, $r_4 = 0$, $r = 2$, $s = 1$, $m = 2$, and

$$f_0 = | x - x^*(t, \xi)|^p + | u_1 |^p + | u_2 |^p, \qquad f^{(2)} = (u_1, u_2).$$

Here $f_0 \geqslant 0$ and thus f_0 satisfies condition (α). For any sequence x_k, $k = 1, 2, \ldots$, of elements $x_k \in \Lambda_0 \in \{x\}_{\Omega_0}$, we certainly have

$$\| x_k \|_p \leqslant K_0, \qquad \| u_{1k} \|_p \leqslant K_0, \qquad \| u_{2k} \|_p = \| \partial x_k/\partial \xi \|_p \leqslant K_0$$

for some constant K_0. Hence,

$$\| \partial x_k/\partial t \|_p = \| b^{-1}(u_{1k} - x_k) - V(\partial x_k/\partial \xi)\|_p \leqslant (2C_2^{-1} + C_1)K_0 .$$

Thus, there is a sequence, still labelled $[k]$, and an element $x \in W_p{}^1(G)$ so that $x_k \to x$ weakly in $W_p{}^1(G)$, and a further subsequence, say still $[k]$, such that the boundary values $x_k(t, 0)$, $0 \leqslant t \leqslant T$, converge strongly in $L_p[0, T]$ to $x(t, 0)$. (Actually, $p > 2$, or $0 < 1 - 2/p$, and, by Sobolev's imbedding theorems, $x_k(t, 0) \to x(t, 0)$ uniformly in $[0, T]$.) In any case, from $x_k(t, 0) = x(t_0)$ we derive $x(t, 0) = x_0(t)$.

Moreover, the operator $L^{(2)}$ has the weak convergence property and the weak closure property on S, and M has the strong convergence property and strong closure property; hence convergence in measure property. Finally, the sets

$$\tilde{Q}(t, \xi, x) = [(z^0, z^1, z^2) \mid z^0 \geqslant | x - x^*(t, \xi)|^p + | u_1 |^p + | u_2 |^p, \\ z^1 = u_1, z^2 = u_2]$$

[for $(u_1, u_2) \in E^2$] have property (Q) with respect to x. Theorem 8.1 applies, and the functional under consideration has an absolute minimum.

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