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Nemitsky's Operators and Lower Closure Theorems[1]

L. CESARI[2] AND M. B. SURYANARAYANA[3]

Abstract. This paper focuses on certain analytic criteria given by the authors in earlier works, for the geometric property of upper semicontinuity of set-valued functions, used in the proofs of lower closure theorems, and hence in existence theory. In particular, it is observed that, under Filippov-type condition (namely, when the set of controls is bounded in measure or in norm), mere Carathéodory-type continuity of the relevant functions f is sufficient to guarantee a weak form of property (Q), and in turn the lower closure theorems.

Key Words. Lower closure theorems, Nemitsky operators, upper semicontinuity, control theory, existence theorems, orientor fields, functional analysis.

1. Introduction

The role of upper semicontinuity properties of sets, in relation to closure and lower closure theorems in optimal control theory, has been studied by several authors such as Filippov (Ref. 1), Cesari (Refs. 2–6), Cesari and Suryanarayana (Refs. 7–8), Olech (Ref. 9), Lasota and Olech (Ref. 10), Bidaut (Ref. 11), Berkovitz (Refs. 12–14), and others. Recently, such semicontinuity conditions have been drastically reduced or eliminated by the use of simple geometric properties of the sets (Cesari, Ref. 5, and Suryanarayana, Ref. 15), or by the use of analytic conditions on the relevant functions f_0 and f (Cesari and Suryanarayana, Refs. 7–8), or by a combination of the two ideas. The purpose of this paper is to highlight certain remarks concerning analytic criteria which are found in earlier work of the authors and have not been explicitly stated (Refs. 4, 5, 7, 8). These remarks have recently gathered considerable interest in applications; as such, this

[1] This work has been partially supported by Research Project AFOSR-71-2122 at the University of Michigan, Ann Arbor, Michigan.

[2] Professor, Department of Mathematics, University of Michigan, Ann Arbor, Michigan.
[3] Associate Professor, Department of Mathematics, Eastern Michigan University, Ypsilanti, Michigan.

paper tries to give them their due emphasis and perspective. The proofs remain the same as in our previous papers, though for some statements also alternate proofs are given here.

First, we shall point out here that closure and lower closure theorems under only usual Carathéodory continuity conditions on f_0 and f are essentially proved in our papers under the sole hypothesis that the sequence of minimizing controls are bounded in the L_1-norm. Indeed, convergence in measure to zero of the usual differences

$$\delta_k(t) = f(t, x_k(t), u_k(t)) - f(t, x(t), u_k(t))$$

implies the weak form of property (Q) which we mentioned in Ref. 8 and which is sufficient to ensure closure and lower closure theorems.

It was also noticed in Ref. 5 that, in the case of Mayer optimal control problems, that is, when

$$Q(t, x) = f(t, x, U(t)) \subset E^r, \qquad \text{with } U(t) \subset E^m,$$

the sets $Q(t, x)$ have property (K) provided f is continuous and $|f(t, x, u)| \to \infty$ as $|u| \to \infty$ uniformly for $(t, x) \in A$ (more detailed and less demanding statements can be found in Ref. 5). However, this growth condition is not needed if we know that the sequence of minimizing controls are bounded in the L_1-norm. The same holds for Lagrange problems under usual lower-bound conditions on the integrand f_0 of the cost functional (e.g., $f_0 \geq 0$). Several examples are given to illustrate our statements.

2. Preliminaries

Let G be a given measurable subset of the t-space E^ν of finite measure, $t = (t^1, \ldots, t^\nu)$; for every $t \in G$, let $A(t)$ be a given nonempty subset of the x-space E^n, $x = (x^1, \ldots, x^n)$, and let

$$A = \{(t, x) \mid t \in G, x \in A(t)\}.$$

For each $(t, x) \in A$, let $\tilde{Q}(t, x)$ be a given subset of the $\tilde{z} = (z^0, z)$-space E^{1+r}, $z = (z^1, \ldots, z^r)$, and let $Q(t, x)$ be its projection on the z-space E^r. Let $H_0 \subset G$ denote a subset of G of measure zero. We assume that $A(t)$ is closed for $t \in G - H_0$. As usual, we say that the sets $Q(\bar{t}, x)$, $x \in A(\bar{t})$, for a fixed $\bar{t} \in G$, satisfy property (K) with respect to x at $\bar{x} \in A(\bar{t})$ if

$$Q(\bar{t}, \bar{x}) = \bigcap_{\delta > 0} \operatorname{cl} \cup \{Q(\bar{t}, x) \mid |x - \bar{x}| \leq \delta, x \in A(\bar{t})\}; \tag{1}$$

and we say that they satisfy property (Q) if (1) holds with cl replaced by cl co.

In the proofs of closure and lower closure theorems, the following weaker form of property (Q) has been used (see Refs. 4, 5, 7, 8). We say that, for fixed $\bar{t} \in G$, the sets $Q(t, x)$, $x \in A(t)$ satisfy property (Q^-) at $\bar{x} \in A(\bar{t})$ if, for every sequence of points $x_k \in A(\bar{t})$, $x_k \to \bar{x}$ as $k \to \infty$, and, for every sequence of points z_k, $k = 1, 2, \ldots$, with $z_k \in Q(\bar{t}, x_k)$, there exists a subsequence z_{k_s} such that

$$Q(\bar{t}, \bar{x}) \supset \bigcap_{h=1}^{\infty} \text{cl co} \cup \{z_{k_s}, s \geq h\}. \tag{2}$$

The following criterion for property (Q^-) is of interest.

Lemma 2.1. Let $\bar{t} \in G$ and $\bar{x} \in A(\bar{t})$, and let $Q(\bar{t}, \bar{x})$ be closed and convex. Let us assume that, for every sequence (x_k, z_k), $k = 1, 2, \ldots$, with $x_k \in A(\bar{t})$ and $z_k \in Q(t, x_k)$, $x_k \to \bar{x}$, there is another sequence $\bar{z}_k$, $k = 1, 2, \ldots$, of points $\bar{z}_k \in Q(\bar{t}, \bar{x})$, and a subsequence k_s such that $z_{k_s} - \bar{z}_{k_s} \to 0$ as $s \to \infty$. Then, the sets $Q(\bar{t}, x)$, $x \in A(\bar{t})$, satisfy property (Q^-) at $\bar{x}$.

Proof. The subsequence z_{k_s}, guaranteed by the hypothesis, satisfies Eq. (2). Indeed, let z^* be any point of the right-hand side of relation (2). Then

$$z^* = \lim_{l \to \infty} \sum_{i=1}^{n_l} (c_{li} z_{ks_i}),$$

where

$$c_{li} \geq 0 \qquad \text{and} \sum c_{li} = 1.$$

Since $z_{k_s} - \bar{z}_{k_s} \to 0$, we also have

$$\theta_l = \sum_{i=1}^{n_l} c_{li} \bar{z}_{ks_i} \to z^* \qquad \text{as } l \to \infty.$$

Here $\theta_l \in Q(\bar{t}, \bar{x})$, since $Q(\bar{t}, \bar{x})$ is convex, and $\bar{z}_{k_s} \in Q(\bar{t}, \bar{x})$. Then, $z^* \in Q(\bar{t}, \bar{x})$, since $\theta_l \to z^*$ and $Q(\bar{t}, \bar{x})$ is closed. This proves (2) and property (Q^-) at $\bar{x} \in A(\bar{t})$.

Theorem 2.1. Let G, $A(t)$, H_0 be as above. For $t \in G$, let $U(t) \subset E^m$ be a nonempty subset of the u-space E^m. Let

$$M = \{(t, x, u) \mid t \in G, x \in A(t), u \in U(t)\} \subset E^{\nu+n+m}.$$

Let $f: M \to E^r$ be a given function satisfying the following Carathéodory-type condition:

(C) Given $\epsilon > 0$, there exists a compact subset $K \subset G - H_0$ with $|G-K| < \epsilon$ and such that the sets

$$M_K = \{(t, x, u) \in M \mid t \in K\} \quad \text{and} \quad A_K = \{(t, x) \in A \mid t \in K\}$$

are closed and f is continuous on M_K.

Let

$$Q(t, x) = \{z \mid z = f(t, x, u), u \in U(t)\}.$$

Let $x_k(t)$, $u_k(t)$, $x(t)$, $t \in G$, $k = 1, 2, \ldots$, be a.e. finite, measurable functions on G with $x_k(t) \to x(t)$ in measure in G, and such that the functions

$$\delta_k(t) = f(t, x_k(t), u_k(t)) - f(t, x(t), u_k(t))$$

converge in measure to zero. Let us also assume that the sets $Q(t, x(t))$ are closed and convex for almost all $t \in G$. Then, there is a subsequence $[k_s]$, such that $x_{k_s}(t) \to x(t)$ pointwise a.e. in G as $s \to \infty$; and, for almost all $t \in G$, we also have

$$Q(t, x(t)) \supset \bigcap_{h=1}^{\infty} \operatorname{cl} \operatorname{co} \{f(t, x_{k_s}(t), u_{k_s}(t)) \mid s \geq h\}. \tag{3}$$

Proof. The proof is found in Ref. 8. In that paper, the functions $\delta_k(t)$ were assumed to converge strongly to zero, but the proof in Ref. 8 uses only the convergence in measure.

Remark 2.1. See Example 5.4 for the case where $\delta_k \to 0$ weakly, but above relation (3) does not hold.

Remark 2.2. Statements analogous to the ones above are also valid for the sets

$$\tilde{Q}(t, x) = \{(z^0, z) \mid z^0 \geq f_0(t, x, u), z = f(t, x, u), u \in U(t)\}.$$

More precisely, let the sets $U(t)$ depend only on t, let $\tilde{f} = (f_0, f) = \tilde{f}(t, x, u)$ satisfy condition (C), let $x(t)$, $x_k(t)$, $u_k(t)$, $t \in G$, $k = 1, 2, \ldots$, be measurable functions, let the sets $\tilde{Q}(t, x(t))$ be closed and convex for almost all $t \in G$, and let $\tilde{\delta}_k(t) = (\delta_k^0(t), \delta_k(t))$ be defined as usual by

$$\tilde{\delta}_k(t) = \tilde{f}(t, x_k(t), u_k(t)) - \tilde{f}(t, x(t), u_k(t)).$$

If both $\tilde{\delta}_k \to 0$ and $x_k \to x$ in measure in G as $k \to \infty$, then there is a subsequence k_s, $s = 1, 2, \ldots$, such that $x_{k_s}(t) \to x(t)$ pointwise a.e. in G as

$s \to \infty$; and, for almost all $t \in G$, we also have

$$\tilde{\tilde{Q}}(t, x(t)) \supset \bigcap_{h=1}^{\infty} \text{cl co}\{(z^0, z) \mid z^0 \geq f_0(t, x_{k_s}(t), u_{k_s}(t)),$$

$$z = f(t, x_{k_s}(t), u_{k_s}(t)), x \geq h\}.$$

Following is an analogue of a Nemitsky's theorem (see Lemma 2.1, p. 20, Ref. 16, and Theorem 17.4, p. 355, Ref. 17).

Theorem 2.2. Let $G \subset E^\nu$, $H_0 \subset G$, $A(t)$, $U(t)$, M and f as in Theorem 2.1. In particular, f satisfies property (C); and, for $t \in G - H_0$, $A(t)$ is a closed subset of E^n. Let $x(t)$, $x_k(t)$, $u_k(t)$, $t \in G$, $k = 1, 2, \ldots$, be a.e. finite, measurable functions on G with $x_k \to x$ in measure in G as $k \to \infty$, and $u_k \in (L_1(G))^m$, $\|u_k\|_1 \leq L$, $k = 1, 2, \ldots$, for some constant L. Then, the functions

$$\delta_k(t) = f(t, x_k(t), u_k(t)) - f(t, x(t), u_k(t))$$

converge to zero in measure in G as $k \to \infty$.

Proof. Given $\epsilon > 0$ and $\eta > 0$, we have to prove that there exists $N_0(\epsilon, \eta)$ such that $k > N_0(\epsilon, \eta)$ implies that

$$\text{meas}\{t \in G \mid |\delta_k(t)| > \eta\} < \epsilon.$$

Since $x(t)$ is a.e. finite, there is an $N_1 > 0$ such that, whenever $N > N_1$, the set

$$S(x, N) = \{t \mid |x(t)| \leq N\}$$

satisfies the condition

$$|G - S(x, N)| < \epsilon/4.$$

Since $\|u_k\|_1 \leq L$, for each $N > 0$, the set

$$S(u_k, N) = \{t \in G \mid |u_k(t)| \leq N\}$$

satisfies

$$|G - S(u_k, N)| \leq L/N.$$

Let $N_2 > 0$ be such that $L/N_2 < \epsilon/4$.

Let $K \subset G$ be a compact subset of G such that $|G - K| < \epsilon/4$ and f restricted to the set M_k be continuous. Then, the set

$$M(K, N_1, N_2) = \{(t, x, u) \in M \mid t \in K, |x| \leq N_1 + 1, |u| \leq N_2\}$$

is compact and f is continuous, hence uniformly continuous, on $M(K, N_1, N_2)$. Let $0 < \delta < 1$ be so chosen that

$$|(t, x, u) - (t', x', u')| \leq \delta$$

implies that

$$|f(t, x, u)-f(t', x', u')|<\eta$$

for $(t, x, u), (t', x', u')\in M(K, N_1, N_2)$.

Since $x_k(t)\to x(t)$ in measure, there exists $N_3(\epsilon, \delta)$ such that the set

$$S(x_k, x, \delta)=\{t\,|\,|x_k(t)-x(t)|\leqslant\delta\}$$

satisfies

$$|G-S(x_k, x, \delta)|<\epsilon/4$$

for $k>N_3(\epsilon, \delta)$. Now, for $k>N_3(\epsilon, \delta)$ and for

$$t\in S=K\cap S(x_k, x, \delta)\cap S(x, N_1)\cap S(u_k, N_2),$$

we have

$$|x_k(t)-x(t)|\leqslant\delta$$

and

$$|x|\leqslant N_1, \qquad |x_k|\leqslant N_1+1, \qquad |u_k|\leqslant N_2,$$

and

$$|f(t, x_k(t), u_k(t))-f(t, x(t), u_k(t))|<\eta.$$

But,

$$|G-S|<\epsilon/4+\epsilon/4+\epsilon/4<\epsilon,$$

and Theorem 2.2 is thereby proved.

Remark 2.3. In the above theorem, we need less than L_1-boundedness of the controls u_k. Indeed, we need only boundedness in measure; that is, given $\epsilon>0$, there is $L_\epsilon>0$ such that

$$\text{meas}\,\{t\in G\,|\,|u_k(t)|\geqslant L_\epsilon\}<\epsilon \qquad \text{for all } k=1, 2, \ldots .$$

3. Closure Theorems

The following closure theorems for orientor fields (Theorem 3.1) and Mayer problems (Theorem 3.2) are valid in view of the remarks in Ref. 4 (see also Ref. 7). Alternate direct proofs are omitted here, since these theorems can be considered as particular cases of the lower closure theorems of the next section.

Theorem 3.1. Let $G \subset E^\nu$, $|G| < \infty$, $H_0 \subset G$, $|H_0| = 0$, and let $A(t) \subset E^n$, $Q(t, x) \subset E^r$ be given nonempty subsets for $t \in G$, $x \in A(t)$. Let $\xi(t)$, $x(t)$, $\xi_k(t)$, $\bar{\xi}_k(t)$, $\delta_k(t)$, $k = 1, 2, \ldots$, be measurable functions such that

$$\xi, \xi_k \in (L_1(G))^r, \qquad x(t) \in A(t), \qquad \bar{\xi}_k(t) \in Q(t, x(t)),$$
$$\delta_k(t) = \xi_k(t) - \bar{\xi}_k(t), \qquad t \in G \text{ a.e.}, \qquad k = 1, 2, \ldots,$$

and $\xi_k \to \xi$ weakly in $(L_1(G))^r$ and $\delta_k \to 0$ in measure in G as $k \to \infty$; and finally, let the sets $Q(t, x(t))$ be closed and convex for $t \in G - H_0$. Then,

$$\xi(t) \in Q(t, x(t)), \qquad t \in G \text{ a.e.}$$

Remark 3.1. An aspect of the above theorem is highlighted by introducing the notion of *approximate weak convergence.* We shall say that a sequence $\bar{\xi}_k$ in $(L_1(G))^r$ converges to $\xi \in (L_1(G))^r$ *approximately weakly,* if there exists a sequence $\xi_k(t)$ in $(L_1(G))^r$ such that (i) $\bar{\xi}_k - \xi_k$ converges to zero in measure and (ii) ξ_k converges to ξ weakly in $(L_1(G))^r$.

With this notion, the above theorem can now be considered as a closure theorem under approximate weak convergence; thus, if $\bar{\xi}_k(t) \in Q(t, x(t))$ and $\bar{\xi}_k \to \xi$ approximately weakly in $(L_1(G))^r$, then $\xi(t) \in Q(t, x(t))$ a.e.

Remark 3.2. It is to be noted that there are no constraints on $\xi_k(t)$, except those required by the definition of approximate weak convergence. In view of this, we obtain the following corollary.

Corollary 3.1. Let G, H_0, $A(t)$, $\xi(t)$, $x(t)$, $\xi_k(t)$, $\bar{\xi}_k(t)$, and $Q(t, x(t))$ be as in Theorem 3.1. Further, let there exist functions $\varphi_k(t)$, $\psi_k(t)$, $\phi(t)$, $\psi(t)$ in $(L_1(G))^r$ such that (i) $\xi_k(t) - \bar{\xi}_k(t) = \varphi_k(t) + \psi_k(t)$, $k = 1, 2, \ldots$, and (ii) $\varphi_k(t) \to \varphi(t)$ in measure in G and $\psi_k \to \psi$ weakly in $(L_1(G))^r$. Then,

$$\xi(t) - \varphi(t) - \psi(t) \in Q(t, x(t)), \quad t \in G \text{ a.e.}$$

Proof. Let $\xi'_k(t) = \xi_k(t) - \psi_k(t) - \varphi(t)$ and $\delta'_k(t) = \varphi_k(t) - \varphi(t)$, $k = 1, 2, \ldots$. Then, $\xi'_k \to \xi - \psi - \phi$ weakly in $(L_1(G))^r$ and $\delta'_k \to 0$ in measure as $k \to \infty$. Thus, by Theorem 3.1, $\xi(t) - \psi(t) - \varphi(t) \in Q(t, x(t))$, $t \in G$ a.e.

Remark 3.3. In the above corollary, $\xi_k - \bar{\xi}_k$ may not converge to zero in measure, even if $\varphi(t) = \psi(t) = 0$.

The following closure theorem for Mayer problems follows directly from Theorem 3.1.

Theorem 3.2. Let G, A, M, f, H_0, $Q(t, x)$ be as in Theorem 2.1. Let $x_k(t)$, $x(t)$, $\xi_k(t)$, $\xi(t)$, $u_k(t)$, $\delta_k(t)$, $t \in G$, $k = 1, 2, \ldots$, be measurable functions, such that

$$\xi, \xi_k \in (L_1(G))^r, \qquad x_k(t) \in A(t), \qquad u_k(t) \in U(t),$$
$$\xi_k(t) = f(t, x_k(t), u_k(t)),$$
$$\delta_k(t) = f(t, x_k(t), u_k(t)) - f(t, x(t), u_k(t)), \qquad t \in G, k = 1, 2, \ldots,$$

and let

$$x_k \to x \text{ and } \delta_k \to 0 \text{ in measure in } G \text{ and } \xi_k \to \xi \text{ weakly in } (L_1(G))^r \text{ as } k \to \infty.$$

Let $Q(t, x(t))$ be convex and closed for $t \in G$ a.e. Then, there exists a measurable function $u(t)$, $t \in G$, such that

$$x(t) \in A(t), \qquad u(t) \in U(t), \qquad \xi(t) = f(t, x(t), u(t)), \qquad t \in G \text{ a.e.}$$

Remark 3.4. In the above theorem, we have more information on the sequence $\xi_k(t)$, which can be used to obtain convergence (to zero in measure) of $\delta_k(t)$ under the assumption of boundedness (in norm or in measure) of the sequence u_k of controls. Also, in the above theorem, we have $x_k(t) \in A(t)$, $x_k \to x$ in measure. By a suitable choice of subsequences and omitting sets of measure zero, we obtain $x_k(t) \to x(t)$ for almost all t. Since $A(t)$ is closed, it follows that $x(t) \in A(t)$ a.e.

4. Lower Closure Theorems

The following theorem is valid in view of Theorem 2.1 above, Remark 7 of Ref. 4, and the remarks immediately preceding Theorem (7.iv) of Ref. 4. However, we present here for convenience a direct proof which essentially is the proof of Theorem (5.i) of Ref. 4.

Theorem 4.1. Let $G \subset E^\nu$ be of finite measure, and let $A(t) \subset E^n$, $\tilde{Q}(t, x) \subset E^{1+r}$ be given nonempty subsets for $t \in G$, $x \in A(t)$. Let $x(t)$, $\xi_k(t)$, $\xi(t)$, $\eta_k(t)$, $\bar{\xi}_k(t)$, $\bar{\eta}_k(t)$, $\lambda_k(t)$, $\lambda(t)$ be a.e. finite, measurable functions on G. Let

$$\xi, \xi_k, \bar{\xi}_k \in (L_1(G))^r, \qquad x(t) \in A(t), \qquad (\bar{\eta}_k(t), \bar{\xi}_k(t)) \in \tilde{Q}(t, x(t)),$$
$$t \in G \text{ a.e.}, \quad k = 1, 2, \ldots,$$

and let $\tilde{\delta}_k$ denote (δ_k^0, δ_k) with

$$\delta_k = \xi_k - \bar{\xi}_k \qquad \text{and } \delta_k^0 = \eta_k - \bar{\eta}_k.$$

Let $\xi_k \to \xi$ weakly in $(L_1(G))^r$ and $\tilde{\delta}_k \to 0$ in measure in G as $k \to \infty$,

$$-\infty < i = \liminf_{k \to \infty} \int_G \eta_k(t)\, dt < \infty.$$

Let $\eta_k(t) \geq \lambda_k(t)$, λ, $\lambda_k \in L_1(G)$, $\lambda_k \to \lambda$ weakly in $L_1(G)$ as $k \to \infty$. Let $\tilde{\tilde{Q}}(t, x(t))$ be closed and convex for almost all $t \in G$. Then, there exists a function $\eta(t)$, $t \in G$, $\eta \in L_1(G)$, such that

$$(\eta(t), \xi(t)) \in \tilde{\tilde{Q}}(t, x(t)), \qquad t \in G \text{ a.e.}, \qquad \text{and} \int_G \eta(t)\, dt \leq i.$$

Proof. Let

$$j_k = \int_G \eta_k(t)\, dt,$$

so that

$$i = \liminf_{k\to\infty} j_k.$$

By choosing a suitable subsequence, we may assume that $j_k \to i$ as $k \to \infty$. Thus, there exists a set $H_1 \subset G$ of measure zero such that for $t \in G - H_1$, $x(t) \in A(t)$, $x(t)$ is finite, $\tilde{\tilde{Q}}(t, x(t))$ is convex and closed, and $\tilde{\delta}_k(t) \to 0$ as $k \to \infty$. For each $s = 1, 2, \ldots,$ let us consider the sequences λ_{s+k}, ξ_{s+k}, $k = 1, 2, \ldots,$ which converge weakly to λ and ε as $k \to \infty$ in $L_1(G)$ and $(L_1(G))^r$, respectively. Then, by Mazur theorem, there are convex combinations

$$\lambda_N^{(s)}(t) = \sum_{k=1}^N c_{Nk}^{(s)} \lambda_{s+k}(t), \qquad \xi_N^{(s)}(t) = \sum_{k=1}^N c_{Nk}^{(s)} \xi_{s+k}(t), \qquad t \in G,$$

$N = 1, 2, \ldots,$ such that $\lambda_N^{(s)} \to \lambda$ and $\xi_N^{(s)} \to \xi$ in the L_1-norm as $N \to \infty$. This is true for each s. Hence, there exists a set H_s of measure zero and a sequence of integers $N_h^{(s)}$, $h = 1, 2, \ldots,$ such that, for $t \in G - H_s$, both $\lambda(t)$ and $\xi(t)$ are finite and

$$\lambda_{N_h}^{(s)}(t) \to \lambda(t), \qquad \xi_{N_h}^{(s)}(t) \to \xi(t), \qquad \text{as } h \to \infty.$$

Since

$$\eta_k(t) \geq \lambda_k(t), \qquad \int_G \eta_k(t)\, dt = j_k, \qquad k = 1, 2, \ldots,$$

we have, for $s = 1, 2, \ldots, N = 1, 2, \ldots,$

$$\eta_N^{(s)}(t) \geq \lambda_N^{(s)}(t), \qquad t \in G, \qquad i - \rho_s \leq \int_G \eta_N^{(s)}(t)\, dt \leq i + \rho_s,$$

where $\eta_N^{(s)}(t)$ is constructed by the same convex combination as $\lambda_N^{(s)}(t)$ and

$$\rho_s = \max\{|j_k - i|, k \geq s+1\} \to 0 \qquad \text{as } s \to \infty.$$

For $N=N_h^{(s)}$ and $h\to\infty$, using Fatou's lemma, we have

$$\eta^{(s)}(t)=\liminf_{h\to\infty}\eta_{N_h}^{(s)}(t)\geq\lambda(t),\qquad t\in G \text{ a.e.},$$

$$\int_G \eta^{(s)}(t)\,dt\leq\liminf_{h\to\infty}\int_G \eta_{N_h}^{(s)}(t)\,dt\leq i+\rho_s,\qquad s=1,2,\ldots.$$

Thus, $\eta^{(s)}(t)$ is finite in $G-H_s'$ with $|H_s'|=0$ and is also in $L_1(G)$. Finally, let

$$\eta(t)=\liminf_{s\to\infty}\eta^{(s)}(t),\qquad t\in G,$$

so that again

$$\eta(t)\geq\lambda(t),\qquad t\in G,\qquad \int_G\eta(t)\,dt\leq i.$$

Thus, $\eta(t)$ is finite a.e. in G. Let H_0'' be the set of points t for which $\eta(t)$ is not finite, and hence $|H_0''|=0$. Let H denote the union of H_0, H_1, H_0'', and all sets H_s and H_s'. Then, $|H|=0$. Let $t_0\in G-H$ and $x_0=x(t_0)$. Then, $\tilde{\delta}_k(t_0)\to 0$ as $k\to\infty$. also,

$$\xi_{s+k}(t_0)-\delta_{s+k}(t_0)=\bar{\xi}_{s+k}(t_0)\quad\text{ and }\quad \eta_{s+k}(t_0)-\delta_{s+k}^0(t_0)=\bar{\eta}_{s+k}(t_0),$$

so that, by convexity of $\tilde{\tilde{Q}}(t_0,x_0)$, we have

$$\left(\sum_{k=1}^{N}c_{Nk}^{(s)}(\eta_{s+k}(t_0)-\delta_{s+k}^0(t_0),\ \sum_{k=1}^{N}c_{Nk}^{(s)}(\xi_{s+k}(t_0)-\delta_{s+k}(t_0))\right)\in\tilde{\tilde{Q}}(t_0,x).\qquad(4)$$

Finally, for $N=N_h^{(s)}$ and $h\to\infty$, the points in the first member of (4) form a sequence for which $(\eta^{(s)}(t_0),\xi(t_0))$ is an accumulation point in E^{1+r}. It is to be noted that $\tilde{\delta}_k\to 0$ and that $\eta^{(s)}(t_0)$, $\xi(t_0)$ are both finite. Thus, $(\eta^{(s)}(t_0),\xi(t_0))\in\tilde{\tilde{Q}}(t_0,x_0)$. Finally,

$$\eta(t_0)=\liminf_{s\to\infty}\eta^{(s)}(t_0)$$

is finite, and hence $(\eta(t_0),\xi(t_0))\in\tilde{\tilde{Q}}(t_0,x_0)$. Theorem 4.1 is thereby proved.

Remark 4.1. As in Corollary 3.1, we may have $\tilde{\delta}_k=\tilde{\varphi}_k+\tilde{\psi}_k$ with $\tilde{\varphi}_k=(\varphi_k^0,\varphi_k)$, $\tilde{\psi}_k=(\psi_k^0,\psi_k)$, $\tilde{\varphi}_k\to\tilde{\varphi}$ in measure, and $\tilde{\psi}_k\to\tilde{\psi}$ weakly in $L_1(G)$. Then,

$$(\eta-\varphi^0-\psi^0,\xi-\varphi-\psi)\in\tilde{\tilde{Q}}(t,x(t))\text{ a.e.},$$

$$\int_G(\eta(t)-\varphi^0(t)-\psi^0(t))\,dt\leq i.$$

In particular, some of the coordinates ϕ^i, ψ^i, $i=0,\ldots,r$ may be zero.

The following theorem is a direct consequence of Theorem 4.1.

Theorem 4.2. Let $G \subset E^\nu$, $|G| < \infty$, $H_0 \subset G$, $|H| = 0$, $A(t)$, $t \in G$ be as above. Let $U(t) \subset E^m$ and

$$M = \{(t, x, u) \mid t \in G, x \in A(t), u \in U(t)\}$$

as in Theorem 2.1. Let $\tilde{f} = (f_0, f) : M \to E^{1+r}$ satisfy property (C). Let

$$\tilde{\tilde{Q}}(t, x) = \{(z^0, z) \mid z^0 \geq f_0, z = f\}, \qquad x \in A(t).$$

Let $\xi(t)$, $x(t)$, $\xi_k(t)$, $x_k(t)$, $\eta_k(t)$, $u_k(t)$, $\lambda(t)$, $\lambda_k(t)$, $t \in G$, $k = 1, 2, \ldots$, be a.e. finite, measurable functions such that

$$\xi, \xi_k \in (L_1(G))^r, \qquad \eta_k \in L_1(G), \qquad x_k(t) \in A(t), \qquad u_k(t) \in U(t),$$

$$(\eta_k(t), \xi_k(t)) = \tilde{f}(t, x_k(t), u_k(t)), \qquad \tilde{\delta}_k(t) = \tilde{f}(t, x_k(t), u_k(t)) - \tilde{f}(t, x(t), u_k(t)),$$

where

$$\tilde{\delta}_k = (\delta_k^0, \delta_k) \qquad \text{and } \tilde{f} = (f_0, f).$$

Let $\tilde{\tilde{Q}}(t, x(t))$ be convex and closed for $t \in G - H_0$. Let

$$\xi_k \to \xi \text{ weakly in } (L_1(G))^r, x_k \to x \text{ and } \tilde{\delta}_k \to 0 \text{ in measure,}$$

$$\eta_k(t) \geq \lambda_k(t), \qquad \lambda, \lambda_k \in L_1(G), \qquad \lambda_k \to \lambda \text{ weakly in } L_1(G),$$

$$-\infty < i = \liminf_{k \to \infty} \int_G \eta_k(t)\, dt < \infty.$$

Then, there exists a measurable function $u(t)$, $t \in G$ such that

$$x(t) \in A(t), \qquad u(t) \in U(t), \qquad \xi(t) = f(t, x(t), u(t));$$

and if

$$\eta(t) = f_0(t, x(t), u(t)), \qquad t \in G,$$

then $\int_G \eta(t)\, dt$ exists (finite or $-\infty$) and is $\leq i$.

Remark 4.2. In view of Theorem 2.2, since f satisfies property (C), the functions $\tilde{\delta}_k \to 0$ in measure if $x_k \to x$ in measure and the sequence u_k is bounded in the L_1-norm, say $\|u_k\|_1 \leq L$, for some constant L, $k = 1, 2, \ldots$.

Remark 4.3. We can say that $\eta(t) \in L_1(G)$ provided $f_0 \geq 0$ or f_0 satisfies any one of the weaker conditions (α_0), (β_0), (γ_0), (δ_0) of Remark 11 of Ref. 4.

Remark 4.4. Theorem 4.2 above includes the results presented by Berkovitz in Ref. 13 as well as Ref. 14. In particular, we do not require the norm boundedness of the trajectories x_k. Also, we require the convexity of the sets $\tilde{\tilde{Q}}(t, x)$ only for $x = x(t)$, that is, along the limiting trajectory alone. Another point of departure from Ref. 14 is that we work with L_1-spaces which definitely includes the situation with L_{p_i}-spaces, $p_i \geq 1$, since the domain G is of finite measure.

Remark 4.5. As observed in Ref. 7, no condition on δ_k is needed if ξ_k are known to converge strongly or in measure to some ξ. This is certainly the case in control problems where ξ_k denote the partial derivatives of order less than or equal to some integer, say d, of the trajectories x_k and we assume x_k to converge weakly in some Sobolev space of order larger than d.

Remark 4.6. In Theorem 4.1 (and correspondingly others), we have used a weak form of property (Q) in assuming $\tilde{\delta}_k \to 0$ in measure. In a different approach (see Ref. 5), the same conclusions are obtained using property (Q) merely of the sections $\tilde{\tilde{Q}}(t, x) \cap (E^1 \times V(0, N))$ rather than of the whole sets $\tilde{\tilde{Q}}(t, x)$; here, $V(0, N)$ denotes the ball of radius N around the origin in E^r. This assumption on the sections is satisfied under property (K) and boundedness below. Under this weak form, we need an additional assumption of the form $p(t) \in \tilde{\tilde{Q}}(t, x)$. $x \in A(t)$, $t \in G$, a.e., for some $p \in (L_1(G))^{1+r}$. Variations of this geometric condition are found in Ref. 14.

5. Examples

The following examples illustrate the various aspects of the theorems above. Analogous examples are found in Refs. 2, 4, 5, 7, 8 and collected in Ref. 17.

Example 5.1. In reference to Theorem 2.2, we observe that δ_k may converge to zero in measure even if the sequence $\|u_k\|_1$, $k = 1, 2, \ldots$, is not bounded. Indeed, let

$$G = \{t \mid 0 \leq t \leq 1\},$$

$$A(t) = U(t) = E^1, \qquad t \in [0, 1],$$

$$f(t, x, u) = xu \qquad \text{for } (t, x, u) \text{ in } M = [0, 1] \times E^2,$$

$$x_k(t) = k^{-2}, \qquad t \in G,$$

and

$$u_k(t) = k\varphi_k(t),$$

where

$$\varphi_k(t) = +1 \qquad \text{for } ik^{-1} \leq t \leq (2i+1)(2k)^{-1},$$

and

$$\varphi_k(t) = -1 \qquad \text{for } (2i+1)(2k)^{-1} \leq t \leq (i+1)k^{-1},$$

with $i = 0, \ldots, n-1$, $k = 1, 2, \ldots$. Then, $\|u_k\|_1 = k$ and is clearly not bounded. However, $x_k \to 0$ uniformly, and hence in measure, and also

$$\delta_k(t) = f(t, x_k(t), u_k(t)) - f(t, x(t), u_k(t)) = x_k(t)u_k(t) = k^{-1}\varphi_k(t) \to 0$$

uniformly, and hence in measure.

Example 5.2. Same as above, but with

$$x_k(t) = k \qquad \text{for } 0 \leq t \leq k^{-1},$$

$$x_k(t) = 0 \qquad \text{for } k^{-1} < t \leq 1,$$

$$u_k(t) = k, \qquad t \in [0, 1].$$

Then,

$$x(t) = 0, \qquad 0 \leq t \leq 1,$$

and

$$\delta_k(t) = x_k(t)u_k(t) - 0 = k^2 \qquad \text{for } 0 \leq t \leq k^{-1},$$

$$\delta_k(t) = 0 \qquad \text{for } k^{-1} < t \leq 1.$$

Clearly, $\delta_k(t) \to 0$ in measure but not weakly.

Example 5.3. This is to show that, in the notations of Theorem 3.2 above, if $\delta_k(t)$ converges to zero in measure but not weakly, then Theorem 3.2 can be applied but not Theorem 5.ii of Ref. 7. Let $G = [0, 1]$ and $A(t) = U(t) = E^1$ as above. Let

$$f(t, x, u) = 0 \qquad \text{if } x \geq u,$$

$$f(t, x, u) = u - x \qquad \text{if } u \geq x.$$

Clearly, f is continuous on $M = [0, 1] \times E^2$. Let

$$x_k(t) = u_k(t) = k \qquad \text{for } 0 \leq t \leq k^{-1},$$

$$x_k(t) = u_k(t) = 0 \qquad \text{for } k^{-1} < t \leq 1.$$

Then,

$$f(t, x_k(t), u_k(t)) = 0$$

and $x_k(t) \to x(t)$ in measure with

$$x(t) = 0, \qquad 0 \leq t \leq 1,$$

so that

$$f(t, x(t), u_k(t)) = u_k(t)$$

and

$$\delta_k(t) = -u_k(t),$$

which converges to zero in measure but not weakly. In this case,

$$\xi_k(t) = f(t, x_k(t), u_k(t)) = 0,$$

and thus $\xi_k(t) \to \xi(t)$ weakly with $\xi(t) = 0$. Theorem 3.2 applies, and there exists a measurable $u(t)$ with

$$\xi(t) = f(t, x(t), u(t));$$

indeed, we may choose

$$u(t) = 0, \qquad 0 \leq t \leq 1.$$

Example 5.4. This is to show that, if the sequence $\delta_k(t)$ does not converge to zero in measure, then Theorem 3.2 above cannot be applied, but we can apply Theorem 5.ii of Ref. 7 if $\delta_k \to 0$ weakly in L_1. As before, let

$$G = [0, 1], \qquad A(t) = U(t) = E^1, \qquad f(t, x, u) = xu,$$

and

$$x_k(t) = k^{-1}.$$

Let $u_k(t) = k\varphi_k(t)$ as in Example 5.1 above. Then, $\|u_k\|_1 = k$ and $x(t) = 0$. Also,

$$\delta_k(t) = f(t, x_k(t), u_k(t)) - f(t, x(t), u_k(t)) = f(t, x_k(t), u_k(t))$$
$$= \xi_k(t) = x_k(t)u_k(t) = \varphi_k(t),$$

with $\varphi_k(t)$ as in Example 5.1. Thus, $\delta_k(t) = \xi_k(t) = \varphi_k(t)$ converges weakly in $L_1(G)$ to 0 as $k \to \infty$. Hence, $\xi = 0$, and Theorem 5.ii of Ref. 7 applies. Indeed, $\xi(t) = x(t)u(t)$ with $u(t) = 1$ for $t \in [0, 1]$. However, Theorem 3.2 does not apply since $\delta_k(t)$ does not converge to 0 in measure. Note that Eq. (3) does not hold in this case.

Example 5.5. This is to show that Theorem 3.2 (and similarly Theorem 5.1) can be applied if the sets

$$Q(t, x(t)) = f(t, x(t), u(t))$$

are closed and convex, while $Q(t, x)$ for other x may not be convex. Let

$$G=[0, 1] \quad \text{and } A(t)=E^1, \quad U(t)=\{-1, +1\}.$$

Let

$$f(t, x, u)=xu,$$

which is clearly continuous on $M=[0, 1]\times E^1\times\{-1, +1\}$. Let

$$x_k(t)=1 \quad \text{for } 0\leq t\leq k^{-1}, \quad x_k(t)=0 \quad \text{for } k^{-1}<t\leq 1.$$

Let $u_k(t)=\varphi_k(t)$, where $\varphi_k(t)$ are as in Example 5.1. Here,

$$\xi_k(t)=f(t, x_k(t), u_k(t))=+1 \quad \text{for } 0\leq t<(2k)^{-1},$$
$$\xi_k(t)=-1 \quad \text{for } (2k)^{-1}\leq t<k^{-1},$$
$$\xi_k(t)=0 \quad \text{for } k^{-1}\leq t\leq 1.$$

Also, $x_k(t)\to x(t)$ with $x(t)=0$ for $0\leq t\leq 1$. Thus, $\delta_k=\xi_k$, and both converge to 0 strongly and in measure. Thus, $\xi(t)=0$, and we can apply Theorem 4 above. In fact, $\xi(t)=x(t)u(t)$ with $u(t)=+1$, $0\leq t\leq 1$. Here, the sets $Q(t, x)$ are $\{-x, x\}$ and are not convex, unless $x=0$.

Example 5.6. This is to show that, if δ_k does not converge to 0 either weakly or in measure, then the closure property may not hold. Let

$$G=[0, 1] \quad \text{and } A(t)=U(t)=E^1$$

and

$$f(t, x, u)=xu \quad \text{for all } (t, x, u)\in M=[0, 1]\times E^2.$$

Let

$$\psi_k(t)=t-ik^{-1} \quad \text{for } ik^{-1}\leq t<ik^{-1}+(2k)^{-1},$$
$$\psi_k(t)=(i+1)k^{-1}-t \quad \text{for } (2i+1)(2k)^{-1}\leq t<(i+1)k^{-1},$$

with $i=0,\dots,n-1$, $k=1, 2, \dots$. Let

$$x_k(t)=2\psi_k(t) \quad \text{and } u_k(t)=k, \quad 0\leq t\leq 1.$$

Then,

$$\xi_k(t)=f(t, x_k(t), u_k(t))=2k\psi_k(t)$$

and

$$0\leq \xi_k(t)\leq 1, \qquad \int_0^1 \xi_k(t)\, dt=2^{-1}.$$

Here, $\xi_k \to \xi$ weakly in $L_1(G)$ and $x_k \to x$ uniformly in $[0, 1]$ if $\xi(t) = 2^{-1}$ and $x(t) = 0$, $0 \leq t \leq 1$. In this case, there is no measurable function $u(t)$ such that $\xi(t) = x(t)u(t)$ a.e. in G, because $\xi = 2^{-1}$ and $x = 0$. Also, the sets $Q(t, x(t))$ are $f(t, x(t), U(t)) = \{0\}$ and are clearly closed and convex for all t. In fact, $Q(t, x) = E^1$ for $x \neq 0$ and $Q(t, 0) = \{0\}$. Theorems 3.2 of this paper and Theorem 5.ii of Ref. 7 fail because $\delta_k(t) = \xi_k(t)$ and δ_k does not converge to 0 in measure and δ_k converges weakly to 2^{-1} and not to 0 (see Corollary 3.1).

Example 5.7. This is to show that property (C) of f as a function of (t, x, u) is relevant. Let G, $A(t)$, $U(t)$, and M be as in the above example. Let

$$f(t, x, u) = x \qquad \text{if } u = \pm n\pi/2,$$

$$f(t, x, u) = x \tan u \qquad \text{otherwise.}$$

Let ξ_k, x_k, δ_k be as above. Let $u_k(t) = \arctan(k)$, $0 \leq t \leq 1$. In this case, $x_k \to x$ and $\xi_k \to \xi$ with $x(t) = 0$ and $\xi(t) = 2^{-1}$. Again, $\xi(t)$ cannot be written as $f(t, x(t), u(t))$ for any measurable function $u(t)$, $t \in G$. Here, once again δ_k converges to zero neither in measure nor weakly. The function f does not satisfy property (C).

Example 5.8. This is to show the relevance of convexity of the sets $Q(t, x(t))$. Let

$$G = [0, 1],$$

and let

$$A(t) = \{x : x \geq 0\}, \qquad U(t) = \{1, -1\} \qquad \text{for all } t \in [0, 1].$$

Let

$$x_k(t) = k \qquad \text{for } 0 \leq t \leq k^{-1},$$

$$x_k(t) = 0 \qquad \text{for } k^{-1} < t \leq 1.$$

Let $u_k(t) = \varphi_k(t)$, where φ_k are as in Example 5.1. Then, $x_k \to 0$ in measure and $u_k \to 0$ weakly. Let

$$f(t, x, u) = u + x^{\frac{1}{2}}.$$

Then,

$$\xi_k(t) = f(t, x_k(t), u_k(t)) = u_k(t) + (x_k(t))^{\frac{1}{2}}$$

converges weakly to 0 in $L_1(G)$. Indeed,

$$\int_0^1 (x_k(t))^{\frac{1}{2}}\, dt = k^{-\frac{1}{2}} \to 0 \qquad \text{as } k \to \infty.$$

Also,

$$\delta_k(t) = (u_k(t) + (x_k(t))^{\frac{1}{2}}) - (u_k(t) + 0) = (x_k(t))^{\frac{1}{2}}$$

converges to zero strongly and in measure. Here, $\xi = 0$ and $x = 0$, so that

$$\xi(t) = u(t) + (x(t))^{\frac{1}{2}}$$

only if $u(t) = 0$, $0 \leq t \leq 1$; and, since $U(t) = \{1, -1\}$, this control function is not admissible. Theorem 3.2 and Theorem 5.ii of Ref. 7 do not apply, because $Q(t, x(t)) = \{1, -1\}$ are not convex. It is to be noted that, if $U(t)$ is chosen to be $[-1, 1]$ (the closed interval), then the sets $Q(t, x(t))$ are closed and convex, and Theorem 3.2 applies.

Example 5.9. This is to illustrate the lower bound restrictions on the integrands of cost functionals in the lower closure theorems. Let

$$G = [0, 1], \qquad A(t) = U(t) = E^1, \qquad 0 \leq t \leq 1.$$

For $(t, x, u) \in M = [0, 1] \times E^2$, let

$$f(t, x, u) = 0, \qquad f_0(t, x, u) = ut^{-\frac{1}{2}} \qquad \text{for } 0 < t \leq 1, \text{ and } f_0(0, x, u) = 0.$$

Let

$$x_k(t) = x(t) = 0 \qquad \text{for } 0 \leq t \leq 1,$$

$$u_k(t) = t^{-\frac{1}{2}} \qquad \text{for } 2^{-k-1} \leq t \leq 2^{-k},$$

$$u_k(t) = 0 \qquad \text{otherwise.}$$

Then, $u_k \to u$ strongly in $L_1([0, 1])$ if $u(t) = 0$, $t \in G$. In this case,

$$\int_0^1 f_0(t, x_k(t), u_k(t))\, dt = -\log 2 \qquad \text{and} \int_0^1 f_0(t, x(t), u(t))\, dt = 0.$$

Thus, the function $u(t) = 0$, $t \in [0, 1]$, which is the strong limit of $u_k(t)$, is not the measurable function guaranteed by Theorem 4.2. However, if we choose

$$u(t) = c, \qquad t \in [0, 1], \qquad \text{with } c < -2^{-1} \log 2,$$

then

$$\int_0^1 f_0(t, x(t), u(t))\, dt \leq \liminf \int_0^1 f_0(t, x_k(t), u_k(t))\, dt.$$

It is to be noted that here f_0 is not bounded below (by an L_1-function) uniformly in u.

Example 5.10. This is to illustrate that a choice of $u(t)$ as in the above example may make it inadmissible. Let

$$G=[0,1], \qquad A(t)=U(t)=E^1.$$

For $(t, x, u)\in M=G\times E^2$, let

$$f(t,x,u)=x+u, \qquad f_0(t,x,u)=xut^{-\frac{1}{2}} \qquad \text{for } 0<t\leq 1,$$
$$\text{and } f_0(0,x,u)=0.$$

Let

$$x_k(t)=1 \qquad \text{and } u_k(t)=-t^{-\frac{1}{2}} \qquad \text{for } 2^{-k-1}\leq t\leq 2^{-k},$$
$$x_k(t)=u_k(t)=0 \qquad \text{otherwise}.$$

Then, $x_k(t)\to x(t)$ in measure if $x(t)=0$, $t\in[0,1]$. Also,

$$\xi_k(t)=x_k(t)+u_k(t)\to 0$$

strongly in $L_1([0,1])$, so that $\xi(t)=0$. Now, $\xi(t)=f(t,x(t),u(t))$ is true only if $u(t)=0$, $t\in[0,1]$. But, in this case,

$$f_0(t,x(t),u(t))=0$$

and $\int_0^1 f_0(t,x(t),u(t))\,dt$ is not less than

$$\liminf \int_0^1 f_0(t,x_k(t),u_k(t))\,dt=-\log 2.$$

Here, once again, f_0 is not bounded below uniformly in u by an L_1-function. Moreover, it is not possible to choose a weakly convergent sequence $\lambda_k(t)$ such that

$$\eta_k(t)=f_0(t,x_k(t),u_k(t))\geq\lambda_k(t).$$

It is to be noted, however, that

$$\delta_k=(x_k+u_k)-(x+u_k)\to 0$$

as $k\to\infty$ in measure and

$$\delta_k^0=(x_ku_k-xu_k)t^{-\frac{1}{2}}=-t^{-1} \qquad \text{for } 2^{-k-1}\leq t\leq 2^{-k},$$
$$\delta_k^0(t)=0 \qquad \text{otherwise},$$

and thus $\delta_k^0\to 0$ in measure.

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