Characterization of the Darboux Point for Particular Classes of Problems[1]

P. M. MEREAU[2] AND W. F. POWERS[3]

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Abstract. A minimal sufficient condition for global optimality involving the Darboux point, analogous to the minimal sufficient condition of local optimality involving the conjugate point, is presented. The Darboux point is then characterized for optimal control problems with linear dynamics, cost functionals with a general terminal state term and an integrand quadratic in the state and control, and general terminal conditions. The Darboux point is shown to be the supremum of a sequence of conjugate points. If the terminal state term is quadratic, along with a scalar quadratic boundary condition, then the Darboux point is also the time at which the Riccati matrix becomes unbounded, giving a characterization of the unboundedness of the Riccati matrix at points which are not in general conjugate points.

Key Words. Calculus of variations, optimal control, global sufficient conditions, Darboux point, conjugate point.

1. Introduction and Preliminaries

The concept of a Darboux point was formalized in Ref. 1, and several related properties have been proved for a general class of problems (Refs. 1–2). The Darboux point appears as an extension to global optimality of the conjugate point concept, in the sense that it allows a minimal sufficient condition for global optimality (i.e., the gap between necessary and sufficient conditions is minimal). The following definition was proposed.[4]

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[4] This is Definition 2.1 of Ref. 1.

Definition 1.1.   Let $x^*(t)$, $t \in [t_0, t_f^*]$, be a trajectory which satisfies the minimum principle. A point $t_D \in [t_0, t_f^*)$ is called a *Darboux point* if the following conditions are satisfied[5]:

(i)   for all $t_2 \in (t_D, t_f^*)$, there does not exist an admissible trajectory $x(t)$, $t \in [t_D, t_f]$, with $x(t_2) = x^*(t_2)$, giving a smaller value to the cost functional between t_2 and t_f than $x^*(t)$ between t_2 and t_f^*;

(ii)   for all $t_1 \in [t_0, t_D)$, there exists an admissible trajectory $\hat{x}(t)$, $t \in [t_2, \hat{t}_f]$, with $x(t_1) = x^*(t_1)$, giving a smaller value to the cost functional between t_1 and $\hat{t}_f$ than $x^*(t)$ between t_1 and t_f^*.

It follows that a Darboux point on (t_0, t_f) indicates nonglobal optimality. One can make a distinction between two types of Darboux points: Type-1 Darboux points, indicating only the loss of global optimality; and Type-2 Darboux points, indicating also the loss of the existence of a solution to the problem (see Refs. 1–2 for details and examples).

In this paper, a method is proposed to characterize Darboux points for problems with linear dynamics, quadratic cost functionals, and analytic terminal constraints. A sufficient condition for global optimality will be presented in Section 2, and several formulas for the Darboux point will be obtained in Sections 3–6 for various classes of terminal conditions. Examples will be given to illustrate pertinent aspects of the Darboux point and its characterizations.

We shall consider problems of the following form: minimize the functional

$$J = g(x_f) + \frac{1}{2} \int_{t_0}^{t_f} [x^T C(t) x + u^T E(t) u] \, dt, \tag{1}$$

subject to

$$\dot{x} = A(t)x + B(t)u, \qquad t \in [t_0, t_f], \tag{2}$$

$$x(t_0) = x_0, \tag{3}$$

$$\psi(x_f) = 0, \tag{4}$$

where $x \in R^n$, $u \in R^m$, $m \le n$, $\psi \in R^p$, $p \le n$, t_0 and t_f are prescribed, $A(t)$, $B(t)$, $C(t)$, and $E(t)$ are continuous matrices on $[t_0, t_f]$ of dimensions consistent with the dimensions of x and u. The matrices $C(t)$ and $E(t)$ are symmetric. The following assumptions are made.

(A.1)   The problem is normal (Ref. 3).

(A.2)   The matrix $E(t)$ is positive definite on $[t_0, t_f]$ (strengthened Legendre–Clebsch condition).

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[5] When $t_D = t_0$, condition (ii) can be verified only if the trajectory $x^*(t)$ can be extended on a small interval $[t_0 - \epsilon, t_0)$.

(A.3) The system (A, B) is completely controllable, i.e., the matrix

$$W(t_1, t_2) = \int_{t_1}^{t_2} \phi(t_1, t)B(t)B^T(t)\phi(t_1, t)\, dt$$

is positive definite for all t_1, t_2, with $t_0 \le t_1 < t_2 \le t_f$, where $\phi(t, t_0)$ is the transition matrix of $\dot{x} = A(t)x$.

(A.4) The functions $g(\cdot)$ and $\psi_j(\cdot), j = 1, \ldots, p$, are analytic on the terminal set $\theta \triangleq \{x_f: \psi(x_f) = 0\}$.

Because of Assumption (A.2), there is no loss of generality in assuming no mixed term in the integrand of (1) (see Ref. 4).

First-order necessary conditions for an admissible pair (u, x) [i.e., constraints (2), (3), (4) are satisfied] to be optimal are that there exist an absolutely continuous function $\lambda(t) \in R^n$, $t \in [t_0, t_f]$, and a constant $\nu \in R^p$ such that

$$\dot{\lambda}(t) = -C(t)x(t) - A^T(t)\lambda(t), \qquad t \in [t_0, t_f], \tag{5}$$

$$E(t)u(t) + B^T(t)\lambda(t) = 0, \tag{6}$$

$$\lambda(t_f) = [dg(x_f)/dx_f]^T + [d\psi(x_f)/dx_f]^T \nu. \tag{7}$$

Suppose that we have an extremal pair $(u^*(t), x^*(t))$, $t \in [t_0, t_f]$, and let $\lambda^*(t)$, $t \in [t_0, t_f]$, and ν^* be the associated multiplier function and constant Lagrange multiplier. The definition of a Darboux point given above implies the following sufficient condition for global optimality.

Theorem 1.1. *Sufficient Condition.* An admissible pair $(u^*(t), x^*(t))$, $t \in [t_0, t_f]$, is a globally optimal pair if

(i)   the first-order necessary conditions are satisfied;
(ii) $E(t)$ is positive definite on $[t_0, t_f]$;
(iii) there is no Darboux point on $x^*(t)$ for $t \in [t_0, t_f)$.

This sufficient condition is minimal, since the only difference with necessary conditions lies in (ii), where $E(t)$ is required to be positive semidefinite on $[t_0, t_f]$, and in (iii), where a Darboux point is not allowed on (t_0, t_f) for global optimality. In this paper, we shall develop a characterization of the Darboux point which allows a test of (iii).

Define the $n \times n$ matrix $F(q)$ and the $p \times n$ matrix $T(q)$ as

$$F(q) = 2 \sum_{k=2}^{\infty} (1/k!)(d^k/dx_f^k)[G(x_f^*, \nu^*)](q - x_f^*)^{k-2}, \tag{8}$$

$$T^T(q) = [T_1(q), \ldots, T_p(q)], \tag{9}$$

$$T_j(q) = \sum_{k=1}^{\infty} (1/k!)(d^k/dx_f^k)[\psi_j(x_f^*)](q - x_f^*)^{k-1}, \qquad j = 1, \ldots, p, \tag{10}$$

in tensor notation, $x_f^* = x^*(t_f)$, q is a constant n-dimensional vector with values in the terminal set θ, and $G(x_f, \nu^*)$ is defined as

$$G(x_f, \nu^*) \triangleq g(x_f) + \nu^{*T}\psi(x_f). \tag{11}$$

Note that, for any $q \in \theta$, Assumption (A.4) and Eqs. (8)–(10) imply that

$$G(q, \nu^*) = G(x_f^*, \nu^*) + (d/dx_f)G(x_f^*, \nu^*)(q - x_f^*) + \tfrac{1}{2}(q - x_f^*)^T F(q)(q - x_f^*), \tag{12}$$

$$\psi(q) = \psi(x_f^*) + T(q)(q - x_f^*). \tag{13}$$

Now, let the matrices $S(t, q)$, $R(t, q)$, and $Q(t, q)$ be the solutions of

$$\dot{S} = -SA(t) - A^T(t)S - C(t) + SB(t)E^{-1}(t)B^T(t)S, \qquad S(t_f, q) = F(q), \tag{14}$$

$$\dot{R} = -[A^T(t) - S(t, q)B(t)E^{-1}(t)B^T(t)]R, \qquad R(t_f, q) = T^T(q), \tag{15}$$

$$\dot{Q} = R^T(t, q)B(t)E^{-1}(t)B^T(t)R(t, q), \qquad Q(t_f, q) = 0, \tag{16}$$

and define

$$M(t, q) \triangleq S(t, q) - R(t, q)Q^{-1}(t, q)R^T(t, q). \tag{17}$$

Let t_{sq} and t_{cq} designate the first times when the matrices $S(t, q)$ and $M(t, q)$ become infinite[6] when integrated backward from t_f (t_{sq} and t_{cq} may be equal to $-\infty$). Notice that, when $q = x^*(t_f)$,

$$F(q) = (d^2/dx_f^2)[G(x_f^*, \nu^*)], \qquad T(q) = (d/dx_f)\psi(x_f^*),$$

and the matrices $S(t, q)$, $R(t, q)$, $Q(t, q)$ reduce to the usual S, R, Q matrices used to test for conjugate points (see Ref. 5, Chapter 6). Thus, the first backward conjugate point on $x^*(t)$ is given by t_{cq^*}, where $q^* = x^*(t_f)$.

We shall close this section by presenting a property which will be used later.

Lemma 1.1. Given an extremal trajectory $x^*(t)$, $t \in [t_0, t_f]$, if there exists another extremal trajectory going through $x^*(t_1)$ for some $t_1 \in (t_0, t_f)$ and giving the same value to the cost functional between t_1 and t_f as x^* between t_1 and t_f, then the trajectory x^* is not globally optimal on $[t_0, t_f]$.

A proof of this lemma is given in Ref. 2, Appendix D.

2. Sufficient Condition for Global Optimality

Using the assumptions and notation defined in Section 1, we have the following property.

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[6] A matrix is infinite if at least one of its elements is infinite.

Lemma 2.1. If, for a given $q \in \theta$, the matrix $M(t, q)$ is finite on $[t_0, t_f)$, then the trajectory $x^*(t)$, $t \in [t_0, t_f]$, gives a strictly smaller value to the cost functional than any other admissible trajectory terminating at q [i.e., $x(t_f) = q$].

The proof of this lemma is given in Appendix A. A consequence of Lemma 2.1 is the following sufficient condition for global optimality.

Theorem 2.1. *Sufficient Condition.* An admissible pair $(u^*(t), x^*(t))$, $t \in [t_0, t_f]$, is a proper globally optimal pair if

(i) the first-order necessary conditions are satisfied;
(ii) $E(t)$ is positive definite on $[t_0, t_f]$;
(iii) $t_0 > \sup_{q \in \theta} t_{cq}$.

Proof. Condition (iii) implies that $M(t, q)$ is finite on $[t_0, t_f)$ for all $q \in \theta$, and Theorem 2.1 follows from Lemma 2.1. □

We will see in the following sections that the sufficient condition of Theorem 2.1 is equivalent to the minimal sufficient condition of Theorem 1.1 under certain conditions.

3. Case of Linear Terminal Constraints

Suppose that $g(x_f)$ and $\psi(x_f)$ have the following form:

$$g(x_f) = \tfrac{1}{2} x_f^T F x_f, \tag{18}$$

$$\psi(x_f) = T x_f + \psi_0, \tag{19}$$

where F is a symmetric $n \times n$ matrix, T is a $p \times n$ matrix of maximal rank, and ψ_0 is a p-dimensional constant vector. The problem defined in Section 1 is then a linear–quadratic problem (LQP), and the properties of a conjugate point (Ref. 4), together with Definition 1.1 of a Darboux point, imply the following theorem.

Theorem 3.1. For a linear–quadratic problem, a Darboux point on an extremal trajectory occurs at the first backward conjugate point.

Note that the matrices $F(q)$ and $T(q)$ of Section 1 reduce respectively to F and T, and $M(t, q)$ is independent of q. Thus, we have

$$\sup_{q \in \theta} t_{cq} = t_c = t_D,$$

a Darboux point. This result will be generalized in the next section.

4. Case of n-1 Terminal Constraints

Suppose that $p = n - \omega$ and $g(x_f)$, $\psi_j(x_f)$, $j = 1, \ldots, n - 1$, are arbitrary analytic functions on θ. We make the following assumption:

(A.5) rank$[T(q)] = p$ for all q in θ.

We will now show that $\sup_{q \in \theta} t_{cq}$ is a Darboux point on x^*. Theorem 2.1 indicates that condition (i) of Definition 1.1 is satisfied, and it remains to be shown that condition (ii) is also satisfied. Given $q \in \theta$, consider the following (LQP): minimize

$$I_q = \tfrac{1}{2} y_f^T F(q) y_f + \tfrac{1}{2} \int_{t_0}^{t_f} [y^T C(t) y + v^T E(t) v] \, dt,$$

subject to

$$\dot{y} = A(t)y + B(t)v, \qquad y(t_0) = y_0 \qquad T(q)y_f = 0.$$

This problem is denoted by the abbreviation (LQP)$_q$.

We have the following lemma.

Lemma 4.1. If, for a given $q \in \theta$, the matrix $M(t, q)$ becomes infinite at $t_{cq} \in (t_0, t_f)$, then there exists an extremal trajectory of (LQP)$_q$ going from $y = 0$ at $t = t_{cq}$ to $q - x^*(t_f)$ at $t = t_f$.

Proof. Obviously, t_{cq} is a conjugate point of (LQP)$_q$ on the trajectory $y^* = 0$. It can be shown (Ref. 2, Theorem 3.1) that there exists at least a one-parameter family of extremal trajectories $y_\alpha(t)$, $t \in [t_{cq}, t_f]$, of (LQP)$_q$ going through $y^*(t_{cq}) = 0$ at $t = t_{cq}$. Now, because of Assumption (A.5), the terminal set

$$\theta_q \triangleq \{y_f : T(q)y_f = 0\}$$

of (LQP)$_q$ is an $(n - p)$-dimensional hyperplane in R^n; since $p = n - 1$, $q - x^*(t_f)$, and $y^*(t_f) = 0$ are in θ_q, this set reduces to the line $(0, q - x_f^*)$ in R^n. Thus, the vectors $y_\alpha(t_f) - y^*(t_f)$ and $(q - x_f^*) - y^*(t_f)$ have the same direction and α, which is proportional to the magnitude of $y_\alpha(t_f) - y^*(t_f)$ (see Ref. 2, proof of Theorem 3.1), can always be chosen, say $\alpha = \bar{\alpha}$, so that

$$y_{\bar{\alpha}}(t_f) = q - x_f^*. \qquad \square$$

The trajectory

$$x_q(t) \triangleq y_{\bar{\alpha}}(t) + x^*(t),$$

where $y_{\bar{\alpha}}$ has been defined above, is such that $x_q(t_f) = q$ and is admissible on $[t_{cq}, t_f]$ for the original problem, since both $y_{\bar{\alpha}}(t)$ and $x^*(t)$ satisfy (2) and q satisfies (4). We have, furthermore, the following lemma.

Lemma 4.2. For a given $q \in \theta$, the trajectory $x_q(t)$ defined above gives the same value to the cost functional J (of the original problem) between t_{cq} and t_f as x^* between t_{cq} and t_f.

The proof is given in Appendix A.

Now, we can show that x^* is no longer globally optimal if one of the matrices $M(t, q)$ is infinite on (t_0, t_f).

Theorem 4.1. If there exists $q \in \theta$ such that $t_{cq} \in (t_0, t_f)$, then the trajectory $x^*(t)$ is not globally optimal on $[t_0, t_f]$.

Proof. If the trajectory $x_q(t)$ (defined before Lemma 4.2) is extremal on $[t_{cq}, t_f]$, then the result follows from Lemmas 4.2 and 1.1. If x_q is not extremal on $[t_{cq}, t_f]$, it is certainly not globally optimal from $(x^*(t_{cq}), t_{ca})$, and one can find an admissible trajectory from $(x^*(t_{cq}), t_{ca}))$ giving a smaller value to the cost functional than x_q. Then, Theorem 4.1 follows from Lemma 4.2. $\qquad\square$

Finally, Theorems 2.1, 4.1, and Definition 1.1 imply the following theorem.

Theorem 4.2. For the class of problems defined in Section 1 with $p = n - 1$ terminal constraints, a Darboux point on the trajectory x^* occurs at

$$t_D = \sup_{q \in \theta} t_{cq}.$$

This theorem implies that the sufficient condition of Theorem 2.1 is equivalent to the minimal sufficient condition of Theorem 1.1.

Example 4.1. Consider the case when
$$n = m = 2, \qquad t_0 = 0, \qquad t_f = 1,$$

$$A(t) = C(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B(t) = E(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad g(x_f) = 0, \qquad \psi(x_f) = x_{2f}^3 - x_{2f} + x_{1f} - 1$$

in Eqs. (1)–(4). Actually, this example is a minimum distance problem from the point x_0 to the curve

$$\psi(x_f) = 0.$$

An extremal solution is

$$u^*(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad x^*(t) \begin{bmatrix} t \\ 1-t \end{bmatrix}, \qquad \lambda^*(t) = -u^*(t), \qquad \nu^* = -1,$$

and it can be shown that, for any

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \in \theta,$$

we have

$$S(t, q) = \begin{bmatrix} 0 & 0 \\ 0 & -1/(t-1+1/2q_2) \end{bmatrix},$$

$$M(t, q) = \frac{1}{(1-t)[t-1+(1+(q_2^2-1)^2)/2q_2]}$$

$$\times \begin{bmatrix} t-1+1/2q_2 & (q_2^2-1)/2q_2 \\ (q_2^2-1)/2q_2 & t-1+(q_2^2-1)^2/2q_2 \end{bmatrix}$$

Then,[7]

$$t_{sq} = 1-1/2q_2, \qquad t_{cq} = 1-(1+(q_2^2-1)^2)/2q_2,$$

$$t_D = \sup_{q \in \theta} t_{cq} = \sup_{q_2 \in [0, +\infty)} t_{cq} = 0.523 \text{ (given by } q_2 = 1.1\text{)}.$$

Note that a conjugate point on x^* is given by t_{cq^*}, that is,

$$t_{cq^*} = (t_{cq})_{q_2=0} = -\infty,$$

which illustrates the possibility of the existence of a Darboux point without the existence of a conjugate point.

Example 4.2. Consider the case when

$$n = m = 1, \qquad p = 0,$$

$$A(t) = 0, \qquad C(t) = -1, \qquad B(t) = E(t) = 1, \qquad t \in [t_0, \pi],$$

$$x_0 = 0, \qquad g(x_f) = \tfrac{1}{2} x_f^2 [\exp(-x_f) - 1]$$

in Eqs. (1)–(4). An extremal solution is

$$u^*(t) = x^*(t) = \lambda^*(t) = 0, \qquad t \in [t_0, \pi],$$

and we have

$$F(q) = \exp(-q) - 1.$$

This can be obtained by using the relation

$$\exp(-x_f) = \sum_{k=0}^{\infty} (-1)^k (x_f^k / k!).$$

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[7] By definition, $t_{cq} < t_f = 1$ and we can restrict q_2 to belong to $[0, +\infty)$.

Then,

$$t_{cq} = t_{sq} = \tan^{-1}\left[\frac{\exp(q)}{(1-\exp(q))}\right]$$

and

$$t_D = \sup_{q \in (-\infty, +\infty)} t_{cq} = 3\pi/4.$$

It can be proved that the problem does not have a solution when $t_0 < 3\pi/4$. The choice

$$u = \text{const} = \alpha, \qquad x = \alpha(t - t_0)$$

makes the cost functional as small as desired when α is large enough. This indicates that t_D is a Type-2 Darboux point.

We notice that, in Example 4.1, $\sup t_{cq}$ was attained (given by $q_2 = 1.1$), while it was not attained in Example 4.2. This observation holds in general.

Theorem 4.3. For a Type-1 Darboux point, the supremum of t_{cq} is attained, i.e.,

$$t_D = \max_{q \in \theta} t_{cq}.$$

Proof. Suppose that t_D is a Type-1 Darboux point on x^*. If t_D is a conjugate point on x^*, then $t_D = t_{cq^*}$ (see Section 1) and

$$\sup_{q \in \theta} t_{cq} = t_D = t_{cq^*}$$

is attained with

$$q = q^* = x^*(t_f).$$

If t_D is not a conjugate point on x^*, then a property of Type-1 Darboux points (Ref. 1, Theorem 4.1) implies that there exists a trajectory $\bar{x}(t)$, $t \in [t_D, t_f]$, distinct from x^*, globally optimal on $[t_D, t_f]$. Let $\bar{q} = \bar{x}(t_f)$. We have

$$t_{c\bar{q}} \leq \sup_{q \in \theta} t_{cq} = t_D.$$

But, if $t_{c\bar{q}} < t_D$, Lemma 2.1 implies that x^* gives a smaller value to the cost functional J than $\bar{x}$ between t_D and t_f, which is in contradiction with the optimality of $\bar{x}$ between t_D and t_f. Thus, we have

$$t_{c\bar{q}} = t_D = \sup_{q \in \theta} t_{cq},$$

which completes the proof.   □

Remark 4.1. For a Type-2 Darboux point, the supremum of t_{cq} may be attained (see Section 3) or may not be attained (see Example 4.2).

5. General Case

Suppose now that p is an arbitrary integer, $0 \le p \le n$, that $g(x_f), \psi_j(x_f), j = 1, \ldots, p$, are arbitrary analytic functions on θ, and that Assumption (A.5) is satisfied. We wish to investigate a generalization of the result of Theorem 4.2. The approach used in Section 4 to show that the trajectory x^* is not globally optimal on $[t_0, t_f]$ when $t_0 < t_D$ is no longer valid, since Lemma 4.1 is not necessarily true, as shown in the example below.

Example 5.1. Consider the case when

$$n = m = 2, \qquad p = 0,$$

$$A(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B(t) = -C(t) = E(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad t \in [t_0, \pi],$$

$$x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad g(x_f) = x_{1f}^3 + \tfrac{1}{2} x_{1f}^4$$

in Eqs. (1)–(4). An extremal solution is

$$u^*(t) = x^*(t) = \lambda^*(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad t \in [t_0, \pi],$$

and we have

$$M(t, q) = S(t, q) = \begin{bmatrix} \dfrac{[q_1^2 + 2q_1 + \tan t]}{[1 - (q_1^2 + 2q_1)\tan t]} & 0 \\ 0 & \tan t \end{bmatrix} \qquad \text{for all } q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \in R^2.$$

It follows that

$$t_{cq} = \sup \left\{ \frac{\pi}{2}, \ \tan^{-1}[1/(q_1^2 + 2q_1)] \right\}$$

and

$$\sup_{q \in \theta} = \sup \left\{ \frac{\pi}{2}, \ \sup_{q_1 \in (-\infty, +\infty)} \tan^{-1}[1/(q_1^2 + 2q_1)] \right\} = 3\pi/4.$$

It can be shown (Ref. 2, Example 5.3) that the trajectory $y_\alpha(t)$ in the proof of Lemma 4.1 is

$$y_\alpha(t) = \begin{bmatrix} -\alpha \sin(t - t_{cq}) \\ 0 \end{bmatrix}.$$

It follows that α can be chosen such that $y_{1\alpha}(t_f)$ has a given value $q_1 - x_1^*(t_f)$, but $y_{2\alpha}(t_f)$ cannot be made to have an arbitrary value $q_2 - x_2^*(t_f)$. This shows

that Lemma 4.1 is not necessarily true when the number p of terminal constraints is arbitrary.

Notice, however, that

$$\bar{q} = \begin{bmatrix} q_1 \\ 0 \end{bmatrix}$$

is such that

$$t_{c\bar{q}} = t_{cq}$$

and that $y_{\bar{\alpha}}(t_f)$, where

$$\bar{\alpha} = -q_1/\sin(t_f - t_{cq})$$

satisfies Lemma 4.1 with $q = \bar{q}$. Then, Theorem 4.1 holds with $q = \bar{q}$, and

$$t_{c\bar{q}} = t_{cq}$$

implies that x^* is not globally optimal when $t_0 < t_{cq}$. It follows that the Darboux point is still given by $\sup_{q \in \theta} t_{cq}$. This result can be checked by noticing that the problem considered can be separated into two independent problems of the type considered in Section 4 (i.e., $p = n - 1$).

In general, for a given $q \in \theta$, there does not necessarily exist a $\bar{q}$ as above (i.e., satisfying Lemma 4.1 and $t_{c\bar{q}} = t_{cq}$), and we cannot say *a priori* that the Darboux point is given by $\sup_{q \in \theta} t_{cq}$. For a given q in θ, let $V_i^q, i = 1, \ldots, n - p - 1$, be a set of vectors in the $(n - 1)$-dimensional subspace of R^n orthogonal to the line (q, x_f^*), such that the vectors $T_j(q), j = 1, \ldots, p$, and V_i^q are linearly independent [this choice is always possible, since the vectors $T_j(q)$ are linearly independent, from Assumption (A.5)]. Let $\overline{(LQP)}_q$ be the problem similar to $(LQP)_q$ defined in Section 4, but with the extra terminal conditions

$$V_i^q y_f = 0, \qquad i = 1, \ldots, n - p - 1,$$

and define $\bar{t}_{cq}$ as follows.

Definition 5.1. $\bar{t}_{cq}$ is the first backward conjugate point of $\overline{(LQP)}_q$ on the trajectory $y^*(t) = 0$.

We have the following theorem.

Theorem 5.1. For the class of problems defined in Section 1 with an arbitrary number p, $0 \le p \le n$, of terminal constraints, a Darboux point on the trajectory x^* occurs at

$$t_D = \sup_{q \in \theta} \bar{t}_{cq}.$$

Proof. Suppose that

$$\sup_{q \in \theta} \bar{t}_{cq} \in (t_0, t_f);$$

let

$$t_1 = \sup_{q \in \theta} \bar{t}_{cq} - \epsilon;$$

then, for ϵ small enough, there exists $\bar{q} \in \theta$ such that

$$\bar{t}_{c\bar{q}} = t_1.$$

From Lemma 4.1 [note that $\overline{(LQP)}_q$ has $n-1$ terminal constraints], there exists an extremal trajectory $\bar{y}(t)$ of $\overline{(LQP)}_q$ such that

$$\bar{y}(\bar{t}_{c\bar{q}}) = 0 \quad \text{and } \bar{y}(t_f) = \bar{q} - x^*(t_f).$$

The trajectory

$$\bar{x}(t) \triangleq \bar{y}(t) + x^*(t)$$

is admissible on $[t_1, t_f]$ for the original problem, since $\bar{x}(t)$ satisfies (2) and $\bar{x}(t_f) = \bar{q}$ satisfies (4). Then, Lemma 4.2 and Theorem 4.1 hold, and condition (ii) of Definition 1.1 is satisfied.

Now, let

$$t_2 > \sup_{q \in \theta} \bar{t}_{cq};$$

it follows that the trajectory

$$y^*(t) = 0, \qquad t \in [t_0, t_f],$$

is proper globally optimal on $[t_2, t_f]$ for all $\overline{(LQP)}_q$, $q \in \theta$. Suppose that condition (i) of Definition 1.1 is not satisfied; then, there exists a trajectory $\hat{x}(t)$, admissible on $[t_2, t_f]$, of the original problem giving a smaller value to the cost functional between t_2 and t_f than x^*. The trajectory

$$\hat{y}(t) \triangleq \hat{x}(t) - x^*(t)$$

is obviously admissible on $[t_2, t_f]$ for $\overline{(LQP)}_{\hat{q}}$, where $\hat{q} = \hat{x}(t_f)$, and $\hat{y}$ gives a smaller value to the cost $I_{\hat{q}}$ of $\overline{(LQP)}_{\hat{q}}$ than y^* between t_2 and t_f, which is a contradiction with $t_2 > \bar{t}_{c\hat{q}}$. To show that $\hat{y}$ gives a negative value of $I_{\hat{q}}$ consider

$$\Delta J = G(\hat{q}, \nu^*) - G(x^*, \nu) - \lambda^{*T}(t_f) \Delta x_f + \tfrac{1}{2} \int_{t_2}^{t_f} (\Delta x^T C \Delta x + \Delta u^T E \Delta u) \, dt,$$

where ΔJ is the difference between the values of the cost functional given by $\hat{x}(t)$ and $x^*(t)$ between t_2 and t_f,

$$\Delta x(t) \triangleq \hat{x}(t) - x^*(t), \qquad \Delta u(t) \triangleq \hat{u}(t) - u^*(t)$$

(see Appendix A, proof of Lemma 4.2). Then,

$$\Delta J = \tfrac{1}{2} y_f^T F(\hat{q}) y_f + \tfrac{1}{2} \int_{t_2}^{t_f} (y^T C y + v^T E v)\, dt,$$

where

$$y(t) = \Delta x(t), \qquad v(t) = \Delta u(t)$$

have been used. Therefore, ΔJ is the value between t_2 and t_f of the cost functional $I_{\hat{q}}$ of $\overline{(\text{LQP})}_{\hat{q}}$, and the result follows from $\Delta J < 0$. The proof of Theorem 5.1 is thus complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Remark 5.1. It can be shown (Ref. 2) that $\bar{t}_{cq} \le t_{cq}$ for any $q \in \theta$, and it follows that

$$t_D = \sup_{q \in \theta} \bar{t}_{cq} \le \sup_{q \in \theta} t_{cq}.$$

For the problem of Example 5.1, it can be shown that

$$\bar{t}_{cq} = \sup\left\{ \frac{\pi}{2},\ \tan^{-1}[(q_1^2 + q_2^2)/q_1^2(q_1^2 + 2q_1)] \right\}$$

and that

$$t_D = \sup_{q \in \theta} \bar{t}_{cq} = 3\pi/4 = \sup_{q \in \theta} t_{cq},$$

as expected.

Remark 5.2. $\bar{t}_{cq}$ is the first time when the matrix $M(t, q)$ becomes infinite when integrated backward. $M(t, q)$ is defined as

$$M(t, q) = S(t, q) - R(t, q) Q^{-1}(t, q) R^T(t, q),$$

where $S(t, q)$, $R(t, q)$, and $Q(t, q)$ are solutions of Eqs. (14)–(16), when $R(t_f, q)$ is replaced by

$$R(t_f, q) = [T^T(q) : V^{qT}], \qquad V^{qT} = [V_1^q, \ldots, V_{n-p-1}^q].$$

Thus, for an arbitrary number of terminal constraints, a Darboux point t_D is less than or equal to $\sup_{q \in \theta} t_{cq}$. A case when

$$t_D = \sup_{q \in \theta} t_{cq}$$

will be investigated in the next section. The particular structure is such that a Darboux point is easily calculated and, furthermore, is related to the Riccati matrix of the LQP.

6. Case of One Quadratic Terminal Constraint

Suppose that $g(x_f)$ and $\psi(x_f)$ have the following form:

$$g(x_f) = \tfrac{1}{2} x_f^T F_0 x_f, \tag{20}$$

$$\psi(x_f) = \tfrac{1}{2} x_f^T F_1 x_f + T x_f + \psi_0, \tag{21}$$

where F_0 and f_1 are symmetric $n \times n$ matrices, $F_1 \neq 0$ (i.e., $e^T F_1 e \neq 0$ for all $e \in R^n$, $e \neq 0$), T is a $1 \times n$ vector ($p = 1$), and ψ_0 is a scalar. The matrices $F(q)$ and $T(q)$ are

$$F(q) = F_0 + \nu^* F_1, \qquad \nu^* \in R^1,$$

and

$$T(q) = \tfrac{1}{2}(q + x_f^*)^T F_1 + T,$$

respectively; thus, $S(t, q)$ does not depend on q. Accordingly, we shall designate $S(t, q)$ by $S(t)$ and t_{sq} by t_s.

The major result of this section is that t_s is a Darboux point. In this direction, the following corollary of Theorem 2.1 implies that condition (i) of the Darboux point definition (Definition 1.1) is satisfied.

Corollary 6.1. A sufficient condition for an extremal trajectory $x^*(t)$, $t \in [t_0, t_f]$, to be proper globally optimal is that $t_0 > t_s$.

Proof. It can be shown (Ref. 2, Theorem 3.7) that $t_s \geq t_{cq}$ for all $q \in \theta$. Thus,

$$t_0 > t_s \geq \sup_{q \in \theta} t_{cq},$$

and the result follows from Theorem 2.1.   □

We will now follow several steps to show that condition (ii) of Definition 1.1 is satisfied. Let $\Omega(t, t_0)$ be the $2n \times 2n$ transition matrix of

$$\dot{x} = \begin{bmatrix} A(t) & -B(t)E^{-1}(t)B^T(t) \\ -C(t) & -A^T(t) \end{bmatrix} x,$$

and denote by $\Omega_{ij}(t, t_0)$, $i, j = 1, 2$, *the block of* $\Omega(t, t_0)$ partitioned into four $n \times n$ blocks. Suppose that $t_s \in (t_0, t_f)$. Then, there exists a nonzero vector

$e \in R^n$ in the null space of

$$\bar{S}(t_s, t_f) = \Omega_{22}(t_f, t_s) - S(t_f)\Omega_{12}(t_f, t_s) \tag{22}$$

(see Ref. 2, Section 3.3). Let $(x_\alpha(t), \lambda_\alpha(t))$ be the solution on $[t_s, t_f]$ of Eqs. (2), (5), where (6) and Assumption (A.2) have been used to eliminate u, with initial conditions

$$x(t_s) = x^*(t_s), \qquad \lambda(t_s) = \lambda^*(t) + \alpha e, \tag{23}$$

where α is a scalar parameter. We have

$$x_\alpha(t) = \Omega_{11}(t, t_s)x^*(t_s) + \Omega_{12}(t, t_s)\lambda(t_s) = x^*(t) + \alpha\Omega_{12}(t, t_s)e,$$
$$\lambda_\alpha(t) = \Omega_{21}(t, t_s)x^*(t_s) + \Omega_{22}(t, t_s)\lambda(t_s) = \lambda^*(t) + \alpha\Omega_{22}(t, t_s)e, \tag{24}$$

and the following lemma.

Lemma 6.1. If there exists $\alpha \neq 0$ such that $x_\alpha(t_f) \in \theta$, then the trajectory $x_\alpha(t)$, $t \in [t_s, t_f]$, is extremal.

Proof. Since $(x_\alpha, \lambda_\alpha)$ satisfies Eqs. (2), (4), (5), and (6), we only need to show that (7) is satisfied with $\lambda_\alpha(t_f)$ and $x_\alpha(t_f)$ for some ν_α. Using (24), we have

$$\lambda_\alpha(t_f) - (F_0 + \nu_\alpha F_1)x_\alpha(t_f) - \nu_\alpha T^T = \lambda^*(t_f) - (F_0 + \nu^* F_1)x^*(t_f) - \nu^* T^T$$
$$- (\nu_\alpha - \nu^*)\{F_1[x^*(t_f) + \alpha\Omega_{12}(t_f, t_s)e] + T^T\}$$
$$+ \alpha[\Omega_{22}(t_f, t_s) - (F_0 + \nu^* F_1)\Omega_{12}(t_f, t_s)]e.$$

Then, the satisfaction of (7) with $x_\alpha(t_f)$, $\lambda_\alpha(t_f)$, ν_α follows from the definition of e, the choice $\nu_\alpha = \nu^*$, and the satisfaction of (7) with $x^*(t_f)$, $\lambda^*(t_f)$, ν^*. □

Next, we examine the existence of an $\alpha \neq 0$ such that $x_\alpha(t_f) \in \theta$. This requires, using (24), that

$$0 = \psi[x_\alpha(t_f)] = \tfrac{1}{2}[x^*(t_f) + \alpha\Omega_{12}(t_f, t_s)e]^T F_1[x^*(t_f) + \alpha\Omega_{12}(t_f, t_s)e]$$
$$+ T[x^*(t_f) + \alpha\Omega_{12}(t_f, t_s)e] + \psi_0 = 0,$$

or

$$0 = \psi[x^*(t_f)] + \tfrac{1}{2}\alpha^2[\Omega_{12}(t_f, t_s)e]^T F_1[\Omega_{12}(t_f, t_s)e]$$
$$+ \alpha[x^{*T}(t_f)F_1 + T]\Omega_{12}(t_f, t_s)e,$$

with $\psi[x^*(t_f)] = 0$. Let v represent the vector $\Omega_{12}(t_f, t_s)e$. Then, α must be the solution of

$$\alpha(v^T F_1 v) + 2(x_f^{*T}F_1 + T^T)v = 0.$$

We have four cases:

Case (i): $v^T F_1 v \neq 0$ and $(x_f^{*T} F_1 + T)v \neq 0$; then, the solution is

$$\alpha = -2(v^T F_1 v)^{-1}(x_f^{*T} F_1 + T)v \neq 0.$$

Case (ii): $v^T F_1 v \neq 0$ and $(x_f^{*T} F_1 + T)v = 0$; then, the only solution is

$$\alpha = 0.$$

Case (iii): $v^T F_1 v = 0$ and $(x_f^{*T} F_1 + T)v \neq 0$; then, the only solution is

$$\alpha = +\infty.$$

Case (iv): $v^T F_1 v = 0$ and $(x_f^{*T} F_1 + T)v = 0$; then,

$$\alpha = \text{any real number.}$$

Case (iv) corresponds to the case where θ consists of two hyperplanes (ψ is the product of two affine functions), one of them containing v. Case (iii) corresponds to the case where $x_\alpha(t_f)$ is at infinity (for instance, if θ is a parabola and v is in the direction of the principal axis) and is not realistic, since it means that $x_\alpha(t_f)$ is attained in the finite amount of time $t_f - t_s$ from $x^*(t_s)$, with

$$\|x_\alpha(t_f) - x^*(t_s)\| = \infty.$$

However, we will consider this case and view the corresponding trajectory as the limit of $x_\alpha(t)$ as α goes to infinity. Case (ii) corresponds to the case where t_s is a conjugate point, as shown below.

Lemma 6.2. If there exists a nonzero vector e in the null space of $\bar{S}(t_s, t_f)$ such that

$$(x_f^{*T} F_1 + T)\Omega_{12}(t_f, t_s)e = 0,$$

then t_s is a conjugate point on the trajectory x^*.

Proof. Consider

$$V(\tau) = [\Omega_{22}(\tau, t_s) - M(\tau)\Omega_{12}(\tau, t_s)]e,$$

where

$$M(\tau) = M(\tau, q = x_f^*)$$

is the matrix used to test for conjugate points on x^* (as noted in Section 1). Using (17) and the definition of e, we have

$$V(\tau) = [\Omega_{22}(\tau, t_s) - S(\tau)\Omega_{12}(\tau, t_s)]e + R(\tau)Q^{-1}(\tau)R^T(\tau)\Omega_{12}(\tau, t_s)e,$$

$$V(\tau) = -R(\tau)Q^{-1}(\tau)R^T(\tau)\Omega_{12}(\tau, t_s)e.$$

Define

$$\rho(\tau) \triangleq R^T(\tau)\Omega_{12}(\tau, t_s)e;$$

we have

$$\dot{\rho}(\tau) = \dot{R}^T\Omega_{12}(\tau, t_s)e + R^T\dot{\Omega}_{12}(\tau, t_s)e$$

$$= -R^T(A - BE^{-1}B^TS)\Omega_{12}(\tau, t_s)e$$

$$+ R^T[A\Omega_{12}(\tau, t_s) - BE^{-1}B^T\Omega_{22}(\tau, t_s)]e$$

$$= -R^TBE^{-1}B^T[\Omega_{22}(\tau, t_s) - S(\tau)\Omega_{12}(\tau, t_s)]e = 0,$$

where the expression for

$$\dot{\Omega}(\tau, t_s) = (\partial/\partial\tau)\Omega(\tau, t_s)$$

is easily derived from the definition of $\Omega(t, t_0)$. Thus, we have

$$\rho(\tau) = \rho(t_f) \qquad \text{for all } \tau \in (t_s, t_f],$$

but

$$\rho(t_f) = R^T(t_f)\Omega_{12}(t_f, t_s)e = R^T(t_f, q = x_f^*)\Omega_{12}(t_f, t_s)e = T(q = x_f^*)\Omega_{12}(t_f, t_s)e,$$

or

$$\rho(t_f) = (x_f^{*T}F_1 + T)\Omega_{12}(t_f, t_s)e = 0.$$

Since $R(\tau)Q^{-1}(\tau)$ is finite[8] for all $\tau \in (t_s, t_f)$, it follows that

$$V(\tau) = 0 \qquad \text{for all } \tau \in (t_s, t_f).$$

Then, $M(t_s)$ is infinite (see Ref. 2, Section 3.3), and the result follows, since $M(t_s)$ infinite implies a conjugate point (Ref. 6). □

Remark 6.1. Lemma 6.2 gives a sufficient condition for t_s to be a conjugate point; however, this condition is not necessary, as will be seen in Example 6.2.

Thus far, we have seen that either t_s is a conjugate point on x^* or there exists an extremal trajectory $x_\alpha (\alpha \neq 0, \alpha$ may be unbounded), distinct from x^*, with associated Lagrange multiplier $\nu_\alpha = \nu^*$ and going through $x^*(t_s)$ at $t = t_s$. We have, furthermore, the following lemma.

Lemma 6.3. If two extremal trajectories with the same Lagrange multiplier ν go through the same point at some time t_1, then they give the same value to the cost functional between t_1 and t_f.

---

[8] $R(\tau)Q^{-1}(\tau)$ is finite for all $\tau \in (t_{cq^*}, t_f)$, see Ref. 6, and the result follows from $t_2 \geq t_{cq^*}$.

Proof.   Let $\bar{x}(t)$ and $x^*(t)$, $t \in [t_1, t_f]$, be two extremal trajectories such that

$$\bar{x}(t_1) = x^*(t_1) \qquad \text{and } \bar{\nu} = \nu^*.$$

The argument used at the beginning of the proof of Lemma 4.2 (see Appendix A) is applicable, with $x_q(t)$ replaced by $\bar{x}(t)$, and we have

$$\Delta J = \tfrac{1}{2} \Delta x_f^T F(q = \bar{x}(t_f)) \, \Delta x_f - \tfrac{1}{2} \Delta \lambda^T(t_f) \, \Delta x_f,$$

where ΔJ is the difference between the values of the cost functional between t_1 and t_f given by $\bar{x}$ and x^*,

$$\Delta x(t) \triangleq \bar{x}(t) - x^*(t), \qquad \Delta \lambda(t) \triangleq \bar{\lambda}(t) - \lambda^*(t),$$

$\bar{\lambda}(t)$ being the multiplier function associated with $\bar{x}(t)$. But

$$\bar{\lambda}(t_f) = (F_0 + \bar{\nu} F_1) \bar{x}(t_f) + \bar{\nu} T,$$

$$\lambda^*(t_f) = (F_0 + \nu^* F_1) x^*(t_f) + \nu^* T,$$

$$\bar{\nu} = \nu^*$$

imply

$$\Delta \lambda^T(t_f) \, \Delta x_f = \Delta x_f^T (F_0 + \nu^* F_1) \, \Delta x_f.$$

Then,

$$\Delta J = 0$$

follows from

$$F(q) = F_0 + \nu^* F_1. \qquad \qquad \square$$

Remark 6.2.   In the case (iii) mentioned earlier, $\bar{x}(t)$ is the limit of a sequence of trajectories $x_\alpha(t)$ as α goes to infinity. For each $x_\alpha(t)$, the same argument as in the proof above would give

$$\Delta J_\alpha = -\nu^* \psi(x_\alpha(t_f)).$$

Since $x_\alpha(t_f)$ is not in θ, the term $-\nu^* \psi(x_\alpha(t_f)) \neq 0$ must be added to compensate for $G(x_\alpha(t_f), \nu^*)$ used in ΔJ_α; see proof of Lemma 4.2 in Appendix A. At the limit,

$$\Delta J = \lim_{\alpha \to \infty} \Delta J_\alpha = \lim_{\alpha \to \infty} -\nu^* \psi(x_\alpha(t_f)) = 0,$$

since

$$\lim_{\alpha \to \infty} x_\alpha(t_f) \in \theta.$$

Recapitulating the varous results obtained, it can be seen that the trajectory $x^*(t)$ is not globally optimal on $[t_0, t_f]$ when $t_0 < t_s$. Indeed, either t_s is a conjugate point and x^* is not even locally optimal on $[t_0, t_f]$, or t_s is not a conjugate point and the result follows from Lemmas 6.1, 6.3, and 1.1. Then, Corollary 6.1 and Definition 1.1 imply the following theorem.

Theorem 6.1. For the class of problems considered in this section, a Darboux point occurs on an extremal trajectory at t_s, that is, when the corresponding S-matrix becomes infinite.

Note that we have

$$t_D = t_s = \sup_{q \in \theta} t_{cq}.$$

Indeed, $t_{cq} \leq t_s$ (see Ref. 2) implies

$$\sup_{q \in \theta} t_{cq} \leq t_s,$$

and the result follows from:

$$t_D = \sup_{q \in \theta} \bar{t}_{cq} \leq \sup_{q \in \theta} t_{cq} \leq t_s$$

(see Section 5). We shall now present two illustrative examples.

Example 6.1. Consider the case when

$$n = m = 2, \qquad t_0 = 0, \qquad t_f = 1,$$

$$A(t) = C(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B(t) = E(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$x_0 = \begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix}, \qquad F_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad F_1 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \qquad T = [2, 0], \qquad \psi_0 = -3$$

in Eqs. (1)–(4) and (20)–(21). This example is a minimum-distance problem from a point to a parabola. An extremal solution is

$$u^*(t) = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}, \qquad x^*(t) = \begin{bmatrix} \frac{3}{2}(t - \frac{1}{3}) \\ \frac{3}{2}(t - \frac{1}{3}) \end{bmatrix}, \qquad t \in [0, 1],$$

$$\lambda^*(t) = -u^*(t), \qquad \nu^* = -\frac{3}{4}.$$

We have

$$S(t) = \begin{bmatrix} 0 & 0 \\ 0 & -1/(t - \frac{1}{3}) \end{bmatrix}$$

and

$$M(t, q) = \frac{1}{(1-t)[t-\frac{1}{3}+\frac{1}{6}(q_2+1)^2]} \begin{bmatrix} t-\frac{1}{3} & \frac{1}{3}(q_2+1) \\ \frac{1}{3}(q_2+1) & t-1+\frac{1}{6}(q_2+1)^2 \end{bmatrix}$$

for all

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \in \theta.$$

It follows that

$$t_s = \tfrac{1}{3} \quad \text{and } t_{cq} = \tfrac{1}{3}-\tfrac{1}{6}(q_2+1)^2.$$

Note that

$$\sup_{q \in \theta} t_{cq} = \sup_{q_2 \in (-\infty, +\infty)} t_{cq} = \tfrac{1}{3} = t_s.$$

The Darboux point

$$t_D = t_s = \tfrac{1}{3}$$

is of Type-1. Also, note that there exists a trajectory $\bar{x}$, distinct from x^*, giving the same value to the cost functional between t_D and t_f as x^* between t_D and t_f (see Fig. 1).

Now, suppose that
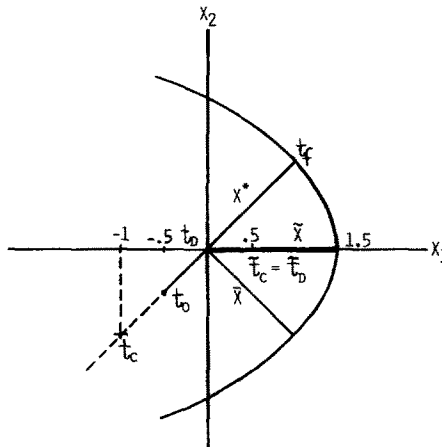
$$x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Fig. 1.   Darboux points in Example 6.1.

An extremal solution is

$$\tilde{u}(t) = \begin{bmatrix} \frac{3}{2} \\ 0 \end{bmatrix}, \qquad \tilde{x}(t) = \begin{bmatrix} \frac{3}{2}t \\ 0 \end{bmatrix}, \qquad \tilde{\lambda}(t) = -\tilde{u}(t), \qquad \tilde{\nu} = -\tfrac{3}{4}.$$

We have

$$\Omega_{22}(\tau, t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \Omega_{12}(\tau, t) = -\begin{bmatrix} \tau - t & 0 \\ 0 & \tau - t \end{bmatrix}.$$

Then,

$$\bar{S}(t_s, t_f) = \begin{bmatrix} 1 & 0 \\ 0 & 1 - \frac{3}{2}(1 - t_s) \end{bmatrix},$$

where $\bar{S}$ is defined in (22). This matrix becomes singular when

$$t_s = \tfrac{1}{3},$$

and a unit vector in its null space is

$$e = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

It follows that

$$(\tilde{x}_f^T F_1 + T)\Omega_{12}(t_f, t_s)e = [2, 0]\begin{bmatrix} -\frac{2}{3} & 0 \\ 0 & -\frac{2}{3} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0.$$

It may be shown that the Darbous point

$$\tilde{t}_D = \tilde{t}_s = \tfrac{1}{3}$$

on $\tilde{x}$ is also a conjugate point, as expected from Lemma 6.2 (see Fig. 1), thus providing an illustration of a Darboux point which is also a conjugate point.

Example 6.2.   Consider the same problem as Example 6.1, but with

$$F_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \qquad T = [0, 0], \qquad \psi_0 = -2, \qquad x_0 = \begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix}.$$

The terminal set θ is now a circle. The pair $(u^*(t), x^*(t))$ of Example 6.1 is also extremal for this example with the same multipliers $\lambda^*(t), \nu^*$. The Darboux point is still

$$t_D = t_s = \tfrac{1}{3},$$

but it is also a conjugate point. We have

$$\bar{S}(t_s, t_f) = \begin{bmatrix} 1 - \frac{3}{2}(1 - t_s) & 0 \\ 0 & 1 - \frac{3}{2}(1 - t_s) \end{bmatrix}.$$

When

$$t_s = \tfrac{1}{3},$$

$\tilde{S}$ becomes identically zero, and any vector $e \in R^2$ is in its null space. In particular, choose

$$e = \begin{bmatrix} 1 \\ 1 \end{bmatrix};$$

it follows that

$$(x_f^{*T} F_1 + T) \Omega_{12}(t_f, t_s) e = [2, 2] \begin{bmatrix} -\tfrac{2}{3} \\ -\tfrac{2}{3} \end{bmatrix} = -\tfrac{8}{3} \neq 0,$$

which shows that the condition of Lemma 6.2 is not necessary for t_s (the Darboux point) to also be a conjugate point.

7. Conclusions

In this paper, characterizations of the Darboux point for particular classes of problems (linear dynamics and quadratic integrand in the cost functional, along with certain analyticity assumptions) have been determined. The Darboux point was shown to be the supremum of a sequence of conjugate points. This approach allows a connection between the conjugate point and the Darboux point on an extremal trajectory, namely, that a conjugate point is a first-order approximation to a Darboux point. When the terminal constraints are limited to a single quadratic constraint, a Darboux point occurs when the Riccati matrix (i.e., the S-matrix) associated with the trajectory becomes unbounded. This notable property makes the Darboux point test very simple for this type of problem. To the best of the authors' knowledge, it is the first time that $S \to \infty$ has been characterized in problems with terminal constraints.

The various results concerning the characterizations of the Darboux point for the classes of problems considered are summarized in Table 1, where t_c is the first backward conjugate point, t_s is the time when the Riccati matrix becomes infinite, and t_{cq}, $\bar{t}_{cq}$, q, θ are defined in the text.

Table 1. Darboux point characterizations.

	Terminal constraints (number and form)			
	$0 \leq p \leq n$, linear	$0 \leq p \leq n-1$, arbitrary	$p = n-1$, arbitrary	$p = 1$, quadratic
Darboux point occurs at	t_c	$\sup\limits_{q \in \theta} \bar{t}_{cq}$	$\sup\limits_{q \in \theta} t_{cq}$	t_s

8. Appendix A: Proof of Lemmas 2.1 and 4.2

Proof of Lemma 2.1.[9]   Given q, let $\bar{x}(t)$, $t \in [t_0, t_f]$, be an admissible trajectory starting at $\bar{x}(t_0) = x_0$ and terminating at $\bar{x}(t_f) = q$. It can be shown from (1) that

$$\bar{J} - J^* = g(\bar{x}_f) - g(x_f^*) + \tfrac{1}{2} \int_{t_0}^{t_f} (\Delta x^T C \, \Delta x + \Delta u^T E \, \Delta u) \, dt$$

$$+ \int_{t_0}^{t_f} (x^{*T} C \, \Delta x + u^{*T} E \, \Delta u) \, dt,$$

where $\bar{J}$ and J^* are the values of the cost functional given respectively by the trajectories $\bar{x}$ and x^*,

$$\Delta x(t) \triangleq \bar{x}(t) - x^*(t), \qquad \Delta u(t) \triangleq \bar{u}(t) - u^*(t),$$

$\bar{u}(t)$, $t \in [t_0, t_f]$, being the control associated with $\bar{x}$. Add the identically zero quantity

$$0 \equiv \nu^{*T} (\psi(\bar{x}_f) - \psi(x_f^*)) + \tfrac{1}{2} \int_{t_0}^{t_1} \Delta x^T M(t, q)(A \, \Delta x + B \, \Delta u - \Delta \dot{x}) \, dt$$

$$+ \int_{t_1}^{t_f} [\tfrac{1}{2} \Delta x^T S(t, q)$$

$$+ \nu^T R^T(t, q)](A \, \Delta x + B \, \Delta u - \Delta \dot{x}) \, dt + \tfrac{1}{2} \nu^T [Q(t_1, q) + \int_{t_1}^{t_f} \dot{Q}(t, q) \, dt] \nu,$$

$$(25)$$

where $\nu \in R^p$ is a constant to be chosen later and $t_1 \in (t_0, t_f)$ is such that the matrix $S(t, q)$ is defined on $[t_1, t_f]$. The existence of such t_1 sufficiently close to t_f is implied by the existence of $S(t_f)$. It follows from (15) and (16) that $R(t, q)$ and $Q(t, q)$ are defined on $[t_1, t_f]$. Thus, (25) is well defined, since $M(t, q)$ is finite on $[t_0, t_1]$. After integrating by parts and using [through necessary conditions (5), (6), (7), and Eq. (11)]

$$\int_{t_0}^{t_f} (x^{*T} C \, \Delta x + u^{*T} E \, \Delta u) \, dt = -\int_{t_0}^{t_f} (d/dt)(\lambda^{*T} \, \Delta x) \, dt = -\lambda^{*T}(t_f) \, \Delta x_f$$

$$= -(d/dx_f) G(x_f^*, \nu^*) \, \Delta x_f,$$

---

[9] This proof was suggested by similar proofs due to McReynolds (Ref. 7) and Wood (Ref. 8).

we have

$$
\begin{aligned}
\bar{J} - J^* = {} & G(\bar{x}_f, \nu^*) - G(x_f^*, \nu^*) - (d/dx_f)G(x_f^*, \nu^*)\,\Delta x_f - \tfrac{1}{2}\Delta x_1^T M(t_1, q)\,\Delta x_1 \\
& -\tfrac{1}{2}\Delta x_f^T S(t_f, q)\,\Delta x_f \\
& -\nu^T R^T(t_f, q)\,\Delta x_f + \tfrac{1}{2}\Delta x_1^T S(t_1, q)\,\Delta x_1 + \nu^T R^T(t_1, q)\,\Delta x_1 \\
& +\tfrac{1}{2}\nu^T Q(t_1, q)\nu \\
& +\tfrac{1}{2}\int_{t_0}^{t_1} [\Delta x^T(C + MA + A^T M + \dot{M})\Delta x + \Delta x^T MB\,\Delta u + \Delta u^T B^T M \Delta x \\
& +\Delta u^T E\,\Delta u]\,dt \\
& +\tfrac{1}{2}\int_{t_1}^{t_f} [\Delta x^T(C + SA + A^T S + \dot{S})\,\Delta x + \Delta x^T SB\,\Delta u + \Delta u^T B^T S\,\Delta x \\
& +\Delta u^T E\,\Delta u \\
& +2\nu^T(R^T A + \dot{R}^T)\,\Delta x + 2\nu^T R^T B\,\Delta u + \nu^T \dot{Q}\nu]\,dt,
\end{aligned}
$$

where $\Delta x_1 = \Delta x(t_1)$ and M, S, R, Q in the integrands stand for $M(t, q), S(t, q)$, etc. Then, using (14)–(16), we have

$$
\begin{aligned}
\bar{J} - J^* = {} & G(q, \nu^*) - G(x_f^*, \nu^*) - (d/dx_f)G(x_f^*, \nu^*)\,\Delta x_f - \tfrac{1}{2}\Delta x_f^T F(q)\,\Delta x_f \\
& -\nu^T T^T(q)\,\Delta x_f \\
& +\tfrac{1}{2}\Delta x_1^T R(t_1, q)Q^{-1}(t_1, q)R^T(t, q)\,\Delta x_1 + \nu^T R^T(t_1, q)\,\Delta x_1 \\
& +\tfrac{1}{2}\nu^T Q(t_1, q)\nu \\
& +\tfrac{1}{2}\int_{t_0}^{t_1} [\Delta u + E^{-1}B^T M(t, q)\,\Delta x]^T E[\Delta u + E^{-1}B^T M(t, q)\,\Delta x]\,dt \\
& +\tfrac{1}{2}\int_{t_0}^{t_f} \{\Delta u + E^{-1}B^T[S(t, q)\,\Delta x + R(t, q)\nu]\}^T E\{\Delta u \\
& +E^{-1}B^T[S(t, q)\,\Delta x + R(t, q)\nu]\}\,dt.
\end{aligned}
$$

Now, using Assumption (A.4), Eqs. (12)–(13), the condition

$$
\psi(q) = \psi^*(x_f^*) = 0,
$$

which implies

$$T^T(q)\Delta x_f = 0,$$

and choosing

$$\nu = -Q^{-1}(t_1, q)R^T(t_1, q)\,\Delta x(t_1),$$

we are left with the two integrals appearing in the above expression of $J - J^*$. Then, Assumption (A.2) implies

$$J - J^* \geq 0.$$

Also, the first integral above, between t_0 and t_1, cannot be zero; this would imply that

$$\Delta u(t) = -E^{-1}B^T M(t, q)\,\Delta x(t) \qquad \text{a.e. on } [t_0, t_1],$$

which, together with (2), $M(t, q)$ finite on $[t_0, t_1]$, and $\Delta x(t_0) = 0$ would imply

$$\Delta x(t) \equiv 0, \qquad t \in [t_0, t_1],$$

a contradiction. Hence the result of Lemma 2.1 follows.   □

Proof of Lemma 4.2.   Let

$$\Delta J \triangleq J_q[t_{cq}, t_f] - J^*[t_{cq}, t_f]$$

be the difference between the values of the cost functional given by the trajectories x_q and x^* between t_{cq} and t_f, and define

$$\Delta x(t) \triangleq x_q(t) - x^*(t), \qquad \Delta u(t) \triangleq u_q(t) - u^*(t),$$

where $u_q(t)$ is the control associated with $x_q(t)$. Note that Δx and Δu are the quantities $y_{\bar{a}}$ and $v_{\bar{a}}$ in the proof of Lemma 4.1. since $x_q(t_f) = q$ and $x^*(t_f)$ are in θ, we can write

$$\Delta J = G(q, \nu^*) - G(x_f^*, \nu^*) + \tfrac{1}{2}\int_{t_{cq}}^{t_f} (\Delta x^T C\,\Delta x + \Delta u^T E\,\Delta u)\,dt$$

$$+ \int_{t_{cq}}^{t_f} (x^{*T} C\,\Delta x + u^{*T} E\,\Delta u)\,dt,$$

where $G(x_f, \nu^*)$ was defined in (11).

By construction, $\Delta x(t)$ satisfies the first-order necessary conditions for optimality of $(LQP)_q$ with some multiplier function $\Delta\lambda(t)$. It follows that

$$\Delta u(t) = -E^{-1}(t)B^T(t)\,\Delta\lambda(t)$$

and

$$\Delta x^T C\,\Delta x + \Delta u^T E\,\Delta u = -(d/dt)[\Delta\lambda^T\,\Delta x],$$

$$x^{*T} C\,\Delta x + u^{*T} E\,\Delta u = -(d/dt)[\lambda^{*T}\,\Delta x].$$

Then,

$$\Delta J = G(q, \nu^*) - G(x_f^*, \nu^*) - \lambda^*(t_f)\, \Delta x_f - \tfrac{1}{2} \Delta \lambda^T(t_f)\, \Delta x_f;$$

and, using (7), (11), and (12), we have

$$\Delta J = \tfrac{1}{2} \Delta x_f^T F(q)\, \Delta x_f - \tfrac{1}{2} \Delta \lambda^T(t_f)\, \Delta x_f.$$

Using

$$\Delta \lambda(t_f) = F(q)\, \Delta x(t_f) + T^T(q)\nu,$$

for some ν, yields

$$\Delta J = -\tfrac{1}{2}\nu^T T(q)\, \Delta x_f = -\tfrac{1}{2}\nu^T T(q)(q - x_f^*) = 0. \qquad \square$$

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