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TECHNICAL NOTE

An Elementary Proof of an Equivalence Theorem Relevant in the Theory of Optimization

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Abstract. The authors give an elementary proof of an equivalence theorem of analysis which is often used in optimization theory. The theorem asserts that certain conditions are equivalent to weak convergence in L_1. One is the Dunford–Pettis condition concerning absolute integrability. Two others are expressed in terms of Nagumo functions, and can be thought of as growth properties. The original proofs of the various parts of the theorem are scattered in different and specialized mathematical publications. The authors feel it useful to present here a straightforward proof of the various parts in terms of standard Lebesgue integration theory.

Key Words. Weak convergence in L_1, absolute integrability, equiabsolute integrability, Nagumo functions, absolute continuity, equiabsolute continuity, weak relative compactness in L_1, Ascoli's theorem, Lusin's theorem.

1. Introduction

Recently, an equivalence theorem of real analysis has been often used in optimization theory. The theorem establishes conditions equivalent to weak convergence in L_1 and was reported without proof, e.g., in the two books of Ekeland and Temam (Ref. 1, p. 223) and Cesari (Ref. 2, p. 329). Since the proofs of the various parts of the theorem are scattered in different and specialized mathematical publications, we feel it proper to give here a plain and elementary proof of the various parts of the theorem.

The theorem can be stated for Lebesgue measures in R^n, or for abstract measure spaces, finite or σ-finite, with or without atoms. We prefer to

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present it here in its simplest and typical form, namely for Lebesgue measure and a finite interval in R^1, and the proofs are based on sole Lebesgue integration for functions of one real variable.

Theorem 1.1. *Equivalence Theorem.* Let $\{f(t), a \leq t \leq b\}$ be a family of real-valued L-integrable functions on a fixed finite interval $[a, b]$. The following statements are equivalent:

(a) The family $\{f\}$ is sequentially weakly relatively compact in $L_1[a, b]$.

(b) The family $\{f\}$ is equiabsolutely integrable on $[a, b]$.

(c) There is a constant M and a real-valued function $\phi = \phi(\xi)$, $0 \leq \xi < +\infty$, bounded below, such that $\phi(\xi)/\xi \to +\infty$, as $\xi \to +\infty$, and

$$\int_a^b \phi(|f(t)|)\, dt \leq M, \qquad \text{for all } f \in \{f\}.$$

(d) There is a real-valued function $\psi = \psi(\xi)$, $0 \leq \xi < +\infty$, bounded below, such that $\psi(\xi)/\xi \to +\infty$, as $\xi \to +\infty$, and the family $\{\psi(|f|), f \in \{f\}\}$ is equiabsolutely integrable on $[a, b]$.

In (c), (d), it is not restricting to assume ϕ, ψ nonnegative, strictly increasing, continuous, and convex in $[0, +\infty)$. Functions ϕ or ψ, as above, are often called *Nagumo functions.*

The equivalence of (a) and (b) was proved by Dunford and Pettis (see, e.g., Edwards, Ref. 3, p. 274). The implication (b)$\Rightarrow$(c) was proved by de la Vallée-Poussin (for the statement, see, e.g., Natanson, Ref. 4, p. 164). The implication (b)$\Rightarrow$(c)$\cup$(d) was also proved directly by Candeloro and Pucci (Ref. 5). The implication (c)$\Rightarrow$(b) was proved by Tonelli (Ref. 6, Vol. II, p. 283), for some particular ϕ, and then by Nagumo in the general case (see, e.g., McShane, Ref. 7, p. 176). Here, (d)$\Rightarrow$(c) is trivial, and we shall prove directly that (c)$\Rightarrow$(d).

In the theory of optimization, say for functionals $\int_a^b G(t, u(t), u'(t))\, dt$, several parts of the above theorem are often used in proving the existence of an absolute minimum. For instance, if $G(t, x, p) \geq \phi(|p|)$, with ϕ as in (c), then the conclusion (c)$\Rightarrow$(b) is used to guarantee that, for any minimizing sequence $[u_n]$ in a bounded domain, the sequence of derivatives $[u_n']$ is equiabsolutely integrable; hence, the sequence $[u_n]$ itself is equiabsolutely continuous, and Ascoli's theorem applies. For instance, the conclusion (a)$\Rightarrow$(d) is often used to guarantee that if, for a minimizing sequence $[u_n]$, the sequence of derivatives $[u_n']$ is weakly convergent in L_1, then the same sequence $[u_n']$ also possesses property (d), which by closure theorems leads directly to the existence of the minimum.

For the convenience of the reader, we recall here briefly the main concepts which are dealt with in the equivalence theorem.

A sequence $[f_n(t)]$ of L_1-integrable functions in $[a, b]$ is said to be weakly relatively compact in L_1 provided there is an L_1-integrable function $f(t)$ in $[a, b]$ and a sequence $[n_k]$ such that $\int_a^b (f_{n_k}(t)-f(t))g(t)\,dt \to 0$, as $k\to\infty$, for every measurable and bounded function $g(t)$. The family $\{f\}$ of L_1-integrable functions in $[a, b]$ is said to be sequentially weakly relatively compact provided any sequence $[f_n(t)]$ of functions f_n of the family is weakly relatively compact in L_1.

It is well known that any L_1-integrable function $f(t)$ in $[a, b]$ is *absolutely integrable*; that is, given $\epsilon>0$, there is some $\delta>0$, $\delta=\delta(\epsilon, f)$, such that, for any measurable subset B of $[a, b]$ with meas $B<\delta$, we also have $\int_B |f(t)|\,dt<\epsilon$. A family $\{f\}$ of L_1-integrable functions f in $[a, b]$ is said to be equiabsolutely integrable provided, given $\epsilon>0$, there is some $\delta>0$, $\delta=\delta(\epsilon)$, such that, for every function f of the family and for any measurable subset B of $[a, b]$ with meas $B<\delta$, we also have $\int_B |f(t)|\,dt<\epsilon$. In other words, δ may depend on ϵ, but not on the function f of the family.

2. Proof of the Theorem

(a)$\Rightarrow$(b). We assume that (a) holds, and we want to prove (b). The proof is by contradiction. Let us assume that (b) is not true. Then, there are a number $\epsilon>0$, measurable sets $E_k\subset[a, b]$, and functions $f_k\in\{f\}$ such that, for every $k=1, 2, \ldots,$

$$\int_{E_k} |f_k(t)|\,dt>3\epsilon \text{ and } |E_k|\to 0, \qquad \text{as } k\to+\infty.$$

By (a), there is a subsequence, say still (k) for the sake of simplicity, and an L-integrable function f such that $f_k\rightharpoonup f$ weakly in L_1. Since f is a fixed L-integrable function and $|E_k|\to 0$, then, for k sufficiently large, we have $\int_{E_k} |f(t)|\,dt<\varepsilon$, and hence $\int_{E_k} |f_k(t)-f(t)|\,dt>2\epsilon$. By relabeling and replacing f_k-f by f_k, we have a sequence (f_k) with $f_k\rightharpoonup 0$ weakly in L_1 and such that $\int_{E_k} |f_k(t)|\,dt>2\epsilon$, for all k, and $|E_k|\to 0$, as $k\to+\infty$. If E_k^+ and E_k^- denote the subsets of E_k where f_k is nonnegative and f_k is negative, respectively, we have either $\int_{E_k^+} f_k(t)\,dt>\epsilon$ or $\int_{E_k^-} (-f_k(t))\,dt>\epsilon$. One of the two cases must occur infinitely many times, say the first case. By extraction and relabeling, we may always assume

$$\int_{E_k^+} f_k(t)\,dt>\epsilon, \text{ for all } k, \qquad E_k^+|\to 0, \text{ as } k\to+\infty.$$

Since f_1 is by itself absolutely integrable, there is some $\sigma_1>0$ such that $S\subset[a, b]$, $|S|<\sigma_1$ implies $\int_S |f_1(t)|\,dt<\epsilon/4$. Thus, by discarding elements and relabeling, we may well assume $|E_s^+| < \sigma_1/2^{s-1}$, $s = 2, 3, \ldots$, and we take

$$E_1' = E_1^+ - \bigcup_{s=2}^{\infty} E_s^+,$$

so that $|\bigcup_{s=2}^{\infty} E_s^+| < \sigma_1(2^{-1} + 2^{-2} + \cdots) = \sigma_1$, and

$$\int_{E_1'} f_1(t)\, dt = \int_{E_1^+} f_1(t)\, dt - \int_{\bigcup_{s=2}^{\infty} E_s^+} f_1(t)\, dt > \epsilon - \epsilon/4 = 3\epsilon/4.$$

Now, we keep f_1, E_1' fixed, and we note that $\int_{E_1'} f_s(t)\, dt \to 0$, as $s \to +\infty$. By dropping terms, we can well assume that

$$\left| \int_{E_1'} f_s(t)\, dt \right| < \epsilon/2^3, \qquad \text{for all } s = 2, 3, \ldots.$$

Since f_2 is by itself absolutely integrable, there is some $\sigma_2 > 0$ such that $S \subset [a, b]$, $|S| < \sigma_2$ implies $\int_S |f_2(t)|\, dt < \epsilon/4$. Thus, by discarding elements and relabeling, we may well assume $|E_s^+| < \sigma_2/2^{s-2}$, $s = 3, 4, \ldots$, and we take

$$E_2' = E_s^+ - \bigcup_{s=3}^{\infty} E_s^+,$$

so that $|\bigcup_{s=3}^{\infty} E_s^+| < \sigma_2(2^{-1} + 2^{-2} + \cdots) = \sigma_2$, and

$$\int_{E_2'} f_2(t)\, dt = \int_{E_2^+} f_2(t)\, dt - \int_{\bigcup_{s=3}^{\infty} E_s^+} f_2(t)\, dt > \epsilon - \epsilon/4 = 3\varepsilon/4.$$

Now, we keep f_1, f_2, E_1', E_2' fixed, and we note that $\int_{E_1'} f_s(t)\, dt \to 0$, $\int_{E_2'} f_s(t)\, dt \to 0$, as $s \to +\infty$. By dropping terms, we can well assume that

$$\left| \int_{E_1'} f_s(t)\, dt \right| < \epsilon/2^3, \quad \left| \int_{E_2'} f_s(t)\, dt \right| < \epsilon/2^4, \qquad \text{for all } s = 3, 4, \ldots.$$

We can repeat this process. At the kth step, we still have $\int_{E_k^+} f_k(t)\, dt > \epsilon$. Since f_k is by itself absolutely integrable, there is some $\sigma_k > 0$ such that $S \subset [a, b]$, $|S| < \sigma_k$ implies $\int_S |f_k(t)|\, dt < \epsilon/4$. Thus, by discarding elements and relabeling, we may always assume $|E_s^+| < \sigma_k/2^{s-k}$, $s = k+1, k+2, \ldots$, and we take

$$E_k' = E_k^+ - \bigcup_{s=k+1}^{\infty} E_s^+,$$

so that $|\bigcup_{s=k+1}^{\infty} E_s^+| < \sigma_k(2^{-1} + 2^{-2} + \cdots) = \sigma_k$, and

$$\int_{E_k'} |f_k(t)|\, dt = \int_{E_k^+} f_k(t)\, dt - \int_{\bigcup_{s=k+1}^{\infty} E_s^+} f_k(t)\, dt > \epsilon - \epsilon/4 = 3\epsilon/4. \tag{1}$$

Now, we keep $f_1, \ldots, f_k$, $E'_1, \ldots, E'_k$ fixed, and we note that $\int_{E'_\sigma} f_s(t)\, dt \to 0$, as $s \to +\infty$, for $\sigma = 1, 2, \ldots, k$. By dropping terms, we shall assume that

$$\left| \int_{E'} f_s(t)\, dt \right| < \epsilon/2^{\sigma+2}, \qquad \sigma = 1, \ldots, k,\ s = k+1, k+2, \ldots. \tag{2}$$

Now, the entire sequence (f_k) is determined together with the corresponding sets E'_k, and the sets E'_k, $k = 1, 2, \ldots$, are disjoint in $[a, b]$.

Note that the present sequence (f_k) is a subsequence of the original sequence (f_k), but for the displacement $f_k \to f_k - f$ and the deletion of many terms, we have $f_k \rightharpoonup 0$, weakly in L_1.

Now we take $g = g(t)$, $a \leq t \leq b$, as follows:

$$g(t) = 1, \quad \text{for } t \in E'_s,\ s = 1, 2, \ldots, \qquad g(t) = 0, \quad \text{otherwise.}$$

For any k, we have

$$\int_a^b f_k(t) g(t)\, dt = \sum_{s=1}^{\infty} \int_{E'_s} f_k(t) g(t)\, dt$$
$$= \left[\sum_{s=1}^{k-1} \int_{E'_s} + \int_{E'_k} + \int_{\bigcup_{s=k+1}^{\infty} E'_s} \right] f_k(y)\, dt,$$

where from (1) we have

$$\int_{E'_k} f_k(t)\, dt > 3\epsilon/4.$$

Since $E'_s \subset E^+_s$, $|\bigcup_{s=k+1}^{\infty} E^+_s| < \sigma_k$, we also have

$$\int_{\bigcup_{s=k+1}^{\infty} E'_s} f_k(t)\, dt \leq \int_{\bigcup_{s=k+1}^{\infty} E'_s} |f_k(t)|\, dt < \epsilon/4.$$

Finally, by (2), we also have

$$\left| \int_{E'_s} f_k(t)\, dt \right| < \epsilon/2^{s+2}, \qquad s = 1, \ldots, k-1.$$

Thus,

$$\int_a^b f_k(t) g(t)\, dt > 3\epsilon/4 - \epsilon/4 - (\epsilon/2^3 + \epsilon/2^4 + \cdots + \varepsilon/2^{k+1}) > \epsilon/4,$$

and this holds for all k, a contradiction, since $\int_a^b f_k(t) g(t)\, dt \to 0$, as $k \to +\infty$.

(b)$\Rightarrow$(a). Since $\{f\}$ is equiabsolutely integrable, given $\varepsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that, for any measurable set $E \subset [a, b]$, with $|E| < \delta$, we have $\int_E |f(t)|\, dt < \epsilon$, for any $f \in \{f\}$. Now, we consider the family $\{F\}$ of the functions $F(t)=\int_a^t f(\tau)\,d\tau$, for $f\in\{f\}$. The functions F are equiabsolutely continuous. Indeed, given $\epsilon>0$, for every finite system $\{[\alpha_i,\beta_i], i=1,\ldots,N\}$ of nonoverlapping intervals in $[a,b]$, with $\sum_{i=1}^N(\beta_i-\alpha_i)<\delta$, we have, for $E=\bigcup_{i=1}^N[\alpha_i,\beta_i]$,

$$\sum_{i=1}^N |F(\beta_i)-F(\alpha_i)|\leq\int_E |f(t)|\,dt<\epsilon. \tag{3}$$

Moreover, if N is any integer with $N>(b-a)/\delta$, then there is a subdivision of $[a,b]$ into parts $a=\alpha_0<\alpha_1<\cdots<\alpha_N=b$, each part of length $\alpha_i-\alpha_{i-1}<\delta$, such that

$$\int_a^b |f(t)|\,dt=\sum_{i=1}^N\int_{\alpha_{i-1}}^{\alpha_i}|f(t)|\,dt<N\epsilon.$$

Thus, $\int_a^b|f(t)|\,dt<M$, for some constant M and all $f\in\{f\}$. Consequently, $|F|<M$, as well as $V(F)<M$, for all $F\in\{F\}$. This last point can also be proved by noting that, for any further subdivision $\alpha_{i-1}=\alpha_{i_0}<\alpha_{i_1}<\cdots<\alpha_{i_{N'}}=\alpha_i$, we certainly have

$$\sum_{s=1}^{N'}|F(\alpha_{i_s})-F(\alpha_{i_{s-1}})|<\epsilon. \tag{4}$$

Thus, F has a total variation in $[\alpha_{i-1},\alpha_i]$ which is less or equal to ϵ and has a total variation $V(F)\leq\epsilon$, in $[a,b]$.

Let (f_k) be any sequence of functions $f_k\in\{f\}$, and let F_k be the corresponding functions F. Then, the functions F_k are equibounded and equicontinuous; and, by Ascoli's theorem (cf., e.g., Ref. 7), there is a subsequence, say still (k) for the sake of simplicity, such that (F_k) converges uniformly in $[a,b]$ toward a continuous function $F_0=F_0(t)$, $a\leq t\leq b$. Actually, relation (3) holds for all $F=F_k$. By a passage to the limit, the same relation holds for F_0; that is, F_0 is absolutely continuous. Also $|F_k(t)|<M$, for all k and t; hence, $|F_0(t)|\leq M$. Note that, for any interval $[\alpha_{i-1},\alpha_i]$ and any subdivision into parts $\alpha_{i-1}=\alpha_{i_0}<\alpha_{i_1}<\cdots<\alpha_{i_{N'}}=\alpha_i$, relation (4) holds for every F_k. Then, at the limit, as $k\to+\infty$, the same relation holds for F_0. In other words, F_0 has a total variation less or equal to ϵ in $[\alpha_{i-1},\alpha_i]$ and has a total variation $V(F_0)\leq N\epsilon$, in $[a,b]$. Thus, as before, we have $|F_0(t)|\leq M$, $V(F_0)\leq M$.

Let $f_0=F_0'$; thus, f_0 is L-integrable, and we shall prove that (f_k) converges weakly to f_0 in L_1.

Let $g=g(t)$, $a\leq t\leq b$, be any bounded measurable function in $[a,b]$. We must prove that $\int_a^b(f_k(t)-f_0(t))g(t)\,dt\to 0$, as $k\to+\infty$. Let M_0 be a number such that $|g(t)|\leq M_0$, for a.a. $t\in[a,b]$.

Again, let $\epsilon > 0$ be any given number, take $\epsilon_0 = \epsilon(12M_0)^{-1}$, and let $\delta = \delta(\epsilon_0)$ be the number δ above relative to the equiabsolute integrability of the functions $f \in \{f\}$.

By Lusin's theorem, we know that there is a compact subset K of $[a, b]$ such that g is continuous on K and $|K| > b - a - \delta$. Then, the open set $G = [a, b] - K$ is the union of at most countably many intervals I_j, $j = 1, 2, \ldots$, whose total length is $|G| = \sum_{j=1}^{\infty} |I_j| < \delta$. Note that, for any finite system $G_N = \bigcup_{j=1}^{N} I_j$, we certainly have $|G_N| < \delta$ and

$$V(F_k, G_N) = \int_{G_N} |f_k(t)| \, dt < \epsilon_0;$$

and, by the lower semicontinuity property of the total variation (cf., e.g., Ref. 8),

$$V(F_0, G_N) \leq \liminf_k V(F_k, G_N) \leq \epsilon_0, \quad \text{or} \quad \int_{G_N} |f_0(t)| \, dt \leq \epsilon_0.$$

As $N \to +\infty$, we have now

$$V(F_k, G) = \int_G |f_k(t)| \, dt \leq \epsilon_0, \qquad V(F_0, G) = \int_G |f_0(t)| \, dt \leq \varepsilon_0.$$

Let $\sigma = \epsilon(6M)^{-1}$.

The function g is continuous on the compact set K, hence uniformly continuous on K, and there is some $\delta_1 > 0$ such that $|g(t) - g(\tau)| < \sigma$, for all $t, \tau \in K$, $|t - \tau| < \delta_1$.

Let $a = \alpha_0 < \alpha_1 < \cdots < \alpha_N = b$ be any finite subdivision of $[a, b]$ into equal parts, each of length less than or equal to $\min\{\delta, \delta_1\}$; take $E_i = [\alpha_{i-1}, \alpha_i] - K$, $K_i = [\alpha_{i-1}, \alpha_i] \cap K$, and choose any point $c_i \in K_i$, $i = 1, \ldots, N$. If, for some i, the set K_i is empty, let us drop the corresponding term in the sum below. Take k_0 large enough so that $|F_k(t) - F_0(t)| < \epsilon(6NM_0)^{-1}$, for all $t \in [a, b]$ and $k > k_0$. Now,

$$\begin{aligned}
\int_a^b (f_k(t) - f_0(t)) g(t) \, dt &= \sum_{i=1}^{N} \int_{\alpha_{i-1}}^{\alpha_i} (f_k(t) - f_0(t)) g(c_i) \, dt \\
&\quad + \sum_{i=1}^{N} \int_{K_i} (f_k(t) - f_0(t))(g(t) - g(c_i)) \, dt \\
&\quad + \sum_{i=1}^{N} \int_{E_i} (f_k(t) - f_0(t))(g(t) - g(c_i)) \, dt \\
&= i_1 + i_2 + i_3,
\end{aligned}$$

with $|g(c_i)| \leq M_0$; and, for $k > k_0$, also

$$|i_1| = \left| \sum_{i=1}^{N} g(c_i)[F_k(\alpha_i) - F_k(\alpha_{i-1}) - F_0(\alpha_i) + F_0(\alpha_{i-1})] \right|$$

$$\leq M_0 \sum_{i=1}^{N} [|F_k(\alpha_i) - F_0(\alpha_i)| + |F_k(\alpha_{i-1}) - F_0(\alpha_{i-1})|]$$

$$< 2M_0 N\epsilon(6NM_0)^{-1} = \epsilon/3.$$

Also, $|g(t) - g(c_i)| < \sigma$, for $t \in K_i = [\alpha_{i-1}, \alpha_i] \cap K$; hence,

$$|i_2| = \left| \sum_{i=1}^{N} \int_{K_i} (f_k(t) - f_0(t))(g(t) - g(c_i))\, dt \right|$$

$$\leq \sigma \int_a^b (|f_k(t)| + |f_0(t)|)\, dt < 2M\sigma = \epsilon/3.$$

Finally, $|g(t)| \leq M_0$, a.e. in E_i, and $\bigcup_{i=1}^{\infty} E_i = G$, $|G| < \delta$, with $\int_G |f_k(t)|\, dt < \epsilon_0$, $\int_G |f_0(t)|\, dt \leq \epsilon_0$. Hence,

$$|i_3| = \left| \sum_{i=1}^{N} \int_{E_i} (f_k(t) - f_0(t))(g(t) - g(c_i))\, dt \right|$$

$$\leq 2M_0 \sum_{i=1}^{N} \int_{E_i} (|f_k(t)| + |f_0(t)|)\, dt$$

$$= 2M_0 \int_G (|f_k(t)| + |f_0(t)|)\, dt < 2M_0 2\epsilon_0 = 4M_0\epsilon(12M_0)^{-1} = \epsilon/3.$$

Thus, for $k > k_0$, we have

$$\left| \int_a^b (f_k(t) - f_0(t))g(t)\, dt \right| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

We have proved that $f_k \rightharpoonup f_0$, weakly in L_1.

(b)$\Rightarrow$(c). As before, we know that the family $\{f\}$ is equiabsolutely integrable; in other words, given $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that, for any set $S \subset [a, b]$, with $|S| < \delta$, we have $\int_S |f(t)|\, dt < \epsilon$, for every $f \in \{f\}$. As before, we conclude that there is some $M > 0$ such that $\int_a^b |f(t)|\, dt < M$, for all $f \in \{f\}$.

Note that, if $E(\lambda, f)$ denotes the set $E(\lambda, f) = \{t \in [a, b]: |f(t)| \geq \lambda\}$, then $\lambda|E| \leq \int_E |f(t)|\, dt < M$; hence, $|E(\lambda, f)| < M\lambda^{-1}$, for all $\lambda > 0$ and $f \in \{f\}$. Thus, given $\epsilon > 0$, we can take $\lambda = \lambda(\epsilon) > 0$ such that $M\lambda^{-1} \leq \delta$, and then $|E(\lambda, f)| < \delta$, and $\int_E |f(t)|\, dt < \epsilon$. We have proved that, given $\epsilon > 0$, there is $\lambda = \lambda(\varepsilon) > 0$ such that $\lambda|E(\lambda, f)| \leq \int_{E(\lambda, f)} |f(t)|\, dt < \epsilon$, for all $f \in \{f\}$.

We conclude that there are numbers $\lambda_s > 0$, such that, for $E_s = E(\lambda_s, f)$, we have

$$\int_{E_s} |f(t)|\, dt < 2^{-s}, \qquad \lambda_{s+1}/\lambda_s > 2,$$

and hence $\lambda_s|E_s| \leq \int_{E_s} |f(t)|\, ds < 2^{-s}$, and $|E_s| < 2^{-s}\lambda_s^{-1}$.

Now, we define $\phi = \phi(\xi)$, $0 \leq \xi < +\infty$, by taking $\phi(\xi) = 0$, for $\xi = \lambda_0 = 0$, $\phi(\xi) = s\lambda_s$, for $\xi = \lambda_s$, and let ϕ be linear in each interval $[0, \lambda_1]$, $[\lambda_1, \lambda_2], \ldots, [\lambda_{s-1}, \lambda_s]$, $s = 1, 2, \ldots$.

Note that $\lambda_{s+1}/\lambda_s > 2$ implies that $\lambda_{s+1}(\lambda_{s+1} - \lambda_s)^{-1} < 2$; in other words, for the slope h_s of ϕ between λ_s and λ_{s+1}, we have

$$h_s = [(s+1)\lambda_{s+1} - s\lambda_s]/(\lambda_{s+1} - \lambda_s) = s + \lambda_{s+1}/(\lambda_{s+1} - \lambda_s);$$

hence, $s < h_s < s+2$. Moreover, the inequality

$$s + \lambda_{s+1}/(\lambda_{s+1} - \lambda_s) < s + 1 + \lambda_{s+2}/(\lambda_{s+2} - \lambda_{s+1})$$

is equivalent to

$$\lambda_{s+1}/\lambda_s - 1 > 2 - 1 = 1 > 1 - \lambda_{s+1}/\lambda_{s+2}.$$

In other words, $h_s < h_{s+1}$.

Thus, ϕ is nonnegative, continuous, strictly increasing, and convex. Moreover, for $\lambda_s \leq \xi < \lambda_{s+1}$, we have $h_s > s$ and

$$\phi(\xi)/\xi = [s\lambda_s + h_s(\xi - \lambda_s)]/(\lambda_s + \xi - \lambda_s) > s, \qquad \lambda_s \leq \xi < \lambda_{s+1}.$$

This proves that $\phi(\xi)/\xi \to +\infty$, as $\xi \to +\infty$.

Finally, if $F_0 = \{t \in [a, b]: |f(t)| < \lambda_1\}$, and $F_s = \{t \in [a, b]: \lambda_s \leq |f(t)| < \lambda_{s+1}\}$, $s = 1, 2, \ldots$, then

$$\int_a^b \phi(|f(t)|)\, dt = \left(\int_{F_0} + \sum_{s=1}^{\infty} \int_{F_s} \right) \phi(|f(t)|)\, dt,$$

with $0 \leq \phi(\xi) < \lambda_1$, for $0 \leq \xi < \lambda_1$, and $\phi(\xi) = s\lambda_s + h_s(\xi - \lambda_s)$, for $\lambda_s \leq \xi < \lambda_{s+1}$. Thus,

$$\begin{aligned}\int_a^b \phi(|f(t)|)\, dt &< \lambda_1(b-a) + \sum_{s=1}^{\infty} s\lambda_s|F_s| + \sum_{s=1}^{\infty} \int_{F_s} h_s(|f(t)| - \lambda_s)\, dt \\ &< \lambda_1(b-a) + \sum_{s=1}^{\infty} s\lambda_s(2^{-s}\lambda_s^{-1}) + \sum_{s=1}^{\infty} (s+2)2^{-s} \\ &= \lambda_1(b-a) + \sum_{s=1}^{\infty} s2^{-s} + \sum_{s=1}^{\infty} (s+2)2^{-s} = M_1.\end{aligned}$$

Thus, $\int_a^b \phi(|f(t)|)\, dt < M_1$, for all $f \in \{f\}$, where M_1 is a fixed constant. For the argument above, cf. Ref. 5.

Remark 2.1. In (c) and (d), we can always choose ϕ and ψ functions which are nonnegative, strictly increasing, continuous, and convex.

First, let us show that we can take $\phi \geq 0$ in (c). Indeed, if ϕ has not this property, let us take $-L_0 = \inf \phi(\xi)$, and take $\phi_1(\xi) = L_0 + \phi(\xi)$, so that $\phi_1(\xi) \geq 0$, and

$$\begin{aligned}\int_a^b \phi_1(|f(t)|)\,dt &= L_0(b-a) + \int_a^b \phi(|f(t)|\,dt \\ &\leq L_0(b-a) + M = M_1 < +\infty.\end{aligned}$$

Now, assume $\phi(\xi) \geq 0$. By induction, we shall define numbers $0 = h_1 < h_2 < \cdots$, $0 = \lambda_1 < \lambda_2 < \cdots$, $h_n \to +\infty$, $\lambda_n \to +\infty$, $h_{n+1} \geq h_n + 1$, and take

$$\tilde{\phi}(\xi) = \Lambda_{n-1} + h_n(\xi - \lambda_{n-1}), \qquad \lambda_{n-1} \leq \xi < \lambda_n,$$

$$\lambda_0 = 0, \qquad \Lambda_0 = 0, \qquad \Lambda_1 = 0, \qquad \Lambda_n = \Lambda_{n-1} + h_n(\lambda_n - \lambda_{n-1}).$$

Let

$$h_n = \inf\{[\phi(\xi) - \Lambda_{n-1}]/(\xi - \lambda_{n-1}),\ \xi \geq \lambda_{n-1}\},$$

so that

$$\phi(\xi) \geq \Lambda_{n-1} + h_n(\xi - \lambda_{n-1}), \qquad \text{for all } \xi \geq \lambda_{n-1},$$

and take λ_n so large that

$$\phi(\xi)/\xi \geq h_n + 1 + (\Lambda_{n-1} - h_n\lambda_{n-1} - \lambda_n)/\xi, \qquad \text{for all } \epsilon \geq \lambda_n.$$

Then, for $\xi \geq \lambda_n$, we also have

$$\begin{aligned}\phi(\xi) &\geq (h_n + 1)\xi + \Lambda_{n-1} - h_n\lambda_{n-1} - \lambda_n \\ &= \Lambda_{n-1} + h_n(\xi - \lambda_{n-1}) + \xi - \lambda_n \\ &= \Lambda_{n-1} + h_n(\lambda_n - \lambda_{n-1}) + (h_n + 1)(\xi - \lambda_n) \\ &= \Lambda_n + (h_n + 1)(\xi - \lambda_n),\end{aligned}$$

and hence

$$h_{n+1} = \inf\{[\phi(\xi) - \Lambda_n]/(\xi - \lambda_n),\ \xi \geq \lambda_n\} \geq h_n + 1,$$

that is, $h_{n+1} \geq h_n + 1$. Now, $\tilde{\phi}$ is a nonnegative, continuous, strictly increasing, convex function. Since $h_n \to +\infty$, given $k > 0$, there is some n such that $h_n \leq k < h_{n+1} < \cdots$; and then, for all ξ sufficiently large, say $\lambda_m \leq \xi \leq \lambda_{m+1}$, $m > n$, since $h_{n+1} < h_{n+2} < \cdots < h_{m+1}$, we have

$$\begin{aligned}\tilde{\phi}(\xi) &= \Lambda_m + h_{m+1}(\xi - \lambda_m) \\ &= \Lambda_n + h_{n+1}(\lambda_{n+1} - \lambda_n) + \cdots + h_m(\lambda_m - \lambda_{m-1}) + h_{m+1}(\xi - \lambda_m) \\ &\geq \Lambda_n + h_{n+1}(\xi - \lambda_n) = h_{n+1}\xi + \Lambda_n - h_{n+1}\lambda_n.\end{aligned}$$

Thus, $\tilde{\phi}(\xi)/\xi \geq h_{n+1} + (\Lambda_n - h_{n+1}\lambda_n)\xi^{-1} > k$, for all ξ sufficiently large. Thus, $\phi(\xi)/\xi \to +\infty$, as $\xi \to +\infty$.

Analogous argument holds for ψ in (d).

(c)$\Rightarrow$(b). From the remark above, we can assume ϕ to be nonnegative, and we know that $\int_a^b \phi(|f(t)|)\, dt \leq M$, for some constant M and all $f \in \{f\}$, and $\phi(\xi)/\xi \to +\infty$, as $\xi \to +\infty$. Given $\epsilon > 0$, let $\sigma = 2\epsilon^{-1}M$, and let $\lambda_0 > 0$ be such that $\phi(\lambda)/\lambda > \sigma$, for all $\lambda > \lambda_0$. Let E denote the set of all $t \in [a, b]$ with $|f(t)| > \lambda_0$. Then,

$$\int_E |f(t)|\, dt < \sigma^{-1} \int_E \phi(|f(t)|)\, dt \leq \sigma^{-1} \int_a^b \phi(|f(t)|)\, dt \leq \sigma^{-1}M = \epsilon/2.$$

Now, let B be any measurable subset of $[a, b]$ of measure $|B| < \epsilon/2\lambda$. Then, for any $f \in \{f\}$ and any fixed $\lambda > \lambda_0$, we have

$$\begin{aligned}\int_B |f(t)|\, dt &= \int_{B\cap E} |f(t)|\, dt \\ &\quad + \int_{B-E} |f(t)|\, dt < \epsilon/2 + \lambda|B| < \epsilon/2 + \lambda(\epsilon/2\lambda) = \epsilon,\end{aligned}$$

that is, the family $\{f\}$ is equiabsolutely integrable on $[a, b]$.

(d)$\Rightarrow$(c). Indeed, by (d), we easily have $\int_a^b \psi(|f(t)|)\, dt \leq M$, for some constant M and all $f \in \{f\}$.

(c)$\Rightarrow$(d). In view of the remark above, we assume heretofore that ϕ is nonnegative, strictly increasing, continuous, and convex, that $\int_a^b \phi(|f(t)|)\, dt \leq M$, for all $f \in \{f\}$, and that $\phi(\xi)/\xi \to +\infty$, as $\xi \to +\infty$.

For any $\xi \geq 0$, take $\psi(\xi) = \{\phi(\tau), \tau\phi(\tau) = \xi^2\}$; cf. Ref. 9, p. 227. Then $\tau \to +\infty$, as $\xi \to +\infty$; τ describes $[0, +\infty)$ as ξ describes $[0, +\infty)$; and

$$\psi(\xi)/\xi = \phi(\tau)/(\tau\phi(\tau))^{1/2} = (\phi(\tau)/\tau)^{1/2}.$$

Moreover, $\phi(\tau)/\tau = (\xi/\tau)^2$; hence, $\tau/\xi \to 0$, as $\xi \to +\infty$, and $\psi(\xi)/\xi \to +\infty$, as $\xi \to +\infty$. Now, define K by taking $\phi(\xi) = K(\phi(\tau)) = K(\psi(\xi))$, so that

$$\phi(\xi)/\phi(\tau) = K(\phi(\tau))/\phi(\tau),$$

where $\phi(\xi)/\phi(\tau) \to +\infty$, $K(\phi(\tau))/\phi(\tau) \to +\infty$, as $\xi \to +\infty$, or $K(\lambda)/\lambda \to +\infty$, as $\lambda \to +\infty$. Now

$$\int_a^b K(\psi(|f(t)|))\, dt = \int_a^b \phi(|f(t)|)\, dt \leq M.$$

The implication (c)$\Rightarrow$(b) already proved shows that $\{\psi(|f|), f \in \{f\}\}$ is equiabsolutely integrable on $[a, b]$. $\square$

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