JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 89, No. 3, pp. 771-776, JUNE 1996

TECHNICAL NOTE

Optimal Ordering Rule for a Stochastic Sequencing Model[1]

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Communicated by D. L. Brito

Abstract. In this note, necessary and sufficient conditions are derived for the optimality of a sequencing rule for a class of stochastic sequential models. The optimal sequential rule generalizes the deterministic results, given in Refs. 1-2, for situations when some of the parameters of the problem are random variables. Two cases are given to demonstrate the usefulness of the results.

Key Words. Optimal sequential rule, multicharacteristic inspection, random parameters.

1. Introduction

In this note, the problem of sequencing objects in a stochastic sequencing model are studied. The stochastic sequencing model is a generalization of the deterministic sequencing model stated in Ref. 2. The stochastic model is obtained by allowing some of the input variables to be random. A notable example of the deterministic sequencing model is the multicharacteristic inspection model (Refs. 1, 3, 4). In such a problem, components with several characteristics are supplied by vendors that have to be inspected. The characteristics have different defective rates and cost of inspection. The cost

[1]The author would like to acknowledge the support provided by King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia, during his sabbatical leave. Also, the Industrial and Operations Engineering Department, University of Michigan, is acknowledged for hosting the author.
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depends on the type of test and equipment needs in carrying out the quality assurance. The probability of characteristic i being defective is $P_i > 0$, and its cost of inspection is $C_i > 0$. Inspectors commit two types of errors: Type I error θ_i; and Type II error β_i. Type I error is the probability of rejecting a nondefective characteristic, and Type II error is the probability of accepting a defective characteristic. The inspections of characteristics are independent, and inspection will be carried out until one characteristics is rejected or all characteristics pass the inspection. The component is rejected if one characteristic is observed defective. In the deterministic multicharacteristic inspection problem, all parameters such as characteristic defective rate or cost of inspection are assumed to be known and fixed constants. If this assumption is relaxed by allowing the cost of inspection and the defective rates to be random variables, the stochastic version of the multicharacteristic inspection problem is obtained. Next, the stochastic sequencing model is defined and the main result of this note is given.

2. Stochastic Sequencing Model

The stochastic sequencing problem can be stated as follows. There are n objects with no precedence relation associated with the objects. Each object i is associated with a positive random weight w_i and a nonzero random variable P_i, both with finite expectations and independent. Also, a random cost function $f_i: R^N \leftrightarrow R_+$, where R^N includes all ordered subsets of the n objects [for example, $(1, 2, 3, 4)$ is different from $(1, 2, 4, 3)$]. All random variables are assumed to be independent. The objective is to find an ordering $\pi = (j_1, j_2, \ldots, j_n)$ of the n objects to minimize the expected total cost,

$$E[TC(\pi)] = \sum_{i=1}^{n} E[f_{j_i}(j_1, j_2, \ldots, j_i)]. \tag{1}$$

It is assumed that for each object i, f_i is independent of specific ordering of the objects $(j_1, j_2, \ldots, j_{i-1})$. Consider a partial sequence B of the objects and an object l, such that $l \notin B$; the following relation is assumed for each function f_i, $i = 1, 2, \ldots, n$:

$$\begin{aligned} &f_{j_i}(j_1, j_2, \ldots, j_a, l, j_{a+1}, \ldots, j_i) - f_{j_i}(j_1, j_2, \ldots, j_a, j_{a+1}, \ldots, j_i) \\ &= P_l w_{j_i} H(j_1, j_2, \ldots, j_a, j_{a+1}, \ldots, j_{i-1}), \end{aligned} \tag{2}$$

where H is a random set function defined on all subsets of the n objects with a positive expectation and $H(\phi) = 1$. Note that the set function H does not depend on any specific objects ordering and has finite expectation.

Taking the expected-value operator of each side of Equation (2), we get

$$E[f_{j_i}(j_1, j_2, \ldots, j_a, l, j_{a+1}, \ldots, j_i) - f_{j_i}(j_1, j_2, \ldots, j_a, j_{a+1}, \ldots, j_i)]$$
$$= E[P_l]E[w_{j_i}]E[H(j_1, j_2, \ldots, j_a, j_{a+1}, \ldots, j_{i-1})]. \tag{3}$$

Theorem 2.1. For any stochastic sequencing problem defined above, a sequence π will minimize the expected total cost if and only if

$$h(j_1) \leq h(j_2) \leq \cdots h(j_n), \tag{4a}$$

$$h(j_i) = E[P_{j_i}]/E[w_{j_i}], \qquad \text{for } i = 1, 2, \ldots, n. \tag{4b}$$

Proof. Let π be a sequence of n objects and let π^1 be a sequence obtained from π by changing the position of the ith and $(i+1)$th objects. Then, we have

$$\begin{aligned} E[TC(\pi)] - E[TC(\pi^1)] &= E[f_{j_{i+1}}(B, j_i, j_{i+1}) - f_{j_{i+1}}(B, j_{i+1})] \\ &= E[f_{j_i}(B, j_{i+1}, j_i) - f_{j_i}(B, j_i)] \\ &= E[H(B)]E[P_{j_i}]E[w_{j_{i+1}}] - E[H(B)]E[P_{j_{i+1}}]E[w_{j_i}] \\ &= E[H(B)][E[P_{j_i}]E[w_{j_{i+1}}] - E[P_{j_{i+1}}]E[w_{j_i}]. \end{aligned}$$

Since $E[H(B)]$ and $E[w_i]$ are positive, then $E[TC(\pi)] < E[TC(\pi^1)]$ if and only if $h(j_i) \leq h(j_{i+1})$ and this is independent of $H(B)$. Since the value of h for each specific object i is equal to

$$h(i) = E[P_i]/E[w_i]$$

and is positive, then it satisfies the transitivity relation; i.e., for three objects i, j, k, if $h(i) \leq h(j)$ and $h(j) \leq h(k)$, then $h(i) \leq h(k)$. It follows from the transitivity of h that π is an optimal sequence if and only if

$$h(j_1) \leq h(j_2) \leq \cdots \leq h(j_n).$$ □

3. Examples of Stochastic Models

3.1. Stochastic Multicharacteristic Inspection Problem. Consider a component with n characteristics such that each characteristic has to be inspected separately. It is assumed that the inspections of the characteristics are independent from each other. Each characteristic i has a cost of inspection $C_i > 0$; C_i is not known exactly and is a random variable with finite expectation. Also, it has a probability of rejection R_i; R_i is a random variable between zero and one; it is a function of the defective rate P_i and the inspectors errors θ_i and β_i. All the R_i's and C_i's are independent random variables.

The objective is to find the optimal sequence (order the characteristics for inspections) to minimize the expected total cost,

$$TC(\pi)=C_{j_1}+\sum_{r=2}^{n} C_{j_r}\prod_{t=1}^{r-1}(1-R_{j_t}). \tag{5}$$

Since all random variables are independent, the expected total cost is given by

$$E[TC(\pi)]=E[C_{j_1}]+\sum_{r=2}^{n} E[C_{j_r}]\prod_{t=1}^{r-1}(1-E[R_{j_t}]), \tag{6}$$

where $\pi=(j_1, j_2, \ldots, j_n)$ is an ordering policy (a sequence) and j_r is the rth characteristic to be inspected.

The multicharacteristic inspection problem with random inspection cost and rejection rate is a special case of Theorem 2.1. It can be shown easily if we define

$$\begin{aligned}
&w_{j_i}=C_{j_i}, \qquad P_{j_i}=-R_{j_i},\\
&E[w_{j_i}]=E[C_{j_i}], \qquad E[P_{j_i}]=-E[R_{j_i}],\\
&f_{j_1}=C_{j_1}, \qquad E(f_{j_1})=E(C_{j_1}),\\
&f_{j_i}(j_1, j_2, \ldots, j_i)=C_{j_i}\left(\prod_{t=1}^{i-1}(1-R_{j_t})\right), \qquad i=2, \ldots, n,\\
&E[f_{j_i}(j_1, j_2, \ldots, j_i)]=E[C_{j_i}]\left(\prod_{t=1}^{i-1}(1-E[R_{j_t}])\right), \qquad H(\phi)=1.
\end{aligned}$$

All the conditions of Theorem 2.1 are satisfied and π is the optimal sequence if and only if

$$E[C_{j_1}]/E[R_{j_1}]\le E[C_{j_2}]/E[R_{j_2}]\le\cdots\le E[C_{j_n}]/E[R_{j_n}]. \tag{7}$$

3.2. Optimization of Jobs with Random Processing Time on a Single Machine. This is a generalization of the problem in Refs. 2, 5 obtained by making some of its variable random. Consider a set of n independent single-operation jobs to be processed by a single machine. Once the processing of a job starts, it continues until the processing of all jobs finish. No preemption is assumed. Let t_i, $F_i=t_1+\cdots+t_i$, $f_i(t)=\alpha_i\exp(t)$ be the processing time, flow time (completion time), and cost function of job i, $i=1, 2, \ldots, n$, respectively. Assume that t_i is a positive random variable and α_i is a constant. Therefore, F_i and $f_i(t)$ are also random variables, since they are functions of t. The random variables t_i are independent.

The objective is to find an optimal ordering (optimal sequencing) to minimize the expected total cost. The total cost is given by

$$TC(\pi) = \sum_{i=1}^{n} f_{j_i}(F_{j_i}). \tag{8}$$

The expected total cost is given as

$$\begin{aligned} E[TC(\pi)] &= \sum_{i=1}^{n} E[f_{j_i}(F_{j_i})] \\ &= \sum_{i=1}^{n} \alpha_i E[\exp(t_{j_1} + t_{j_2} + \cdots + t_{j_i})]. \end{aligned} \tag{9}$$

Let w_i, P_i, t_i, and H for all jobs be as follows:

$$w_i = \alpha_i \exp(t_i), \qquad P_i = \exp(t_i) - 1,$$

$$f_{j_i}(j_1, j_2, \ldots, j_i) = \alpha_{j_1} \exp(F_{j_i}), \qquad H(j_1, j_2, \ldots, j_i) = \exp(F_{j_{i-1}}),$$

for $i = 1, 2, \ldots, n$ and $H(\phi) = 1$. Then, the conditions of Theorem 2.1 are satisfied and π is optimal if and only if

$$h(j_1) \leq h(j_2) \leq \cdots \leq h(j_n), \tag{10a}$$

$$h(j_i) = E[\exp(t_{j_i})] - 1/\alpha_{j_i} E[\exp(t_j)], \qquad i = 1, 2, \ldots, n. \tag{10b}$$

In the above problem, if we assume that the cost function is linear instead of exponential [i.e., $f_i(t) = \alpha_j t$], then the stochastic version of the problem in Ref. 6 is obtained. The expected total cost is given as

$$\begin{aligned} E[TC(\pi)] &= \sum_{i=1}^{n} \alpha_{j_i} E[t_{j_1} + t_{j_2} + \cdots + t_{j_i}] \\ &= \sum_{i=1}^{n} \alpha_{j_i} \{E[t_{j_1}] + E[t_{j_2}] + \cdots + E[t_{j_i}]\}. \end{aligned} \tag{11}$$

If w_i, P_i, f_i, and H are defined for all jobs as

$$w_i = \alpha_i, \qquad P_i = -t_i,$$

$$f_{j_i}(j_1, j_2, \ldots, j_i) = \alpha_{j_1}(F_{j_i}) = \alpha_{j_i}(t_{j_1} + t_{j_2} + \cdots + t_{j_i}),$$

$$H(j_1, j_2, \ldots, j_i) = 1,$$

the conditions of Theorem 2.1 are met, and a sequence π is optimal if and only if

$$h(j_1) \leq h(j_2) \leq \cdots \leq h(j_n), \tag{12a}$$

$$h(j_i) = E[t_{j_i}]/\alpha_{j_i}, \qquad i = 1, 2, \ldots, n. \tag{12b}$$

4. Conclusions

In this note, a stochastic sequencing problem has been defined. The necessary and sufficient conditions for finding the optimal sequence have been derived. The stochastic sequencing problem generalizes the deterministic sequencing problem (Ref. 2) by considering some of the input variables as random. The stochastic sequencing problem represents many practical situations such as sequencing characteristic for inspection, jobs processing. When the random variables are assumed constants in the stochastic sequencing model, the results of this paper specialize to the results in Refs. 1, 2.

References

1. Duffuaa, S. O., and Raouf, A., *An Optimal Sequence in Multicharacteristic Inspection*, Journal of Optimization Theory and Applications, Vol. 67, No. 1, pp. 79–86, 1990.
2. Alidaee, B., *On Optimal Ordering Policy of a Sequential Model*, Journal of Optimization Theory and Applications, Vol. 83, No. 1, pp. 199–205, 1994.
3. Raouf, A., Jain, J. K., and Sathe, P. T., *A Cost Minimization Model for Multicharacteristic Component Inspection*, IIE Transactions, Vol. 15, No. 2, pp. 187–194, 1983.
4. Lee, H. L. *On the Optimality of a Simplified Multicharacteristic Inspection Model*, IIE Transactions, Vol. 20, No. 4, pp. 392–398, 1988.
5. Lawler, E. L., and Sivazlian, B. D., *Minimization of Time-Varying Costs in Single-Machine Scheduling*, Operations Research, Vol. 26, No. 4, pp. 563–569, 1978.
6. Smith, W. E., *Various Optimizers for Single-Stage Production*, Naval Research Logistics Quarterly, Vol. 3, No. 1, pp. 59–66, 1956.