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THE MEAN AND STANDARD DEVIATION OF THE DISTRIBUTION OF GROUP ASSEMBLY SUMS*

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An interesting problem in linear programming is the group assembly problem which is mathematically equivalent to the general transportation problem of economics. Computer programs designed for the determination of exact and approximate optimal group assemblies have been available for some time. This paper presents formulas for the mean and squared standard deviation of the distribution of all possible group assembly sums. Computational techniques are presented and the results are related to those of the analysis of variance of a k-factor problem with n levels of each factor.

Suppose that a group, or team, or crew, consists of k men each of whom is trained for only one of the k different positions of the group. Suppose further that there are n men available for each of the k positions. Then one set of men $\{i_1, i_2, \cdots, i_j, \cdots, i_k\}$, where $1 \leq i_j \leq n$, constitutes one of the groups under consideration and there are n groups. In the personnel classification problem [5] for instance, where a modified interpretation identifies the n men (i_1) with the n jobs (i_2), the number of assignments of men to jobs is n^2.

The score which a group makes in attaining its objective is a group score, $g_{i_1 \cdots i_k}$. It must be measured in units which indicate the effectiveness of the group such as points, percentage of targets hit, etc. The ideal group score may be a high one, as in bowling, or a low one, as in golf. In the following development we assume that the group score is known. It is sometimes difficult to satisfy this assumption by empirical determination of the group score exactly (i) because of the sampling error of such results and (ii) because of the practical impossibility of determining the group score for all the n^k possible groups with n and/or k large. In such cases approximate, or plausible, or calculated, or hypothetical group scores must be provided before the solution can be made. This aspect of the problem is much less serious in an economic version of the problem, the general transportation problem [4], where the transportation costs from origins to destinations through

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intermediate points are presumably known or can be approximated. The results of this paper are applicable to the interpreted variations of the general problem.

A group assembly is a collection of groups such that each of the available nk men is assigned to one and only one group. Combinatorial argument shows that there are $(n!)^{k-1}$ group assemblies since, for each i_1, the $k-1$ sets of $i_2, \cdots, i_k$ may be assigned in $n!$ different ways.

In studying the measure of effectiveness of the assembly, we assume additivity and define the group assembly sum to be the sum of the group scores for the n groups in the assembly. Thus the group assembly sum is given by

$$T = \mathrm{S} g_{i_1 i_2 \cdots i_j \cdots i_k} , \tag{1}$$

where S indicates the summation for the n terms.

The Mean, $\bar{T}$

Since there are $(n!)^{k-1}$ assemblies, the total number of group scores in $\sum T$ is $n(n!)^{k-1}$. Now from symmetry,

$$\sum_{i_1} \cdots \sum_{i_k} g_{i_1 \cdots i_k} = g_{* \cdots *}$$

must factor this result. Since $g_{* \cdots *}$ is the sum of the n^k group scores,

$$\sum T = \frac{n(n!)^{k-1}}{n^k} g_{* \cdots *} = [(n-1)!]^{k-1} g_{* \cdots *} .$$

Then the mean of the distribution of all possible assembly sums is

$$\bar{T} = \frac{1}{(n!)^{k-1}} T = \frac{1}{n^{k-1}} g_{* \cdots *} . \tag{2}$$

The Value of $\overline{T^2}$

In deriving the value of the standard deviation of T, we first obtain the second moment, $\overline{T^2}$. Now T^2, for a given assembly, consists of

$$T^2 = \mathrm{S} g^2_{i_1 \cdots i_k} + \underset{i_j \neq h_j}{\mathrm{S}} g_{i_1 \cdots i_k} g_{h_1 \cdots h_k} . \tag{3}$$

The first term on the right consists of the sum of the squares of the n terms of (1) while the second term on the right consists of the $n(n-1)$ paired products of the n terms of (1). Then the value of $\sum_T \mathrm{S} g^2_{i_1 \cdots i_k}$ is

$$\frac{n(n!)^{k-1}}{n^k} \sum_{i_j} g^2_{i_1 \cdots i_k} = [(n-1)!]^{k-1} \sum_{i_j} g^2_{i_1 \cdots i_k}$$

and the value of $\sum_T \mathrm{S} g_{i_1 \cdots i_k} g_{h_1 \cdots h_k}$ is

$$\frac{n(n-1)(n!)^{k-1}}{[n(n-1)]^k} \sum_{i_j \neq h_j} g_{i_1 \cdots i_k} g_{h_1 \cdots h_k} = [(n-2)!]^{k-1} \sum_{i_j \neq h_j} g_{i_1 \cdots i_k} g_{h_1 \cdots h_k} .$$

Then dividing by $(n!)^{k-1}$ we get

$$\overline{T^2} = \frac{1}{n^{k-1}} \sum_{i_j} g^2_{i_1 \cdots i_k} + \frac{1}{n^{k-1}(n-1)^{(k-1)}} \sum_{i_j \neq h_j} g_{i_1 \cdots i_k} g_{h_1 \cdots h_k} . \tag{4}$$

The next step is the evaluation of the last term of (4) in terms of sums of squares. We use the symbolism of restriction equations in order to accomplish this.

Application of Restriction Equations

We note that

$$\sum_{i \neq h}^{n} f_i f_h = \sum_{i}^{n} \sum_{h}^{n} f_i f_h - \sum_{i=h}^{n} f_i \, f_h$$

can be written symbolically as the restriction equation

$$(i \neq h) = 0 - (i = h),$$

where 0 indicates that there is no restriction on the values of i and h and where $(i = h)$ indicates that the restriction is only that i must equal h. More generally for $f_{i_1 i_2} f_{h_1 h_2}$ and $f_{i_1 i_2 i_3} f_{h_1 h_2 h_3}$ respectively we have

$$(i_1 \neq h_1)(i_2 \neq h_2) = 0 - [(i_1 = h_1) + (i_2 = h_2)] + (i_1 = h_1)(i_2 = h_2),$$

$$\begin{aligned}(i_1 \neq h_1)(i_2 \neq h_2)(i_3 \neq h_3) = 0 &- [(i_1 = h_1) + (i_2 = h_2) + (i_3 = h_3)] \\ &+ [(i_1 = h_1)(i_2 = h_2) + (i_1 = h_1)(i_3 = h_3) \\ &+ (i_2 = h_2)(i_3 = h_3)] - (i_1 = h_1)(i_2 = h_2)(i_3 = h_3),\end{aligned}$$

where $(i_1 = h_1)(i_2 = h_2)$ means the double restriction $(i_1 = h_1)$ and $(i_2 = h_2)$. These can be written more symbolically using product $\prod (i_j = h_j)$, and product type $T_v \prod^r (i_j = h_j)$ which sums all products having r factors. Then the restriction equations appear as

$$\prod^{2} (i_j \neq h_j) = 0 - T_v(i_j = h_j) + \prod^{2} (i_j = h_j),$$

$$\prod^{3} (i_j \neq h_j) = 0 - T_v(i_j = h_j) + T_v \prod^{2} (i_j = h_j) - \prod^{3} (i_j = h_j),$$

and in general

$$\begin{aligned}\prod^{k} (i_j \neq h_j) = 0 &- T_v(i_j = h_j) \\ &+ \sum_{r=2}^{k-1} (-1)^r \, T_v \prod^{r} (i_j = h_j) + (-1)^k \prod^{k} (i_j = h_j).\end{aligned} \tag{5}$$

Application of restriction equations to the summation part of the last term of (4) gives

$$\sum_{i_j \neq h_j} g_{i_1 i_2} g_{h_1 h_2} = g^2_{**} - \sum_{i_1} g^2_{i_1 *} - \sum_{i_2} g^2_{* i_2} + \sum_{i_1, i_2} g^2_{i_1 i_2} , \tag{6}$$

$$\sum_{i_j \neq h_j} g_{i_1 i_2 i_3} g_{h_1 h_2 h_3} = g^2_{***} - \sum_{i_1} g^2_{i_1 **} - \sum_{i_2} g^2_{* i_2 *} - \sum_{i_3} g^2_{** i_3} \tag{7}$$
$$+ \sum_{i_1, i_2} g^2_{i_1 i_2 *} + \sum_{i_1, i_3} g^2_{i_1 * i_3} + \sum_{i_2, i_3} g^2_{* i_2 i_3} - \sum_{i_1, i_2, i_3} g^2_{i_1 i_2 i_3} ,$$

and in general

$$\sum_{i_j \neq h_j} g_{i_1 \cdots i_k} g_{h_1 \cdots h_k} = g^2_{* \cdots *} - T_v \sum g^2_{i_1 ***} \tag{8}$$
$$+ \sum_{r=2}^{k-1} (-1)^r T_v \sum g^2_{i_1 \cdots i_r ***} + (-1)^k \sum g^2_{i_1 \cdots i_k} .$$

The Value of σ_T^2

Defining $\sigma_T^2 = \overline{T^2} - (\bar{T})^2$ we obtain from (2), (4), (6), (7), (8) for $k =$ 2, 3, 4, 5, k

$$\sigma_T^2 = \frac{1}{n-1} \sum g^2_{i_1 i_2} - \frac{1}{n(n-1)} [\sum g^2_{i_1 *} + \sum g^2_{* i_2}] + \frac{1}{n^2(n-1)} g^2_{**} , \tag{9}$$

$$\sigma_T^2 = \frac{n-2}{n(n-1)^2} \sum g^2_{i_1 i_2 i_3} + \frac{1}{n^2(n-1)^2} [-T_v \sum g^2_{i_1 **} \tag{10}$$
$$+ T_v \sum g^2_{i_1 i_2 *}] + \frac{2n-1}{n^4(n-1)^2} g^2_{***} ,$$

$$\sigma_T^2 = \frac{n^2 - 3n + 3}{n^2(n-1)^3} \sum g^2_{i_1 i_2 i_3 i_4} + \frac{1}{n^3(n-1)^3} [-T_v \sum g^2_{i_1 ***} \tag{11}$$
$$+ T_v \sum g^2_{i_1 i_2 **} - T_v \sum g^2_{i_1 i_2 i_3 *}] + \frac{3n^2 - 3n + 1}{n^6(n-1)^3} g^2_{****} ,$$

$$\sigma_T^2 = \frac{(n-2)(n^2 - 2n + 2)}{n^3(n-1)^4} \sum g^2_{i_1 i_2 i_3 i_4 i_5} + \frac{1}{n^4(n-1)^4} [-T_v g_{i_1 ****} \tag{12}$$
$$+ T_v \sum g^2_{i_1 i_2 ***} - T_v \sum g^2_{i_1 i_2 i_3 **} + T_v \sum g^2_{i_1 i_2 i_3 i_4 *}]$$
$$- \frac{(2n-1)(n^2 - 2n + 2)}{n^8(n-1)^4} g^2_{*****} ,$$

and in general

$$\sigma_T^2 = \frac{(n-1)^{k-1} + (-1)^k}{n^{k-1}(n-1)^{k-1}} \sum g^2_{i_1 \cdots i_k} + \frac{1}{n^{k-1}(n-1)^{k-1}} \tag{13}$$
$$\cdot \left[\sum_{r=1}^{k-1} (-1)^r T_v \sum g^2_{i_1 \cdots i_r ***} \right] + \frac{n^{k-1} - (n-1)^{k-1}}{n^{2k-2}(n-1)^{k-1}} g^2_{* \cdots *} .$$

The formula for $k = 2$ (personnel assignment problem) was given essentially by Votaw and Dailey [6] in a report prepared for the Air Force in 1952. The more general formulas appeared in a report by the author [3] prepared for the Air Force in 1956.

Direct Calculation of σ_T^2

The formulas are suitable for direct calculation. It is necessary to calculate the quantities $g_{i_1 \cdots i_r ***}$. Then the squares can be calculated and accumulated. This process is identical with the preliminary calculations for a nonreplicated factorial analysis of variance with k factors and n levels for each factor. Machine programs may be used in obtaining the values of $g^2_{i_1 \cdots i_r ***}$ or they may be obtained quite easily with desk calculator if n and k are not too large.

Calculation Using Deviates

The formulas are simplified with the use of a preliminary transformation to deviates. This transformation is very feasible with automatic calculation and might be considered by users of desk calculators in situations, such as this, where the main effects may be ignored. The technique is first applied to the $k = 2$ case. We define

$$d_{i_1 i_2} = g_{i_1 i_2} - \bar{g}_{i_1 *} - \bar{g}_{* i_2} + \bar{g}_{**} = g_{i_1 i_2} - (g_{i_1 *}/n) - (g_{* i_2}/n) + (g_{**}/n^2)$$

and note that σ_T^2 is not changed by replacing $g_{i_1 i_2}$ by $d_{i_1 i_2}$ since each assembly sum is decreased by the constant

$$\sum_{i_1} \bar{g}_{i_1 *} + \sum \bar{g}_{* i_2} - n\bar{g}_{**} .$$

Furthermore we note that

$$\sum_{i_1} d_{i_1 i_2} = 0 = \sum_{i_2} d_{i_1 i_2} = \sum_{i_1} \sum_{i_2} d_{i_1 i_2}$$

since

$$\sum_{i_1} d_{i_1 i_2} = g_{* i_2} - (g_{**}/n) - g_{* i_2} + (g_{**}/n).$$

Then (9) becomes the simple

$$\sigma_T^2 = \frac{1}{n - 1} \sum d^2_{i_1 i_2} . \tag{14}$$

Similar treatment when $k = 3$ with

$$d_{i_1 i_2 i_3} = g_{i_1 i_2 i_3} - \bar{g}_{i_1 **} - \bar{g}_{* i_2 *} - \bar{g}_{** i_3} + 2\bar{g}_{***}$$

leads to

$$\sigma_T^2 = \frac{n - 2}{n(n - 1)^2} \sum d^2_{i_1 i_2 i_3} + \frac{1}{n^2(n - 1)^2} [\sum d^2_{i_1 i_2 *} + \sum d^2_{i_1 * i_3} + \sum d^2_{* i_2 i_3}]. \tag{15}$$

In the general case with

$$d_{i_1\cdots i_k} = g_{i_1\cdots i_k} - T_v\bar{g}_{i_1*\cdots*} + (k-1)\bar{g}_{*\cdots*}$$

we have

$$\sigma_T^2 = \frac{(n-1)^{k-1} + (-1)^k}{n^{k-1}(n-1)^{k-1}} \sum d^2_{i_1\cdots i_k} + \frac{1}{n^{k-1}(n-1)^{k-1}} \sum_{r=2}^{k-1} (-1)^r T_v \sum d^2_{i_1\cdots i_r ***} . \tag{16}$$

Calculation Using Large Variances

Applying the concept of the large variance ([1], p. 302) to the d's and noting that the number of d_{i_1}, d_{i_1*}, d_{i_1**}, $\cdots$ is n, the number of $d_{i_1i_2}$, $d_{i_1i_2*}$, $d_{i_1i_2**}$, $\cdots$ is n^2, the number of $d_{i_1i_2i_3*}$, $\cdots$ is n^3, etc. and that $d_{i_1*\cdots*} = 0$, we define

$$\begin{aligned} L'(1*\cdots*) &= nT_v \sum d^2_{i_1*\cdots*} = 0, \\ L'(11*\cdots*) &= n^2T_v \sum d^2_{i_1i_2*\cdots*}), \\ L'(111*\cdots*) &= n^3T_v \sum d^2_{i_1i_2i_3*\cdots*}), \quad \text{etc.} \end{aligned} \tag{17}$$

Then for $k = 2$ and $k = 3$ we have

$$\sigma_T^2 = \frac{1}{n^2(n-1)} L'(11), \tag{18}$$

$$\sigma_T^2 = \frac{1}{n^4(n-1)^2} [(n-2)L'(111) + L'(11*)], \tag{19}$$

and for general k

$$\sigma_T^2 = \frac{1}{n^{2(k-1)}(n-1)^{k-1}} \left[\frac{(n-1)^{k-1} + (-1)^k}{n} L'(1^k) + (-1)^{k-r} n^{r-1} L'(1^{k-r}*^r) \right]. \tag{20}$$

Thus for $k = 4$ and $k = 5$ we have respectively

$$\sigma_T^2 = \frac{1}{n^6(n-1)^3} [(n^2 - 3n + 3)L'(1111) + nL'(11**) - L'(111*)]. \tag{21}$$

$$\sigma_T^2 = \frac{1}{n^8(n-1)^4} [(n^3 - 4n^2 + 6n - 3)L'(11111) + n^2L'(11***) - nL'(111**) + L'(1111*)]. \tag{22}$$

The values of L' are easily calculated from the sums of the squares of the d's as is illustrated below for the case with $k = 3$ and $n = 3$.

Calculation Using I Terms

The values of the L' are the values of the nondeviate L with

$$L_{i_1 * \cdots *} = L_{* i_2 * \cdots *} = \cdots = 0.$$

Since the conventional relation of the sum of squares expressed in terms of L's is

$$L_{i_1 i_2} = L_{i_1 *} + L_{* i_2} + I_{i_1 i_2},$$

$$L_{i_1 i_2 i_3} = L_{i_1 **} + L_{* i_2 *} + L_{** i_3} + I_{i_1 i_2 *} + I_{i_1 * i_3} + I_{* i_2 i_3} + I_{i_1 i_2 i_3},$$

where the value of I is n^k times the corresponding conventional sum of squares, we have

$$\begin{aligned} L'(11) &= I_{i_1 i_2} = I(11), \\ L'(11*) &= L'_{i_1 i_2 *} + L'_{i_1 * i_3} + L'_{* i_2 i_3} \\ &= I_{i_1 i_2 *} + I_{i_1 * i_3} + I_{* i_2 i_3} = I(11*), \\ L'(111) &= L'_{i_1 i_2 i_3} = I(11*) + I(111). \end{aligned} \tag{23}$$

Then the formulas for $k = 2$ and $k = 3$ become respectively

$$\sigma_T^2 = \frac{1}{n^2(n-1)} I(11), \tag{24}$$

$$\sigma_T^2 = \frac{1}{n^4(n-1)^2} [(n-2)I(111) + (n-1)I(11*)]. \tag{25}$$

In particular when $n = 3$, (25) becomes

$$\sigma_T^2 = \frac{1}{324} [I(111) + 2I(11*)]. \tag{26}$$

When $k = 4$ we find that

$$\begin{aligned} L'(1111) &= I(1111) + I(111*) + I(11**) \\ L'(111*) &= \phantom{I(1111) +{}} I(111*) + 2I(11**) \\ L'(11**) &= \phantom{I(1111) + I(111*) + 2{}} I(11**) \end{aligned}$$

so that

$$\sigma_T^2 = \frac{1}{n^6(n-1)^3} [(n^2 - 3n + 3)I(1111) + (n-2)(n-1)I(111*) + (n-1)^2 I(11**)]. \tag{27}$$

For $k = 5$ we find

$$\begin{aligned} L'(11111) &= I(11111) + I(1111*) + I(111**) + I(11***) \\ L'(1111*) &= I(1111*) + 2I(111**) + 3I(11***) \\ L'(111**) &= I(111**) + 3I(11***) \\ L'(11***) &= I(11***) \end{aligned}$$

so that

$$\sigma_T^2 = \frac{1}{n^8(n-1)^4}[(n^3-4n^2+6n-4)I(11111)+(n^2-3n+3)(n-1) \tag{28}$$
$$\cdot I(1111*)+(n-2)(n-1)^2I(111**)+(n-1)^3I(11***)].$$

We note that for $k = 2, 3, 4, 5$ at least, the coefficient of $(1^{k-r}*^r)$ in the bracket is $(n-1)^r$ times the coefficients of $I(1^{k-r})$ in the bracket. We prove that this is true in general and that the coefficient in the bracket of

$$I(1^{k-r}*^r) = \left[\frac{(n-1)^{k-r-1}+(-1)^{k-r}}{n}\right](n-1)^r.$$

The coefficient of $I(1^{k-r'-s}*^{r'+s})$ in the expansion of $L'(1^{k-r'}*^{r'})$ is

$$\binom{k}{r'}\binom{k-r'}{s} \div \binom{k}{r'+s} = \binom{r'+s}{s}.$$

When $r' = 0$, this coefficient is $\binom{s}{s} = 1$. Then for a fixed $r = r' + s$ the coefficient of $I(1^{k-r}*^r)$ in the bracket, using (20) with r replaced by $r' = r - s$, is

$$\begin{aligned}
&\frac{(n-1)^{k-1}+(-1)^k}{n} + \sum_{s=0}^{r-1}(-1)^{k-r+s}n^{r-s-1}\binom{r}{s}\\
&= \frac{(n-1)^{k-1}+(-1)^k}{n} + \frac{(-1)^{k-r}}{n}\sum_{s=0}^{r-1}(-1)^s\binom{r}{s}n^{r-s}\\
&= \frac{(n-1)^{k-1}+(-1)^k}{n} + \frac{(-1)^{k-r}}{n}[(n-1)^r+(-1)^{r+1}]\\
&= \frac{(n-1)^{k-r-1}+(-1)^{k-r}}{n}[n-1]^r.
\end{aligned} \tag{29}$$

We can then write, $k \geq 2$,

$$\sigma_T^2 = \frac{1}{n^{2(k-1)}(n-1)^k}\sum_{r=0}^{k-2}\left[\frac{(n-1)^{k-r-1}+(-1)^{k-r}}{n}\right](n-1)^r I(1^{k-r}*^r). \tag{30}$$

The value

$$\frac{(n-1)^{k-r-1}+(-1)^{k-r}}{n}$$

is simply a polynomial of degree $k - r - 2$ with binomial coefficients of degree $k - r - 1$ and alternating sign.

Calculation Using Sums of Squares

Now

$$I(1^{k-r}*^r) = n^k S(1^{k-r}*^r),$$

where $S(1^{k-r}*^r)$ is the sum of the conventional sum of squares which contribute to the interaction term of order r. Thus

$$S(11**) = S(i_1, i_2) + S(i_1, i_3) + S(i_1, i_4) + S(i_2, i_3) + S(i_2, i_4) + S(i_3, i_4)$$

features the second-order sums of squares appearing in a conventional factorial analysis with $n = k = 4$ and no replication. The general formula then is

$$\sigma_T^2 = \frac{1}{n^{k-2}(n-1)^{k-1}} \sum_{r=0}^{k-2} \left[\frac{(n-1)^{k-r-1} + (-1)^{k-r}}{n}\right](n-1)^r S(1^{k-r}*^r). \tag{31}$$

Special cases for $k = 2, 3, 4$ are

$$\sigma_T^2 = S(11)/(n-1), \tag{32}$$

$$\sigma_T^2 = [(n-2)S(111) + (n-1)S(11*)]/n(n-1)^2, \tag{33}$$

$$\begin{aligned}\sigma_T^2 = [(n^2 - 3n + 3)S(1111) &+ (n-2)(n-1)S(111*) \\ &+ (n-1)^2 S(11**)]/n^2(n-1)^3, \quad \text{etc.}\end{aligned} \tag{34}$$

This formula is suitable for the application of the results of a computing program for analysis of variance of an unreplicated factorial experiment. It is only necessary to obtain the sum of squares for the various sources by machine and apply the multipliers indicated in (31).

The formula (31) has more than computational importance since it shows which interaction terms are responsible for the magnitude of σ_T^2. It is worthy of note that, in every case, the coefficient of $S(1^{k-r}*^r)$ in the bracket is of order n^{k-2}. Hence, unless n is small, the weighting of every individual sum of squares, no matter what the order, is approximately the same. This makes possible the direct comparison of the different sums of squares so that one can see which sources are responsible for the size of σ_T^2.

It is also worthy of note that $\sigma_T^2 = 0$ if and only if all sums of squares for interaction terms are zero. This situation is satisfied when the group score is a linear function of the scores of the individuals composing the group and is not a function of interaction of individuals. Then all group assembly sums are the same and the groups can be assembled on any convenient basis without reference to the group score. The group assembly problem is designed to handle situations in which there are interactions in the group.

Illustration with $k = 3$ and $n = 3$

Computation with the various formulas is illustrated in Table 1 and Table 2. Table 1 treats original $g_{i_1i_2i_3}$ and illustrates the formulas using $\sum g^2$, $I(\ )$, and $S(\ )$. Table 2 treats $d_{i_1i_2i_3}$ and illustrates the formulas for $\overline{d^2}$, $L'(\ )$, $I(\ )$, and $S(\ )$.

TABLE 1

Value of $g_{i_1i_2i_3}$ with Sums and Analysis

i_3	i_1 \ i_2	1	2	3	$g_{i_1*i_3}$	
1	1	2	4	6	12	
	2	6	4	8	18	
	3	5	7	2	14	
	g_{*i_21}	13	15	16	44	g_{**1}
2	1	3	7	2	12	
	2	4	7	5	16	
	3	2	2	4	8	
	g_{*i_22}	9	16	11	36	g_{**2}
3	1	7	11	7	25	
	2	2	5	6	13	
	3	3	1	4	8	
	g_{*i_23}	12	17	17	46	g_{**3}
*	$g_{i_1i_2*}$	12	22	15	49	
		12	16	19	47	
		10	10	10	30	
	g_{*i_2*}	34	48	44	126	g_{***}

Source	n	Sum	$\Sigma(\ )^2$	Mult.	L	I	I( )	Mult.	S	S( )	Mult.
***	1	126	15876	5	0						
i_1**	3	126	5510	-9	654						
$*i_2*$	3	126	5396	-9	312						
$**i_3$	3	126	5348	-9	168						
i_1i_2*	9	126	1914	9	1350	384			14.22		
i_1*i_3	9	126	1986	9	1998	1176	1674	2	43.56	62.00	2
$*i_2i_3$	9	126	1830	9	594	114			4.22		
$i_1i_2i_3$	27	126	736	27	3996	1188	1188	1	44.00	44.00	1
Num. Den.			4536 324 14				4536 324 14		168.00 12 14.00		

Examination of the I (or S) column in Table 1 and Table 2 shows that the larger contribution to σ_T^2 comes from the $i_1{*}i_3$ and $i_1i_2i_3$ interaction terms. Actually in this case with $n - 1 = 2$ and $n - 2 = 1$, the $i_1{*}i_3$ term, weighted twice, makes the largest contribution. If this term were zero (the other terms remaining the same) we would have $\sigma_T^2 = 6.74$ units^2 which is less than 50 per cent of the original σ_T^2 . This illustrates the use of the results of a conventional analysis of variance in determining the major contribution to the value of σ_T^2 .

TABLE 2

Value of $d_{i_1i_2i_3}$ with Sums and Analysis

i_3	$i_1 \backslash i_2$	1	2	3	$d_{i_1*i_3}$	
1	1	-2.7778	-2.3333	0.1111	-5.0000	
	2	1.4444	-2.1111	2.3333	1.6666	
	3	2.3333	2.7778	-1.7778	3.3333	
	d_{*i_21}	0.9999	-1.6666	0.6666	-0.0001	d_{**1}
2	1	-0.8889	1.5556	-3.0000	-2.3333	
	2	0.3333	1.7778	0.2222	2.3333	
	3	0.2222	-1.3333	1.1111	0.0000	
	d_{*i_22}	-0.3334	2.0001	-1.6667	0.0000	d_{**2}
3	1	2.0000	4.4444	0.8888	7.3332	
	2	-2.7778	-1.3333	0.1111	-4.0000	
	3	0.1111	-3.4444	0.0000	-3.3333	
	d_{*i_23}	-0.6667	-0.3333	0.9999	-0.0001	d_{**3}
4	$d_{i_1i_2*}$	-1.6667	3.6667	-2.0001	-0.0001	
		-1.0001	-1.6666	-2.6666	-0.0001	
		2.6666	-1.9999	-0.6667	0.0000	
		-0.0002	0.0002	-0.0002	-0.0002	d_{***}

Source	n	Sum	$\Sigma(\,)^2$	Mult.	L′	L′( )	Mult.	I	I( )	Mult.	S	S( )	Mult.
$***$	1	0	0		0	0							
i_1**	3	0	0		0	0							
$*i_2*$	3	0	0		0	0							
$**i_3$	3	0	0		0	0							
i_1i_2*	9	0	42.6663	1	384			384			14.22		
i_1*i_3	9	0	130.6637	1	1176	1674	1	1176	1674	2	43.56	62.00	2
$*i_2i_3$	9	0	12.6665	1	114			114			4.22		
$i_1i_2i_3$	27	0	105.9989	3	2862	2862	1	1188	1188	1	44.00	44.00	1
Num. Den.			503.99 36 14.00			4536 324 14			4536 324 14			168.00 12 14.00	

In Table 1 the multipliers of the $\sum g^2$ obtained from (10) are the numerators of the fractions having $n^4(n - 1)^2$ as the denominator with $n = 3$. Decimal equivalents of the coefficients in (10) could be used if preferred. In Table 2 the computed values of L' are rounded to the nearest integer since it is known that each L' is an integer. In most cases the methods lead to the exact answer, 14 units^2, though in some cases a decimal approximation results.

The various formulas are provided so that the worker may select the one which seems most suitable when considering other desired treatment of the data, the values of k and n, and available resources such as computational equipment and assistants.

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