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A SET OF INEQUALITIES IN FACTOR ANALYSIS

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Inequalities relating the communalities to the multiple-correlation coefficients are derived. They are stronger than the well-known inequalities which have played an important role in factor analysis for the past thirty years. Necessary and sufficient conditions for equality are obtained.

1. *Introduction*

Let $\mathbf{\Sigma} = [\sigma_{ij}]$ denote the correlation matrix of $\mathbf{x} = [x_1\ x_2\ \cdots\ x_p]'$. We shall suppose that $\mathbf{\Sigma}$ is nonsingular and therefore positive definite. (If, to the contrary, $\mathbf{\Sigma}$ is of rank r $(< p)$ then there are only r linearly independent variables $x_1, x_2, \cdots, x_r$, say, and the others are redundant.)

Next let

$$\mathbf{\Sigma} = \mathbf{\Gamma} + \mathbf{\Delta} \tag{1}$$

where

$$\mathbf{\Delta} = \begin{bmatrix} \delta_1^2 & 0 & \cdots & 0 \\ 0 & \delta_2^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \delta_p^2 \end{bmatrix} \qquad 0 \le \delta_i^2 \le 1, \qquad 1 \le i \le p,$$

and $\mathbf{\Gamma} = \mathbf{\Sigma} - \mathbf{\Delta}$ is positive semi-definite. The factor-analysis interpretation of (1) is that

$$\mathbf{x} = \mathbf{y} + \mathbf{z},$$

where $\mathbf{y} = [y_1\ y_2\ \cdots\ y_p]'$ has covariance matrix $\mathbf{\Gamma}$ and is uncorrelated with $\mathbf{z} = [z_1\ z_2\ \cdots\ z_p]'$ which has covariance matrix $\mathbf{\Delta}$, so that z_i is uncorrelated with z_j, $1 \le i \le j \le p$. The variable y_i is called the common-factor component of x_i and $\text{var}(y_i) = 1 - \delta_i^2$ is the communality of x_i while z_i is the specific factor of x_i and $\text{var}(z_i) = \delta_i^2$ is the uniqueness of x_i.

Let ρ_i denote the multiple-correlation coefficient of x_i with the remaining $p - 1$ variables. Then

$$\rho_i < 1 \qquad 1 \le i \le p, \tag{2}$$

because $\mathbf{\Gamma}$ is nonsingular. Roff [3] pointed out and Dwyer [1] proved that

$$1 - \delta_i^2 \geq \rho_i^2 \qquad 1 \leq i \leq p. \tag{3}$$

In this paper we derive a set of stronger inequalities than (3).

2. *The Inequalities*

Write

$$\mathbf{\Sigma} = \begin{bmatrix} 1 & \boldsymbol{\delta}_1' \\ \boldsymbol{\delta}_1 & \mathbf{\Sigma}_{11} \end{bmatrix},$$

where $\mathbf{\Gamma}_{11}$ is the covariance matrix of $\mathbf{x}_1 = [x_2\ x_3 \cdots x_p]'$ and define

$$\boldsymbol{\beta}_1 = \mathbf{\Sigma}_{11}^{-1}\boldsymbol{\delta}_1 .$$

Thus $\boldsymbol{\beta}_1 = [\beta_{12}\ \beta_{13} \cdots \beta_{1p}]'$ is the vector of regression coefficients of x_1 on $\mathbf{x}_1$. Now

$$1 - \rho_1^2 = E[(x_1 - \boldsymbol{\beta}_1'\mathbf{x}_1)^2] = [1 \quad -\boldsymbol{\beta}_1']\mathbf{\Sigma}\begin{bmatrix} 1 \\ -\boldsymbol{\beta}_1 \end{bmatrix}.$$

Therefore

$$1 - \rho_1^2 = [1 \quad -\boldsymbol{\beta}_1']\mathbf{\Gamma}\begin{bmatrix} 1 \\ -\boldsymbol{\beta}_1 \end{bmatrix} + [1 \quad -\boldsymbol{\beta}_1']\mathbf{\Delta}\begin{bmatrix} 1 \\ -\boldsymbol{\beta}_1 \end{bmatrix}. \tag{4}$$

The second term on the right of (4) is $\delta_1^2 + \beta_{12}^2\delta_2^2 + \beta_{13}^2\delta_3^2 + \cdots + \beta_{1p}^2\delta_p^2$ and, since $\mathbf{\Gamma}$ is positive semi-definite, the first term is nonnegative. Applying the same argument to $1 - \rho_2^2, \cdots, 1 - \rho_p^2$, we have the following set of p inequalities

$$1 - \rho_i^2 \geq \delta_i^2 + \sum \beta_{ij}^2\delta_j^2 \qquad 1 \leq i \leq p, \tag{5}$$

where β_{ij} is the coefficient of x_j in the regression of x_i on the remaining $p - 1$ variables.

3. *Conditions for Equality*

Suppose that

$$1 - \rho_1^2 = \delta_1^2 + \sum_{j>1} \beta_{1j}^2\delta_j^2 . \tag{6}$$

Then $\boldsymbol{\alpha}_1'\mathbf{\Gamma}\boldsymbol{\alpha}_1 = 0$, where $\boldsymbol{\alpha}_1' = [1 \quad -\boldsymbol{\beta}_1']$. But, since $\mathbf{\Gamma}$ is positive semi-definite, $\boldsymbol{\alpha}_1'\mathbf{\Gamma}\boldsymbol{\alpha}_1 = 0$ only if $\mathbf{\Gamma}\boldsymbol{\alpha}_1 = 0$. Therefore (6) holds if and only if

$$\begin{bmatrix} 1 - \delta_1^2 & \boldsymbol{\delta}_1' \\ \boldsymbol{\delta}_1 & \mathbf{\Sigma}_{11} - \mathbf{\Delta}_1 \end{bmatrix}\begin{bmatrix} 1 \\ -\boldsymbol{\beta}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{0} \end{bmatrix}, \tag{7}$$

where

$$\mathbf{\Delta}_1 = \begin{bmatrix} \delta_2^2 & 0 & \cdots & 0 \\ 0 & \delta_3^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \delta_p^2 \end{bmatrix}.$$

Equations (7) are

$$1 - \delta_1^2 - \boldsymbol{\delta}_1'\boldsymbol{\beta}_1 = 0$$

$$\boldsymbol{\delta}_1 - \mathbf{\Sigma}_{11}\boldsymbol{\beta}_1 + \mathbf{\Delta}_1\boldsymbol{\beta}_1 = \mathbf{0}.$$

However

$$\boldsymbol{\delta}_1'\boldsymbol{\beta}_1 = \boldsymbol{\delta}_1'\mathbf{\Sigma}_{11}^{-1}\boldsymbol{\delta}_1 = \rho_1^2$$

$$\boldsymbol{\delta}_1 = \mathbf{\Sigma}_{11}\boldsymbol{\beta}_1 .$$

Therefore (6) holds if and only if

$$\delta_1^2 = 1 - \rho_1^2 \tag{8}$$

and

$$\beta_{12}\delta_2^2 = \beta_{13}\delta_3^2 = \cdots = \beta_{1p}\delta_p^2 = 0. \tag{9}$$

Equations (9) state that, for each j, $2 \le j \le p$, either $\delta_j^2 = 0$ or $\beta_{1j} = 0$.

At this stage it is worth noting the connection between the regression coefficients β_{ij} and the matrix $\mathbf{\Sigma}^{-1}$. Writing

$$\mathbf{\Sigma} = \begin{bmatrix} 1 & \boldsymbol{\delta}_1' \\ \boldsymbol{\delta}_1 & \mathbf{\Sigma}_{11} \end{bmatrix},$$

we have

$$\mathbf{\Sigma}^{-1} = (1 - \boldsymbol{\delta}_1'\mathbf{\Sigma}_{11}^{-1}\boldsymbol{\delta}_1)^{-1}\begin{bmatrix} 1 & -\boldsymbol{\delta}_1'\mathbf{\Sigma}_{11}^{-1} \\ -\mathbf{\Sigma}_{11}^{-1}\boldsymbol{\delta}_1 & \mathbf{\Sigma}_{11}^{-1} + \mathbf{\Sigma}_{11}^{-1}\boldsymbol{\delta}_1\boldsymbol{\delta}_1'\mathbf{\Sigma}_{11}^{-1} \end{bmatrix}$$

$$= (1 - \rho_1^2)^{-1}\begin{bmatrix} 1 & -\boldsymbol{\beta}_1' \\ -\boldsymbol{\beta}_1 & \mathbf{\Sigma}_{11}^{-1} + \boldsymbol{\beta}_1\boldsymbol{\beta}_1' \end{bmatrix}.$$

From the form of the first row of $\mathbf{\Sigma}^{-1}$ it follows that

$$\mathbf{\Sigma}^{-1} = \mathbf{D}^{-1}\begin{bmatrix} 1 & -\beta_{12} & -\beta_{13} & \cdots & -\beta_{1p} \\ -\beta_{21} & 1 & -\beta_{23} & \cdots & -\beta_{2p} \\ -\beta_{21} & -\beta_{32} & 1 & \cdots & -\beta_{3p} \\ \vdots & \vdots & \vdots & & \vdots \\ -\beta_{p1} & -\beta_{p2} & -\beta_{p3} & \cdots & 1 \end{bmatrix} = \mathbf{D}^{-1}\begin{bmatrix} \boldsymbol{\alpha}_1' \\ \boldsymbol{\alpha}_2' \\ \boldsymbol{\alpha}_3' \\ \vdots \\ \boldsymbol{\alpha}_p' \end{bmatrix}, \text{ say,} \tag{10}$$

where $\mathbf{D}$ in the diagonal matrix whose ith diagonal element is $1 - \rho_i^2$. Thus $\beta_{ij} = 0$, $i \neq j$, if and only if the (i, j) element of $\mathbf{\Sigma}^{-1}$ is zero.

We see from (8) that there is equality in any member of (5) if and only if there is equality in the corresponding member of (3).

4. *Conditions for $p - m$ Equalities*

In this section we shall obtain necessary and sufficient conditions for equality in the last $p - m$ inequalities of (5) (and therefore of (3)). Guttman [2] obtained sufficient conditions for equality in the last $p - m$ inequalities of (3) (and therefore of (5)). The end results of this section are closely related to Guttman's but the method of analysis is different.

Let

$$S = \{1, 2, \cdots, m\}, \qquad T = \{m + 1, m + 2, \cdots, p\}.$$

Then we know that

$$1 - \rho_i^2 = \delta_i^2 + \sum_{j \neq i} \beta_{ij}^2 \delta_j^2 \qquad i \in T, \tag{11}$$

if and only if

$$\delta_i^2 = 1 - \rho_i^2 \qquad i \in T, \tag{12}$$

and

$$\sum_{j \in S} \beta_{ij}^2 \delta_j^2 + \sum_{j \in T, j \neq i} \beta_{ij}^2 \delta_j^2 = 0, \qquad i \in T. \tag{13}$$

Equation (13), taken in conjunction with (12), holds if and only if

$$\beta_{ij} \delta_j = 0 \qquad i \in T, \quad j \in S, \tag{14}$$

and

$$\beta_{ij} = 0 \qquad i \in T, \quad j \in T, \quad i \neq j. \tag{15}$$

Condition (15) states that the (i, j) element of $\mathbf{\Sigma}^{-1}$ is zero for all $i \in T$, $j \in T$, $i \neq j$. One way of describing this is to say that, given $x_1, x_2, \cdots, x_m$, the variables $x_{m+1}, x_{m+2}, \cdots, x_p$ are uncorrelated (partially). This is clearly a very special situation. (In particular, there is equality in all p inequalities if and only if $x_1, x_2, \cdots, x_p$ are completely uncorrelated, that is $\mathbf{\Sigma} = \mathbf{I}$.) When (12), (14), and (15) hold, $\mathbf{\Gamma} = \mathbf{\Sigma} - \mathbf{\Delta}$ is at most of rank m for

$$\mathbf{\Gamma}\boldsymbol{\alpha}_j = \mathbf{0} \qquad j \in T$$

and, from (10), the vectors $\boldsymbol{\alpha}_{m+1}, \cdots, \boldsymbol{\alpha}_p$ are linearly independent.

So far we have pointed out that (12), (14), and (15) are necessary and sufficient for (11). Now (14) holds in particular if

$$\delta_j = 0 \qquad j \in S. \tag{16}$$

When (12), (15), and (16) are satisfied $\mathbf{\Gamma}$ is exactly of rank m for, using an obvious notation, we have

$$\mathbf{\Sigma} = \begin{bmatrix} \mathbf{\Sigma}_{SS} & \mathbf{\Sigma}_{ST} \\ \mathbf{\Sigma}_{TS} & \mathbf{\Sigma}_{TT} \end{bmatrix} = \begin{bmatrix} \mathbf{\Sigma}_{SS} & \mathbf{\Sigma}_{ST} \\ \mathbf{\Sigma}_{TS} & \mathbf{0} \end{bmatrix}.$$

5. *Discussion*

Guttman [2] proved an important limiting relationship between the communalities $1 - \delta_i^2$ and the multiple-correlation coefficients ρ. Namely that, if q is the rank of $\mathbf{\Gamma}$ and $q/p \to 0$ as $p \to \infty$, then $(1 - \delta_i^2)/\rho_i^2 \to 1$, $i = 1, 2, \cdots, p$. Thus the communality $1 - \delta_i^2$ may be characterised as the squared multiple-correlation coefficient of x_i with an infinite set of "relevant" variables. This property, and the fact that in very special situations it is possible for $1 - \delta_i^2$ to equal ρ_i^2 for some values of i, led Guttman to call ρ_i^2 the *"best possible" systematic estimate* of $1 - \delta_i^2$ in the practical case of a finite number of variables. While it is usually realized that the use of these estimates is strictly illegitimate in the sense that they lead to a $\mathbf{\Gamma}$ which is nonnegative definite and therefore cannot be a covariance matrix, the extent to which they are illegitimate may now be better judged from the amount by which they contradict the p inequalities (5).

In this paper we have only been concerned with helping to demarcate the region of legitimate communalities and not with any criteria which propose a particular point in this region as *the* communality solution. We hope to treat this aspect in a later paper.

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