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ORDMET: A GENERAL ALGORITHM FOR CONSTRUCTING ALL NUMERICAL SOLUTIONS TO ORDERED METRIC STRUCTURES[1]

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The algorithm is applicable to structures such as are obtained from additive conjoint measurement designs, unfolding theory, general Fechnerian scaling, some special types of multidimensional scaling, and ordinal multiple regression. A description is obtained of the space containing all possible numerical representations which can satisfy the structure, the size and shape of which is informative. The Abelson-Tukey maximin r^2 solution is provided.

Introduction

The increased use of ordinal data in the social sciences has generated a demand for improved methods of analyzing and scaling such data. Presented in this paper is a general algorithm ORDMET which provides scale-free tests and constructs all possible solutions for any linear model for a given empirical ordering. The algorithm is applicable to a wide variety of ordinal techniques now used in behavioral sciences, such as additive conjoint measurement, unfolding theory, special types of multidimensional scaling, ordinal multiple regression, and general Fechnerian scaling. Such linear models, when combined with an empirical ordering, generate a system of linear inequalities. Although any standard linear programming technique may be used to test a particular model, such techniques yield only one scale solution. However, if scaling rather than just model-testing is of interest, it is advantageous to obtain a description of the space containing all possible solutions because 1) the size of the space indicates how successfully the scale values and the parameters have been constrained, 2) the shape of the space shows which scale values are adequately constrained and which need additional data, and 3) it is then simple to select a most representative scale solution using such criteria as the Abelson–Tukey [1963] maximin r^2.

The solution space for a system of linear inequalities is a convex cone

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in n-space and the ORDMET algorithm is essentially a procedure for generating that cone and characterizing it by a minimal set of vectors. ORDMET is based upon, and an extension of, previous work by Goode [1964] and Phillips [1971]. Goode [see also Coombs, 1964, pp. 97–102] developed a "delta-method" which produces a solution space for most of the inequalities but is incomplete and in some cases generates numerous side-constraints. Phillips corrected that deficiency in the Goode method by applying a result due to Farkas [1902]. The Phillips algorithm indeed produces a complete solution space with no side-constraints, but the obtained description of such spaces is often far from minimal, in that it contains redundant vectors. A redundant vector is one which can be expressed as a positive linear combination of other vectors in the cone. Phillips states that these redundant vectors can easily be removed by inspection. However, practical applications, particularly when the ordering is incomplete, result in the generation of as many as 10,000 redundant vectors when the minimal set has as few as 20 or 30 vectors. Such large numbers of redundant vectors make computation of final steps in the algorithm very expensive if not impossible, and preclude the calculation of a representative solution using the Abelson–Tukey technique.

In this paper the ORDMET algorithm is completed (or at least improved) by: 1) the addition of a procedure based on an algorithm by Wets and Witzgall [1967] for removing redundant solutions (Section 2) during and after the computation of the basic Goode–Phillips algorithm and 2) the utilization of the Abelson–Tukey maximin r^2 criterion for selecting a representative solution (Section 5). In addition, a formal description (Section 1) is given of the models to which ORDMET is applicable and specific applications, some not considered with respect to this algorithm before, are presented to illustrate 1) algorithm operation (Section 3); 2) the related theoretical development of general Fechnerian scaling (Section 4); and 3) problems of error (Section 6).

1. General Model to which ORDMET Applies

Suppose that an ordering ($\gtrsim$) on a finite set $A = \{A_1, \cdots, A_i, \cdots, A_k, \cdots, A_n\}$ of n stimuli, stimulus differences, or experimental conditions has been empirically established and that there exists a hypothesized linear model for the data. That is, if $S = [S(A_1), \cdots, S(A_i), \cdots, S(A_n)]$ represents the vector of unknown scale values for the stimuli in A, if $R = [r_1, r_2, \cdots, r_h, \cdots, r_p]$ is the unknown vector of parameters, and if V is an $n \times p$ matrix of known coefficients v_{ih} such that v_{ih} represents the number of times the parameter r_h is included in the representation for A_i, then

$$S = VR + B, \tag{1}$$

where B is a constant column vector, is the model to which ORDMET applies. Equation 1 combined with the empirical ordering [$A_j \gtrsim A_k$ implies $S(A_j) \geq S(A_k)$] results in a system of linear inequalities which we denote as $\langle A, V, \gtrsim \rangle$ and call an *ordered metric structure* (OMS). A vector of scale values S is a solution for an OMS if and only if it satisfies (1) and is consistent with the empirical ordering which requires

$$A_j \gtrsim A_k \Rightarrow s_j \geq s_k \Rightarrow \sum v_{jh} r_h \geq \sum v_{kh} r_h \tag{2}$$

where $s_j = S(A_j)$. If S is a solution for the OMS $\langle A, V, \gtrsim \rangle$, then there exists a positive convex polyhedral cone $V^* = [V_1^*, \cdots, V_i^*, \cdots, V_m^*]$ in Re^n where each V_i^* is a column vector such that S is a scale solution if and only if for some non-negative R^* and B

$$S = V^* R^* + B. \tag{3}$$

Thus, finding V^* determines all possible solutions S for $\langle A, V, \gtrsim \rangle$.

Although the scale values S are often of interest in their own right, the values R for the underlying parameters may be more important. This necessitates solving for R in (1). Equation 1 was constructed to be consistent but in most interesting applications it is overdetermined since n, the number of rows in V, is normally greater than p, the number of parameters or columns of V. Thus, if rank of V equals p then (1) may be solved for R by selecting any p linearly independent rows of V and then solving the resulting reduced system of linear equations. However, it is usually more convenient to solve

$$V'S = V'[VR + B] \tag{4}$$

which is the normal equation corresponding to (1) and has the solution

$$R = (V'V)^{-1}V'[S - B]. \tag{5}$$

Substituting the solution for S given in (3) yields

$$R = (V'V)^{-1}V'V^*R^*, \qquad R^* \geq 0, \tag{6}$$

and defines the entire solution space for the parameter vector R. If $V'V$ is not of rank p then V must be reparameterized or the generalized inverse substituted for $(V'V)^{-1}$ in the above. In the latter case however, R will not be unique but rather only certain linear transformations of R will be unique.

Thus, if $\langle A, V, \gtrsim \rangle$ is an OMS, then (3) and (6) show that in order to obtain a complete description of possible scale values for both S and for the underlying parameters R, one need only obtain the convex cone V^*. The next section presents the general ORDMET algorithm for finding V^*.

2. Details of the ORDMET Algorithm

Without the empirical ordering $\gtrsim$ on A, the model of (1) is trivially satisfied by any arbitrary choice of R and B. Requiring a specific ordering on A to be consistent with the model as prescribed by (2) has the effect of constraining the allowable choices for R. The ORDMET algorithm operates by processing those constraints so that a new matrix V^* is constructed which incorporates both the original structure of V and the constraints implied by the ordering on A so that the vector R^* in (3) is again arbitrary. The original model VR is used as the initial trial solution $V^{(0)}R^{(0)}$ and at each step σ, one of the pairwise orderings (*e.g.*, $A_i \gtrsim A_k$) is incorporated into the solution by constructing $V^{(\sigma)}R^{(\sigma)}$ from $V^{(\sigma-1)}R^{(\sigma-1)}$. Since there are at most $\binom{n}{2}$ pairwise orderings on the set A, the algorithm terminates in at most $\binom{n}{2}$ steps. After the redundant columns (*i.e.*, those that are positive linear combinations of other columns) have been removed from the last $V^{(\sigma)}R^{(\sigma)}$ so constructed, it is relabelled as V^*R^* and serves as the complete solution of (3).

The key to the ORDMET algorithm is a result by Farkas [1902] which was first applied to this problem by Phillips [1971]. Farkas has shown that the linear inequality

$$\sum_i a_i f_i - \sum_j b_j g_j \geq 0, \tag{7}$$

where a_i and b_j are known non-negative coefficients, and where f_i and g_j are positive unknowns, has the general solution

$$a_i f_i = w_i + \sum_j x_{ij} \quad \text{and} \quad b_j g_j = \sum_i x_{ij} \tag{8}$$

where w_i and x_{ij} are positive. The basic algorithm consists of repeated applications of the above result to an OMS $\langle A, V, \gtrsim \rangle$, that is, to the linear inequalities generated by the model of (1) and the orderings of (2).

Initialization. Let $V^{(0)} = V$, let $R^{(0)} = R$, and let $\sigma = 0$. (Note: to simplify notation, the σ superscript is omitted in the following, but it should be remembered that v_{jh} and r_h are from the current $V^{(\sigma)}$ and $R^{(\sigma)}$ respectively.)

Step 1. From the ordering on A, select a pairwise ordering (*e.g.*, $A_j \gtrsim A_k$) that has not yet been processed. The order in which the inequalities are selected is arbitrary; the only effect of different processing orders is in the number of redundant columns generated. If there are no inequalities remaining to be processed, go to Step 5 to remove redundant vectors from $V^{(\sigma)}$.

Step 2. Transform the inequality implied by the pairwise ordering

$$A_j \gtrsim A_k \Rightarrow \sum v_{jh} r_h - \sum v_{kh} r_h \geq 0 \Rightarrow \sum (v_{jh} - v_{kh}) r_h \geq 0 \tag{9}$$

into the form of (7) by relabelling the m columns of the current trial solution $V^{(\sigma)}$ so that there are q columns labelled V_α^+ where $a_\alpha = (v_{jh} - v_{kh}) > 0$, and t columns labelled V_β^- where $-b_\beta = (v_{jh} - v_{kh}) < 0$, and $m - q - t$ columns labelled V_γ^0 where $c_\gamma = (v_{jh} - v_{kh}) = 0$. Thus,

$$A_j \gtrsim A_k \Rightarrow \sum_\alpha^q a_\alpha r_\alpha^+ - \sum_\beta^t b_\beta r_\beta^- \geq 0, \tag{10}$$

where r_α^+, r_β^-, and r_γ^0 are the elements of $R^{(\sigma)}$ relabelled in the same manner as their corresponding column in $V^{(\sigma)}$.

Step 3. Check (10) for consistency.

Case i. ($t = 0$) The inequality is already satisfied in the current trial solution $V^{(\sigma)}$, return to Step 1.

Case ii. ($q = 0$, $t \geq 1$) The system of inequalities is inconsistent so $\langle A, V, \gtrsim \rangle$ does not have a solution. Stop or use the procedures for error described in Section 6.

Case iii. ($q \geq 1$, $t \geq 1$) The inequality is appropriate for Farkas' result; go to Step 4.

Step 4. Construct a new matrix $V^{(\sigma+1)}$ from the old matrix $V^{(\sigma)}$ according to the following rules.

i) $V_{\alpha+(\beta-1)q}{}^{(\sigma+1)} = V_\beta^- + \dfrac{b_\beta}{a_\alpha} V_\alpha^+ \qquad \alpha = 1, q; \beta = 1, t$

ii) $V_{qt+\alpha}{}^{(\sigma+1)} = V_\alpha^+ \qquad \alpha = 1, q$

iii) $V_{q(t+1)+\gamma}{}^{(\sigma+1)} = V_\gamma^0 \qquad \gamma = 1, m - q - t$

where V^+, V^-, and V^0 are all old columns of $V^{(\sigma)}$ as defined in Step 2. Increment σ by 1 and let m, the number of columns in $V^{(\sigma)}$, equal $q(t + 1) + m - q - t$. Return to Step 1. If the number of columns of $V^{(\sigma)}$ have increased excessively (as defined by computer storage limitations), do Steps 5 through 11 to reduce $V^{(\sigma)}$ to a manageable size before returning to Step 1.

It is easy to show that the above construction is the general solution for the inequality of (10). Each value v_{jh} in column V_α^+ in the old matrix has been replaced by the value $b_\beta v_{jh}$ in columns $V_{\alpha+(\beta-1)q}{}^{(\sigma+1)}$ for $\beta = 1, t$ and by the value v_{jh} in column $V_{qt+\alpha}{}^{(\sigma+1)}$ in the new matrix; each value in column V_β^- has been replaced by that same value in columns $V_{\alpha+(\beta-1)q}{}^{(\alpha+1)}$ for $\alpha = 1, q$, and all columns V_γ^0 have been transferred directly from the old to the new matrix, so that

$$(11) \qquad \begin{cases} a_\alpha r_\alpha^+ = a_\alpha r_{qt+\alpha}{}^{(\sigma+1)} + \sum_\beta^t b_\beta r_{\alpha+(\beta-1)q}{}^{(\sigma+1)} \\ b_\beta r_\beta^- = \sum_\alpha^q b_\beta r_{\alpha+(\beta-1)q}{}^{(\sigma+1)} \\ c_\gamma r_\gamma^0 = c_\gamma r_{q(t+1)+\gamma}{}^{(\sigma+1)}. \end{cases}$$

Farkas' result shows that (11) is the general solution for the inequality in (10).

The removal of redundant columns of $V^{(\sigma)}$ is accomplished by using an algorithm due to Wets and Witzgall [1967] for determining the frame of a convex polyhedral cone. If $G(V)$ is the convex polyhedral cone formed from V, then the column V_h is redundant if and only if $G(V - \{V_h\}) = G(V)$

or, equivalently,

$$V_h = \sum_{i \neq h} \alpha_i V_i , \qquad \alpha_i \geq 0. \tag{12}$$

Step 5. Row reduce $V^{(\sigma)}$ to canonical form W so that it contains a $k \times k$ permutation matrix with ones as the only non-zero entries and whose columns are called basic; all other columns are non-basic and expressed as linear combinations of the basis (*i.e.*, premultiply $V^{(\sigma)}$ by an appropriate real, non-singular matrix P so that $W = PV^{(\sigma)}$ is in row reduced form). Note that $n - k$ rows of W will be zero and can be omitted from further calculations.

Step 6. Check the following four test criteria to determine whether each column is necessary, redundant, or not yet determined. If any columns are redundant replace them with a zero vector.

Criterion i) If W_j is non-basic and $W_j \geq 0$, then W_j is redundant since it is a positive linear combination of basic columns.

Criterion ii) If W_j is non-basic and has exactly one positive entry w_{ij} , then the basic column W_k for which $w_{ik} = 1$ is redundant since after pivoting W_j into the basis in place of W_k , Criterion i would then apply to W_k .

Criterion iii) If the i-th row of W contains exactly one negative entry w_{ik} , then W_k is necessary since that entry could never be expressed as a positive linear combination of other entries in the i-th row.

Criterion iv) If the i-th row contains exactly one positive entry w_{ik} then W_k is necessary for the same reason as in Criterion iii.

Step 7. If all non-basic columns of W have been classified as either necessary or redundant, go to Step 10. Otherwise, select a negative entry w_{kj} in an unclassified non-basic column (there must be a negative entry or Criterion i would have classified the column as redundant).

Step 8. Find another negative entry w_{kh} in the same row as w_{kj} (there must be another negative entry or Criterion iii would have classified column j as redundant).

Step 9. Find w_{mh} such that

$$0 \leq \frac{w_{mj}}{w_{mh}} \leq \min_i \left\{ \frac{w_{ij}}{w_{ih}} \,\middle|\, w_{ij} \geq 0,\, w_{ih} > 0 \right\} \tag{13}$$

and then use w_{mh} as a pivot in a simplex or row reduction step. This increases the originally negative entry w_{kj} while all non-negative entries in column W_j remain non-negative. Repeat Steps 8 and 9 until either Criterion i or iii classifies W_j . Then return to Step 7.

Step 10. If any unclassified basic columns remain, replace them in the basis with necessary non-basic columns and return to Step 2. Otherwise, go to Step 11.

Step 11. Remove all columns from $V^{(\sigma)}$ and all r_h from $R^{(\sigma)}$ for which the corresponding columns of W have been labelled as redundant. Let $V^* = V^{(\sigma)}$ and $R^* = R^{(\sigma)}$. Stop if no specific numerical representation is desired; otherwise, go to Step 12 described in Section 5.

Since Farkas' method produces a complete general solution at each step, the resulting matrix V^* when used in (3) and (6) generates all possible solutions for both the scale values S and the parameter values R for the OMS $\langle A, V, \gtrsim \rangle$.

3. *ORDMET Applied to a Simple Unfolding Example*

The application of ORDMET to a simple unfolding problem not only provides a convenient illustration, but also gives a more intuitive justification for the method than is evident in Farkas' result. The successive solution matrices produced by the heuristic version below are identical (except for a possible permutation of columns) to those produced by the application of the ORDMET algorithm described in Section 2. Consider the following hypothetical data from an unfolding analysis. Suppose that there are five stimuli in the order A, B, C, D, E on a single dimension and assume the following midpoint ordering has been obtained from the I-scales (see Coombs, [1964], pp. 80–92 for details of that process).

(14) $$AB, AC, AD, BC, AE, BD, CD, BE, CE, DE$$

where XY denotes the midpoint between X and Y on the line from A to E. This midpoint ordering implies the following ordered metric relations on the interstimulus distances

(15) $$\overline{AB} \gtrsim \overline{CD}, \overline{AB} \gtrsim \overline{DE}, \overline{CE} \gtrsim \overline{AB}, \overline{DE} \gtrsim \overline{BC}.$$

Stage 1. The set of all distances $\{\overline{AE}, \overline{AD}, \overline{BE}, \text{etc.}\}$ is taken as the set A of Section 1, the values of the single interval distances $\{\overline{AB}, \overline{BC}, \overline{CD}, \overline{DE}\}$ are used as parameters, and Table 1(a) gives the matrix V which represents the unidimensional distance model (*e.g.*, $\overline{AE} = \overline{AB} + \overline{BC} + \overline{CD} + \overline{DE}$) and serves as the first trial solution, $V^{(0)}$.

Stage 2. The first piece of metric information $\overline{AB} \gtrsim \overline{CD}$ implies

(16) $$S(\overline{AB}) \geq S(\overline{CD}) \Rightarrow r_1 - r_3 \geq 0.$$

Let $r_1 = r_3 + r_5$, where r_5 is a new r column equal to the unknown positive difference between r_1 and r_3. In Table 1(b) then, every entry in column $V_1^{(0)}$ has been replaced by an entry in column $V_3^{(1)}$ and an entry in column $V_5^{(1)}$.

Stage 3. Next, $\overline{AB} \gtrsim \overline{DE}$ implies that $r_3 + r_5 - r_4 \geq 0$. Let us imagine that r_4 is broken into two parts, r_6 and r_7, such that $r_4 = r_6 + r_7$ with $r_3 \geq r_6$ and $r_5 \geq r_7$. Then let $r_3 = r_6 + r_8$ where r_8 is the unknown positive amount that r_3 exceeds r_6 and $r_5 = r_7 + r_9$ where r_9 is the amount that r_5 exceeds r_7. So each r_4 is replaced by an r_6 and an r_7, each r_3 is replaced by an r_6 and an r_8, and finally each r_5 is replaced by an r_7 and an r_9. This has been done in

TABLE 1

Successive V Matrices Produced by the ORDMET Algorithm for a Simple Unfolding Problem

	(a) Stage 1: Initial Model $V^{(0)}$				(b) Stage 2: $\overline{AB} \geq \overline{CD}$ yields $V^{(1)}$				(c) Stage 3: $\overline{AB} \geq \overline{DE}$ yields $V^{(2)}$					(d) Stage 4: $\overline{CE} \geq \overline{AB}$ yields $V^{(3)}$				
	r_1	r_2	r_3	r_4	r_2	r_3	r_4	r_5	r_2	r_6	r_7	r_8	r_9	r_2	r_7	r_8	r_9	r_{10}
$\overline{AE}$	1	1	1	1	1	2	1	1	1	3	2	2	1	1	2	2	4	3
$\overline{AD}$	1	1	1		1	2		1	1	2	1	2	1	1	1	2	3	2
$\overline{BE}$		1	1	1	1	1	1		1	2	1	1		1	1	1	2	2
$\overline{CE}$			1	1		1	1			2	1	1			1	1	2	2
$\overline{AC}$	1	1			1	1		1	1	1	1	1	1	1	1	1	2	1
$\overline{AB}$	1					1		1		1	$\underline{1}$	1	1		1	1	2	1
$\overline{BD}$		1	1		1	1			1	1		1		1		1	1	1
$\overline{DE}$				1			1			1	1				1		1	1
$\overline{BC}$		1			1				1					1				
$\overline{CD}$			1			1				1		1				1	1	1

	(e) Stage 5: $\overline{DE} \geq \overline{BC}$ yields $V^{(4)}$							(f) Stage 8: Final Solution $V^{(*)}$ (normalized)					(g) Stage 9: Maximin r^2 Solution $\tilde{S}$
	r_8	r_{11}	r_{12}	r_{13}	r_{14}	r_{15}	r_{16}	r_8	r_{11}	r_{12}	r_{14}	r_{16}	
$\overline{AE}$	2	3	5	4	2	4	3	1	1	1	1	1	1
$\overline{AD}$	2	2	4	3	1	3	2	1	.67	.8	.5	.67	.75
$\overline{BE}$	1	2	3	3	1	2	2	.5	.67	.6	.5	.67	.54
$\overline{CE}$	1	1	2	2	1	2	2	.5	.33	.4	.5	.67	.46
$\overline{AC}$	1	2	3	2	1	2	1	.5	.67	.6	.5	.33	.54
$\overline{AB}$	1	1	2	1	1	2	1	.5	.33	.4	.5	.33	.46
$\overline{BD}$	1	1	2	2		1	1	.5	.33	.4	0	.33	.28
$\overline{DE}$		1	1	1	1	1	1	0	.33	.2	.5	.33	.31
$\overline{BC}$		1	1	1				0	.33	.2	0	0	.08
$\overline{CD}$	1		1	1		1	1	.5	0	.2	0	.33	.21

Table 1(c). Note that the number of columns has increased from four in $V^{(1)}$ to five in $V^{(2)}$.

Stage 4. Continuing, $\overline{CE} \gtrsim \overline{AB}$ implies that $r_6 - r_9 \geq 0$. So let $r_6 = r_9 + r_{10}$ where r_{10} is the amount r_6 exceeds r_9. This substitution in Table 1(d) guarantees that $S(\overline{CE}) \geq S(\overline{AB})$ in the current matrix representation of the solution $V^{(3)}$.

Stage 5. Finally, $\overline{DE} \gtrsim \overline{BC}$ implies that $r_7 + r_9 + r_{10} - r_2 \geq 0$. In this case r_2 must be decomposed into three parts ($r_2 = r_{11} + r_{12} + r_{13}$) such that $r_7 \geq r_{11}$, $r_9 \geq r_{12}$, and $r_{10} \geq r_{13}$. Then the substitutions $r_7 = r_{11} + r_{14}$, $r_9 = r_{12} + r_{15}$, and $r_{10} = r_{13} + r_{16}$ are made where r_{14}, r_{15}, and r_{16} are the proper non-negative unknown values to convert the respective inequalities to equalities. This has been completed in Table 1(e) to create $V^{(4)}$.

Stage 6. The columns in Table 1(e) form the matrix $V^{(4)}$ which may be used as V^* in (3) and (6) to generate all possible solutions for the orderings in (15) and the model in Table 1(a). However, $V^{(4)}$ may not be minimal so the Wets and Witzgall algorithm for finding a minimal frame for $V^{(4)}$ is applied. Although in this case patient inspection may reveal that $V_{12}^{(4)} =$

$V_8^{(4)} + V_{11}^{(4)}$ and $V_{15}^{(4)} = V_8^{(4)} + V_{14}^{(4)}$ are the only redundant solutions, the formal algorithm will be applied for illustration. First, row reduce $V^{(4)}$ to canonical form W.

$$(17)\qquad W = \begin{array}{c} \begin{array}{ccccccc} W_8 & W_{11} & W_{12} & W_{13} & W_{14} & W_{15} & W_{16} \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Note that there are four rows in W for this four parameter problem and that W_8, W_{11}, W_{13}, and W_{14} form the basis. Columns W_{12} and W_{15} satisfy Criterion i of Step 6 of the ORDMET algorithm in Section 2 and are therefore redundant. The second row satisfies Criterion iii with $w_{ik} = w_{2,16} = -1$, so W_{16} is necessary. With redundant columns W_{12} and W_{15} removed, the first and second rows now satisfy Criterion iv which in this case implies that W_8 and W_{11} are necessary.

Stage 7. Since basic columns W_{13} and W_{14} remain unclassified, the necessary non-basic column W_{16} is moved into the basis and W_{13} out resulting in

$$(18)\qquad W = \begin{array}{c} \begin{array}{ccccc} W_8 & W_{11} & W_{13} & W_{14} & W_{16} \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \end{array}$$

Now Criterion iii applies to the fourth row which implies that W_{13} is necessary. Also, Criterion iv applies to the fourth row and implies that W_{14} is necessary.

Stage 8. All columns are now classified and redundant solutions $V_{12}^{(4)}$ and $V_{15}^{(4)}$ have been removed from $V^{(4)}$ to form the final representation in Table 1(f) of V^*.

4. General Fechnerian Scaling of Dominance Matrices

A Fechnerian scaling model is defined by Luce and Suppes [1965] to consist of a set of binary choice probabilities for which there exist a real-valued function r over a set of stimuli and a cumulative distribution function F such that

$$(19)\qquad \begin{cases} \text{i)} & F(0) = \tfrac{1}{2}, \\ \text{ii)} & \text{for all } a_i, a_j \text{ for which } p(a_i, a_j) \neq 0 \text{ or } 1, \\ & p(a_i, a_j) = F[r(a_i) - r(a_j)], \quad \text{and} \\ \text{iii)} & p(a_i, a_j) + p(a_j, a_i) = 1 \end{cases}$$

where $p(a_i, a_j)$ is the probability that stimulus a_i is chosen over a_j. In this section, we show that the natural ordering (weak stochastic transitivity) on the $p(a_i, a_j)$'s and the model in (19) may be used to construct an OMS. As such, the ORDMET algorithm may then be applied to find the complete general solution giving all possible functions r and F consistent with the particular empirical ordering.

Two well-known instances of Fechnerian scaling are Case V of the Law of Comparative Judgment (CJ-V) and the Bradley–Terry–Luce (BTL) model. In CJ-V, F is the normal integral with mean $r(a_i) - r(a_j)$ and variance equal to one. In the BTL model F is the logistic function

(20) $$p(a_i, a_j) = \frac{1}{1 + \exp -[r'(a_i) - r'(a_j)]}$$

where $r'(a_i) = \log r(a_i)$. Further, Yellott [1971] has shown that Dawkins' [1969] model is a member of this class with

(21) $$p(a_i, a_j) = \int_{-\infty}^{r(a_i)-r(a_j)} \frac{\alpha}{2} e^{-\alpha|x|}\, dx.$$

The advantage of treating general Fechnerian scaling as a special case of the ORDMET algorithm is that no assumptions about the exact form of F are necessary. Instead, it need only be assumed that F is monotonically increasing, which is equivalent to the binary choice probabilities being strictly monotone with the differences in scale values $r(a_i) - r(a_j)$. Furthermore, generality is also obtained in that response measures are not restricted to binary choice probabilities but to any binary response measure which will satisfy the ordinal equivalent of the conditions in (19) and hence be an OMS as in (1) and (2). Condition 19(i) is a scale factor which is free. Conditions 19(ii) and 19(iii) require that the response measure be strictly monotone with the distance and that the ordering of the cells on the two sides of the diagonal be skew-symmetric. Testing this latter condition requires that the complementary observations be empirically independent which is not possible with binary choice probabilities. Candidates for such response measures include latency measures on binary choices and confidence ratings.

If we let the set of all pairs (a_i, a_j) constitute the set A of Section 1 and if we construct V so that

(22) $$S = VR + B \Rightarrow S(a_i, a_j) = r(a_i) - r(a_j) + b$$

and if $p(a_i, a_j) \geq p(a_k, a_l) \geq 1/2$ is considered to imply that $(a_i, a_j) \gtrsim (a_k, a_l)$, then $\langle A, V, \gtrsim \rangle$ is an ordered metric structure. Thus, any general Fechnerian scaling problem converts naturally to the form of an ordered metric structure and the ORDMET algorithm may then be applied to construct all possible solutions.

Of course, the real goal of Fechnerian scaling is determination of R and not of S. However, this is a trivial difference as can be seen by letting the smallest $R(a_j) = 0$ and relabelling that a_j as a_0, so that $R(a_i) = S(a_i, a_0)$ gives the desired description of R.

TABLE 2
Hypothetical Data Matrix for Fechnerian Scaling

(a)

	a_0	a_1	a_2	a_3
a_1	.56			
a_2	.67	.63		
a_3	.74	.70	.58	
a_4	.84	.81	.72	.64

(b)

	a_0	a_1	a_2	a_3
a_1	1			
a_2	5	3		
a_3	8	6	2	
a_4	10	9	7	4

As an example of ORDMET applied to a general Fechnerian scaling problem, consider the hypothetical binary choice probabilities in Table 2(a). Each entry represents the proportion of choices for row over column. The rows and columns have been permuted using triangular analysis [see Coombs, 1964, pp. 352–359] so that all entries below the diagonal are greater than .50, and that entries in each column increase monotonically from top to bottom and the rows increase from right to left. The corresponding rank order (1 = smallest) is given in Table 2(b). This hypothetical empirical ordering and the general Fechnerian model (with $R(a_0)$ set equal to zero to make $(V'V)^{-1}$ non-singular) yield

$$(23)\quad S = \begin{bmatrix} S(a_4, a_0) \\ S(a_4, a_1) \\ S(a_3, a_0) \\ S(a_4, a_2) \\ S(a_3, a_1) \\ S(a_2, a_0) \\ S(a_4, a_3) \\ S(a_2, a_1) \\ S(a_3, a_2) \\ S(a_1, a_0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} R(a_4) \\ R(a_3) \\ R(a_2) \\ R(a_1) \end{bmatrix} + \begin{bmatrix} b \\ b \\ b \\ b \\ b \\ b \\ b \\ b \\ b \\ b \end{bmatrix} = VR + B$$

where the ordering within S from top to bottom reflects the empirical ordering. Formally applying the ORDMET algorithm to the above yields the following general solution for the parameter vector.

$$(24) \qquad R = \begin{bmatrix} R(a_4) \\ R(a_3) \\ R(a_2) \\ R(a_1) \end{bmatrix} = \begin{bmatrix} 3 & 2 & 5 & 4 \\ 2 & 1 & 3 & 3 \\ 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} r_1^* \\ r_2^* \\ r_3^* \\ r_4^* \end{bmatrix}$$

where $r_i^* \geq 0$ and b is arbitrary. At this point we know that the hypothetical data of Table 2 satisfy the general Fechnerian model of (19).

5. A Representative Solution

When the ORDMET algorithm is used for testing a model, successful determination of V^* confirms the fit of the ordinal data to the model and nothing else remains to be done. However, if a specific set of scale values is desired for descriptive purposes or for use as dependent variables in other analyses, then one particular solution must be selected from the convex cone defined by V^*. Unless the solutions given by (3) are tightly constrained (*i.e.* the columns of V^* are all very similar), then a most representative solution must be chosen from the set of solutions. In this section, a set of criteria due to Abelson and Tukey [1963] for finding a maximin linear contrast is applied to this problem. Essentially, the Abelson–Tukey technique is to find that solution $\tilde{S}$ in the convex cone generated by V^* that maximizes the minimum squared correlation between $\tilde{S}$ and any other solution vector in the cone. They showed that this minimum correlation with $\tilde{S}$ must be with one of the column vectors of V^* (*i.e.* on an edge of the convex cone) so that the part of the ORDMET algorithm that removes redundant columns from V^* also greatly simplifies the search for $\tilde{S}$. Such a maximin strategy is conservative in that it protects against the worst state of nature regardless of the likelihood of that state; a better solution could be found if there were additional information about the distribution of the parameters, but this is seldom the case in practical applications of ORDMET.

Let $\tilde{V}$ be a set of column vectors contained in V^*. Then there exists an $\tilde{S}$, unique up to a linear transformation, such that

i) $\tilde{S} = \tilde{V}\tilde{R}^*$, $\tilde{R}^* > 0$,

ii) the squared correlations between $\tilde{S}$ and any other vector in $\tilde{V}$ are all equal to $\tilde{r}^2$, and

iii) the squared correlations between $\tilde{S}$ and any other vector in V^* are at least as great as the common value of $\tilde{r}^2$ in (ii).

The common squared correlation in (ii) is maximin; $\tilde{S}$ is the maximin solution and can serve as the desired representative solution. Abelson and Tukey presented both a "top down" and a "bottom up" approach for constructing $\tilde{S}$ given the set of extreme vectors of V^*. The following is a slight modification of their "top down" approach and is a continuation of the basic ORDMET algorithm presented in Section 2.

Step 12. The matrix W from Step 10 is in canonical form and indicates a set of columns forming a basis and necessarily linearly independent. Take the columns of V^* which correspond to these basic columns as a trial set for $\tilde{V}$.

Step 13. Find a vector of weights Z such that if $\tilde{S} = \tilde{V}Z$, then the r^2 between $\tilde{S}$ and V_i^* is constant for all V_i^* in $\tilde{V}$. Let this constant be $\hat{r}^2$.

Step 14. i) If the r^2 between $\tilde{S}$ and V_i^* for any V_i^* in V^* is less than $\hat{r}^2$, then use W to generate a new set of basic columns (*i.e.* pivot a non-basic column into the basis to replace a previously basic column, making sure that this does not create a set previously tested). Use the columns of V^* corresponding to this new set of basic columns of W as a new trial solution $\tilde{V}$. Go to Step 13.
ii) Otherwise, go to Step 15.

Step 15. i) If the elements z_i of Z are not all non-negative, remove the column from $\tilde{V}$ which corresponds to the lowest $z_i < 0$ and go to Step 13.
ii) If Z is non-negative, stop. $\tilde{S} = \tilde{V}Z$ is the maximin solution and maximin $r^2 = \hat{r}^2$. (Z is simply a particular choice for R^* in (3).)

The above continuation of the ORDMET algorithm will always find the unique maximin solution $\tilde{S}$ for V^*. However, due to its exhaustive search of all possible subsets of linearly independent columns of V^*, it can be rather slow and expensive. Thus, in practice, it is usually better to use the following "bottom up" procedure first even though it is not guaranteed to find the solution, reserving the "top down" algorithm outlined in Steps 12 to 15 for cases where Steps 12′ to 15′ fail.

Step 12′. Find the two columns of V^* which have the smallest r^2. Use these two columns as the first trial solution set $\tilde{V}$.

Step 13′. Same as Step 13 above.

Step 14′. i) If the r^2 between S and V_i^* for any V_i^* in V^* is less than $\hat{r}^2$, then add the V_i^* with the lowest r^2 to the trial solution set $\tilde{V}$ and go to Step 13′.
ii) Otherwise go to Step 15′.

Step 15′. i) If the elements z_i of Z are not all non-negative, the "bottom-up" approach has failed; use the "top-down" procedure.
ii) If $Z \geq 0$, stop. $\tilde{S} = \tilde{V}Z$ is the maximin solution and maximin $r^2 = \hat{r}^2$.

As an example, consider again the unfolding problem in Section 3 (Table 1f). Use of the "bottom-up" algorithm yields the following sequence of calculations:

(12′) The lowest squared correlation is between V_8 and V_{14} and has the value .278.
(13′) The linear combination $\tilde{S} = .487V_8 + .513V_{14}$ has an r^2 with both V_8 and V_{14} of .764.
(14′) However, r^2 for $\tilde{S}$ and V_{11} is .628 which is less than r^2 for $\tilde{S}$ and any other vector in V^*. Therefore, V_{11} is added to the set containing V_8 and V_{14}.
(13′) The linear combination $\tilde{S} = .415V_8 + .234V_{11} + .351V_{14}$ has an r^2 of .749 with each column in $\tilde{V}$.
(14′) The r^2 between $\tilde{S}$ and any other vector in V^* is at least as great.
(15′) The weights in the linear combination defining $\tilde{S}$ are all positive, so the maximum solution has been found, $\hat{r}^2 = .749$. $\tilde{S}$ is displayed as Stage 9 in Table 1(g).

Since there are only four linearly independent subsets [$(V_8, V_{11}, V_{14}, V_{16})$, $(V_8, V_{13}, V_{14}, V_{16})$, $(V_8, V_{11}, V_{13}, V_{16})$, and $(V_8, V_{11}, V_{13}, V_{14})$] as determined from W in (18), the "top down" algorithm of Steps 12 to 15 will terminate in at most four iterations for this example.

The solution $\tilde{S}$, then, is one which is a most representative scale for the set A in a well-defined sense. The rows of V^* used to determine $\tilde{S}$, however, are not linearly independent but are additive compositions of more elemental units, the parameters r_h. The parameter vector R may be of sufficient theoretical interest in its own right to justify some best estimate of the relative values of the components of R. An alternative representative solution, then, is to apply Steps 12 to 15 of the algorithm to the parameter space of (6) to seek directly an Abelson–Tukey maximin r^2 solution for R. Call the set of parameters that results $\tilde{\tilde{R}}$ and the corresponding maximin squared correlation $\hat{\hat{r}}^2$. This is easily accomplished by substituting $V^* = (V'V)^{-1}V'V^*$ for V^* everywhere in Steps 12 to 15 and 12′ to 15′ as well as substituting $\tilde{\tilde{R}}$ for $\tilde{S}$ and $\hat{\hat{r}}^2$ for $\hat{r}^2$. Then

$$\tilde{\tilde{S}} = V^*\tilde{\tilde{R}} \tag{25}$$

is an alternate representative scale solution.

While in general $\tilde{S} \neq \tilde{\tilde{S}}$, $\tilde{R} \neq \tilde{\tilde{R}}$, and $\hat{r}^2 \neq \hat{\hat{r}}^2$, in practice the choice appears to be of little consequence since $\tilde{S}$ and $\tilde{\tilde{S}}$ are very similar (their correlation must be at least as great as $\hat{r}$) and likewise for $\tilde{R}$ and $\tilde{\tilde{R}}$ (their correlation is greater than $\hat{\hat{r}}$). For the example of generalized Fechnerian scaling of Section 4 (see (24)), the two representative parameter solutions are

$$\tilde{R}' = (1.0,\ .64,\ .47,\ .12,\ .00) \quad \text{and} \qquad \hat{\hat{R}}' = (1.0,\ .63,\ .48,\ .11,\ .00); \tag{26}$$

clearly, the distinction is not important in this case.

Since any particular scale solution S is unique at most up to a linear transformation, the solution can be normalized by choosing an origin and unit of measurement to provide a characterization of the solution space as a convex polyhedron in a $(p - 1)$-dimensional subspace of the n-dimensional convex cone. Since each column vector of V^* represents an extreme solution, this normalization is readily accomplished by linearly transforming each column V_h so that for a given row j (preferably the row corresponding to the largest $S(A_j)$) the value of 1 is assigned to v_{hj} for all h, and then requiring that $B = 0$ and that the sum of the r_h^*'s equals 1. It is usually more convenient to consider the normalized solution for R since its rows do not contain the linear dependencies that the rows of S do. Graphs of planes of the convex polyhedron corresponding to R provide a quick check of how successfully given parameter values have been constrained by the ordinal data in relation to other parameters.

TABLE 3

Matrix V* and Maximin r^2 Solution for Successive Scalings of Four Points and Six Interpoint Distances.

		Scaling 1				Scaling 2				Scaling 3 and 4				
Distance	True Value	V*			$\tilde{S}$ ($\hat{r}^2$=.25)	V*			$\tilde{S}$ ($\hat{r}^2$=.856)	V*				$\tilde{S}$ ($\hat{r}^2$=.932)
$\overline{AH}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\overline{CH}$	.85	0	1	1	.63	.75	1	1	.93	.75	1	1	.80	.87
$\overline{EH}$	.56	0	0	1	.37	.50	.50	1	.66	.50	.5	.67	.60	.53
$\overline{AE}$	.44	1	1	0	.63	.50	.50	0	.34	.50	.5	.33	.40	.47
$\overline{CE}$	.29	0	1	0	.26	.25	.50	0	.27	.25	.5	.33	.20	.34
$\overline{AC}$	.15	1	0	0	.37	.25	0	0	.07	.25	0	0	.20	.13

	Scaling 5						Scaling 6				
Distance	V*					$\tilde{S}$ ($\hat{r}^2$=.963)	V*				$\tilde{S}$ ($\hat{r}^2$=.972)
$\overline{AH}$	1	1	1	1	1	1	1	1	1	1	1
$\overline{CH}$	.78	.8	.8	1	.83	.87	.83	1	.83	.87	.91
$\overline{EH}$	.56	.6	.5	.67	.50	.59	.50	.67	.58	.62	.57
$\overline{AE}$	.44	.4	.5	.33	.50	.41	.50	.33	.42	.38	.43
$\overline{CE}$	.22	.2	.3	.33	.33	.29	.33	.33	.25	.25	.33
$\overline{AC}$	.22	.2	.2	0	.17	.13	.17	0	.17	.12	.09

The following unidimensional scaling example illustrates this use of the convex polyhedron as a characterization of the parameter space and also demonstrates how quickly additional ordinal constraints reduce the size of the polyhedron toward a limiting point. Consider the following random points on a line: $A = .00$, $B = .05$, $C = .15$, $D = .32$, $E = .44$, $F = .66$, $G = .79$, $H = 1.00$. These generate the following ordering of interpoint distances from largest to smallest:

$$\begin{aligned}(27)\quad &\overline{AH}, \overline{BH}, \overline{CH}, \overline{AG}, \overline{BG}, \overline{DH}, \overline{AF}, \overline{CG}, \overline{BF}, \overline{EH}, \overline{CF}, \overline{DG}, \overline{AE},\\ &\overline{BE}, \overline{EG}, \overline{FH}, \overline{DF}, \overline{AD}, \overline{CE}, \overline{BD}, \overline{EF}, \overline{GH}, \overline{CD}, \overline{AC}, \overline{FG}, \overline{DE}, \overline{BC}, \overline{AB}\end{aligned}$$

Subsets of these distances were processed by the ORDMET algorithm. The first subset, labelled Scaling 1, consists of only the ordering information for points A, C, E, and H; that is, the ordering of the distances $\overline{AC}$, $\overline{AE}$, and $\overline{AH}$. Scaling 2 consists of all the interpoint distances between A, C, E, and H. Scaling 3 is the same set as Scaling 2 with the addition of the point F and its interpoint distances; Scaling 4 adds point B and its distances; Scaling 5 adds point G and its distances; and finally Scaling 6 consists of all 28 distances between eight points. Table 3 presents the normalized extreme solutions and the maximin r^2 solution for the points of Scaling 1 as determined by successive scalings of the six subsets of distances. Figure 1 depicts the plane of the convex polyhedron for the values of $\overline{AC}$ and $\overline{AE}$ after each scaling.

The area of the admissible solution plane (*i.e.*, a cross-section of the solution polyhedron) generally contracts with additional metric information, but in this case there was no effect from the addition of stimulus B (Scaling 4 compared to Scaling 3). The decrease in area corresponds to the increase in maximin r^2 from .25 for Scaling 1 to .856 for Scaling 2 to .972 for Scaling 6. Note the dramatic increase from .25 to .856, which represents the effect on the uniqueness of the representation by the change from only ordinal information on points to a complete ordered metric on the four points (Scaling 1 to Scaling 2).

6. *Fallible Data*

The description in Section 1 of the general model to which ORDMET applies and the subsequent examples have been based on error-free cases. However, a probabilistic error term can be added to the basic linear model yielding

$$(28)\qquad S = VR + B + E$$

where E is a random vector with mean zero. If the differences between the $S(A_i)$'s are large relative to the components of E (*i.e.*, if E does not alter the ordering $\gtrsim$ on A), then E need not be considered. Thus, an obvious strategy for dealing with error in ordered metric scaling is to use replication

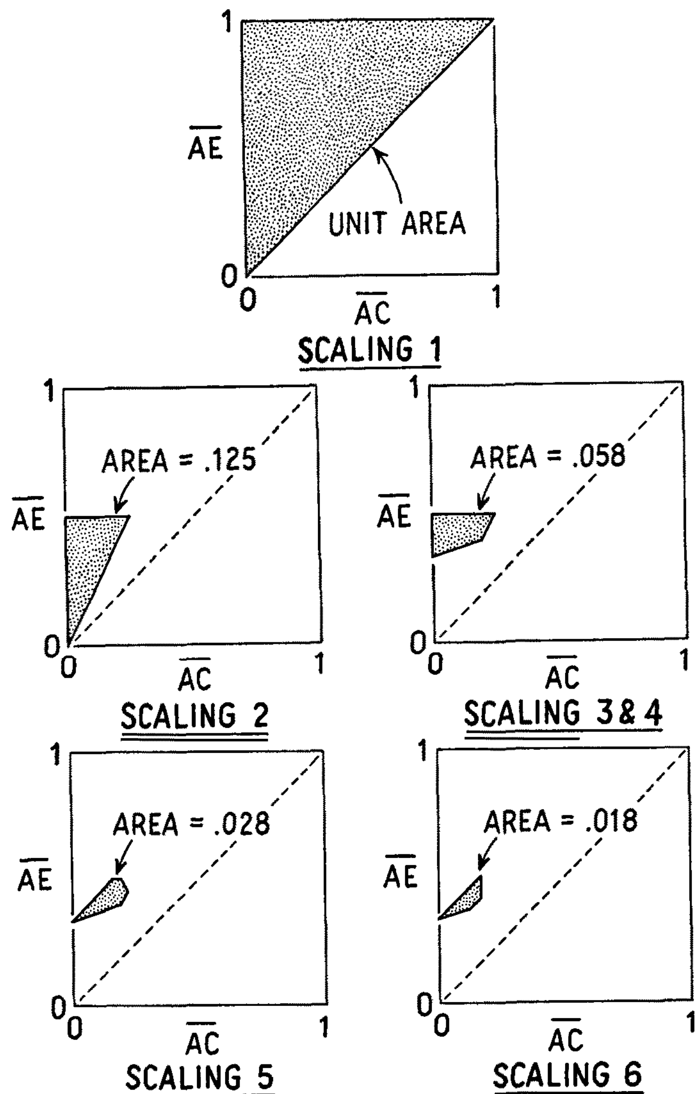

Figure 1
Effect of Ordered Metric Information on the Size and Shape of the Solution Space.

or better control of error-producing factors to reduce the magnitude of E and thereby decrease the likelihood that E will cause the observed ordering to differ from the presumed underlying ordering. There are of course many applications where either extensive replication or control is not feasible or not successful in eliminating error. In this section, two techniques for treating fallible data in ordered metric scaling are presented.

The first is a threshold or criterion technique. While only ordinal information is input to the ORDMET algorithm, the ordering may actually be inferred from other numerical information. For example, we may say $A_i \gtrsim A_k$ if and only if the mean of several observations of A_i exceeds the mean of the observations of A_k or, in the case of Fechnerian scaling, $A_i \gtrsim A_k$ if and only if $p(A_i) \geq p(A_k)$. If we let $T(A_i)$ be the general term for observed numerical information (*i.e.*, the response measure) which is on at least an ordinal scale, then such rules can be expressed as

$$A_i \gtrsim A_k \Leftrightarrow T(A_i) \geq T(A_k). \tag{29}$$

Because of the presence of error and the resulting artificial precision in the measurement of $T(A_i)$, it is sometimes inadvisable to translate very small differences among the $T(A_i)$'s into the ordering on A. The following rule for translating the ordering on T into an ordering on A avoids this problem by establishing a minimal threshold difference that must be exceeded to create an inequality (see Davidson, Suppes, and Siegel [1957] for an application of this to an additive problem,) that is

$$A_i \gtrsim A_k \quad \text{if and only if} \quad T(A_i) - T(A_k) > \theta. \tag{30}$$

A priori assumptions about distributions of the error or prior experience may be used to determine θ, or the ORDMET algorithm may be used to find the minimal θ for a particular model and given observations T. The value of θ need not be constant, but may instead be a function of $T(A_i)$ and $T(A_k)$. Such would be the case if $T(A_i)$ represented a binomial probability and θ were chosen so that an inequality was generated if and only if the two probabilities were significantly different for a given level.

A second technique for dealing with error is to find the minimum number of pairwise reversals that would make the ordering on A consistent with the given linear model. This is equivalent to finding V^* such that the Kendall's τ between the ordering induced on S by V^* and the ordering $\gtrsim$ is maximized. Such a solution is easily found by systematically eliminating subsets of the input inequalities to ORDMET. However, such solutions will not in general be unique, so the complete solution space may consist of the union of several convex cones. Tversky and Zivian [1966] describe their algorithm for finding a solution that maximizes τ, but their algorithm finds only one solution vector in one of the possible cones, with no information as to its uniqueness.

As an example of how these error techniques can be applied to real data, consider the two-factor additivity data in Table 4(a), from a study by Birnbaum [1972]. Subjects made "morality judgements" of all pairs of "immoral behaviors" in a 5 $\times$ 5 design. The additive model for this study is

$$S(A_i , B_j) = R(A_i - A_1) + R(B_i - B_1) + K \tag{31}$$

where A_i and B_j represent descriptions of immoral behaviors, $S(A_i , B_j)$ is

TABLE 4

Morality Judgments* from a 5 × 5 Design

(a) Original Average Values

	B1	B2	B3	B4	B5
A1	1.78	2.28	2.33	2.49	2.65
A2	3.36	4.45	4.77	4.93	5.35
A3	3.90	5.14	5.19	5.14	6.19
A4	4.22	5.29	5.32	5.76	6.92
A5	4.61	6.16	6.16	6.52	8.17

(b) Rank Order

	B1	B2	B3	B4	B5
A1	1	2	3	4	5
A2	6	9	11	12	18
A3	7	13.5	15	13.5	22
A4	8	16	17	19	24
A5	10	20.5	20.5	23	25

* Data from Birnbaum [1972]

the psychological value for the immorality of the pair, $R(A_i - A_1)$ is the psychological difference between stimulus component A_i and A_1, and K is an additive constant equal to $S(A_1, B_1)$. The rank order implied by the values of T (where $T(A_i, B_j)$ is the mean response over 100 subjects on a nine-point rating scale) in Table 4(a) is given in Table 4(b). When the model of (31) and the ordering of Table 4(b) are input to ORDMET an error is detected and no solution is generated. The problem is that $T(A_i, B_3) \leq T(A_i, B_4)$ for all A_i except A_3. This is a violation of the monotonicity (independence) axiom of conjoint measurement [see Krantz, Luce, Suppes, and Tversky, 1971] and must be eliminated if an additive representation is to be found. Setting θ equal to .05 will eliminate the violation caused by $(A_3, B_3) \gtrsim (A_3, B_4)$ and will eliminate several other inequalities [such as $(A_3, B_5) \gtrsim (A_5, B_3)$] as well. When the reduced set of inequalities is input to ORDMET, a solution is found. The maximin r^2 solution and the parameter solution space are presented in Table 5(a). Alternatively, the solution requiring the fewest number of pair reversals may be sought, which in this case yields a solution very similiar to that of the threshold technique. For the data, reversing either of two subsets of orderings will achieve additivity: 1) reversing $(A_i, B_4) \gtrsim (A_i, B_3)$ for all i not equal to 3, or 2) reversing $(A_3, B_3) \gtrsim (A_3, B_4)$. The latter is clearly the minimum necessary reversals and the corresponding maximin r^2 solution and parameter space are presented in Table 5(b).

Because of the relatively low level of θ (less than 0.7 per cent of the range) and because only one pair reversal was necessary to yield a fit to the model, it does not seem that Birnbaum's rejection of the additive model based on an analysis of variance of the data in Table 4(a) was justified. Note also that

TABLE 5

Additive Solution for Data of Table 4

(a) Solution for θ = .05

Parameter	Normalized Parameter Solution Vectors																$\tilde{\tilde{R}}(r^2=.935)$
A5-A1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
A4-A1	1	.85	.80	.80	.87	.80	1	.87	.87	.90	.89	.91	.82	.82	.89	.91	.86
A3-A1	1	.71	.60	.80	.75	.70	.67	.75	.75	.80	.78	.82	.73	.73	.78	.82	.71
A2-A1	1	.57	.60	.60	.62	.60	.67	.62	.62	.60	.67	.64	.64	.64	.67	.64	.71
B5-B1	0	.57	.60	.60	.62	.60	.67	.62	.62	.60	.67	.64	.64	.64	.67	.64	.43
B4-B1	0	.42	.40	.40	.50	.40	.33	.50	.37	.40	.44	.45	.45	.45	.44	.45	.29
B3-B1	0	.42	.40	.40	.50	.40	.33	.37	.37	.40	.33	.36	.36	.45	.44	.45	.29
B2-B1	0	.42	.40	.40	.37	.30	.33	.37	.25	.30	.33	.36	.36	.36	.33	.36	.29

(b) Solution after Eliminating (A3, B3) ≿ (A3, B4)

Parameter	Normalized Parameter Solution Vectors									$\tilde{\tilde{R}}(r^2=.947)$
A5-A1	1	1	1	1	1	1	1	1	1	1
A4-A1	1	.87	.82	.80	.89	1	.91	.87	.89	.95
A3-A1	1	.75	.73	.80	.78	.67	.82	.75	.78	.80
A2-A1	1	.62	.64	.60	.67	.67	.64	.62	.67	.75
B5-B1	0	.62	.64	.60	.67	.67	.64	.62	.67	.46
B4-B1	0	.50	.45	.40	.44	.33	.45	.37	.44	.27
B3-B1	0	.37	.36	.40	.33	.33	.36	.37	.44	.25
B2-B1	0	.37	.36	.40	.33	.33	.36	.25	.33	.25

the pair reversal was also a violation for the alternative model suggested by Birnbaum. Rather the analysis using the ORDMET algorithm shows that the interactions detected by ANOVA were not inherent and could be attributed to a non-linear response scale. As a by-product of this analysis, this non-linearity can be estimated by plotting $\tilde{\tilde{S}}$ versus the original T values, as is done in Figure 2.

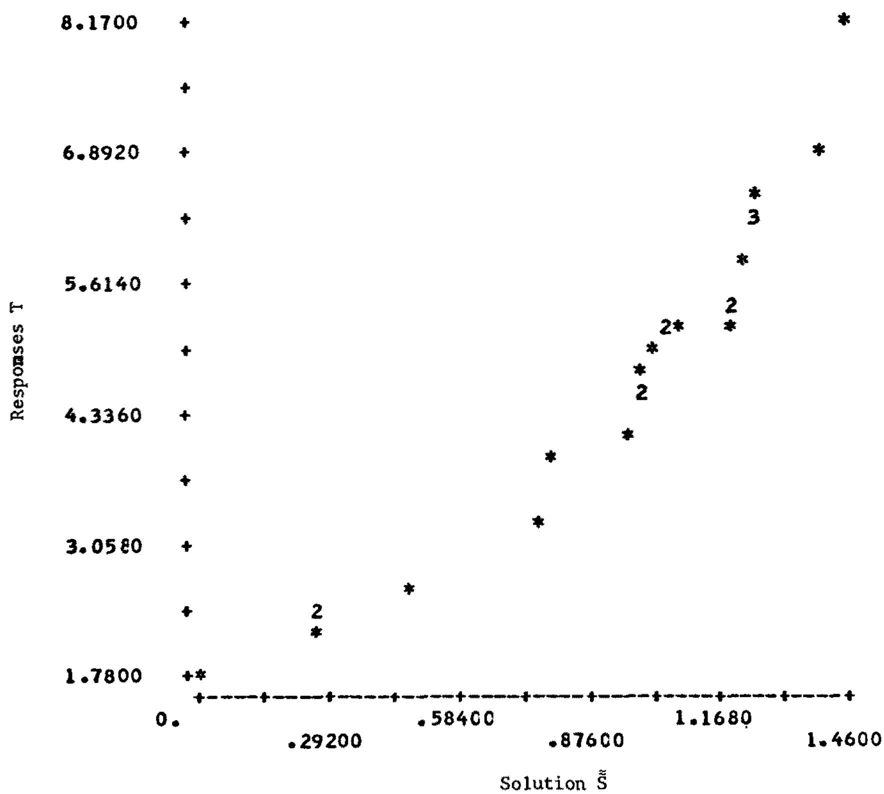

FIGURE 2
Estimation of Non-Linear Response Function.

REFERENCES

Abelson, R. P. and Tukey, J. W. Efficient utilization of non-numerical information in quantitative analysis: General theory and the case of simple order. *Annals of Mathematical Statistics*, 1963, **34**, 1347–1369.

Birnbaum, M. H. Morality judgments: Test of an averaging model. *Journal of Experimental Psychology*, 1972, **93**, 35–42.

Coombs, C. H. *A Theory of Data*. New York: Wiley, 1964.

Davidson, D., Suppes, P., and Siegel, S. *Decision making: An experimental approach*. Stanford: Stanford University Press, 1957.

Dawkins, R. A threshold model of choice behavior. *Animal Behavior*, 1969, **17**, 120–133.

Farkas, J. Veber die Theorie der einfachen Ungleichungen. *Journal fur die Reine and Angewandte Mathematik*, 1902, **124**, 1–27.

Goode, F. M. An algorithm for the additive conjoint measurement of finite data matrices. *American Psychologist*, 1964, **19**, 579. (Abstract)

Krantz, D. H., Luce, R. D., Suppes, P., and Tversky, A. *Foundations of Measurement*, Volume I. New York: Academic Press, 1971.

Luce, R. D. and Suppes, P. Preference, utility, and subjective probability. In R. D. Luce,

R. R. Bush, and E. Galanter (Eds.). *Handbook of Mathematical Psychology*, Vol. III. New York: Wiley, 1965, 249–410.

Phillips, J. P. N. A note on representation of ordered metric scaling. *British Journal of Mathematical and Statistical Psychology*, 1971, **24**, 239–250.

Tversky, A. and Zivian, A. A computer program for additivity analysis. *Behavioral Science*, 1966, **11**, 78–79.

Wets, R. J. B. and Witzgall, C. Algorithms for frames and linearity spaces of cones. *Journal of Research of the National Bureau of Standards*, 1967, **71B**, 1–7.

Yellott, J. I., Jr. Generalized Thurstone representations for three choice theories: Uniqueness results. Paper presented at Mathematical Psychology Meetings, Princeton, New Jersey, September 1–2, 1971.

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