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Dedicated to the seventieth anniversary of R.R. Dogonadze's birth

Meixner Wavelet Transform: A Tool for Studying Stationary Discrete-Time Stochastic Processes

I. B. Grafova[1] and B. M. Grafov[2]

[1]*Department of Economics, University of Michigan, 611 Tappan st., Ann Arbor, Michigan 48104, USA*

[2]*Frumkin Institute of Electrochemistry, Russian Academy of Sciences, Leninskii pr. 31, Moscow, 119071 Russia*

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Abstract—A general approach to analyzing discrete stochastic processes in the context of spectral analysis in the Laplace domain is considered. It is shown that a multichannel algorithm, which may be used for determining a discrete Laplace transform of the correlation function corresponding to a stationary discrete-time stochastic process, may be designed on the basis of Meixner wavelets.

INTRODUCTION

Spectral analysis of stochastic processes and noises plays an important role in multiple scientific, engineering, and technological applications including electrochemistry, electronics, hydrodynamics, econometrics, biomedicine, and finances (see for example [1–13] and references therein).

Traditionally, the spectral density of a stationary stochastic process is determined in the form of a Fourier transform of the correlation function. In other words, only the information value of the imaginary axis in the Laplace plane is used, albeit the information value of this plane substantially exceeds that of a line. In particular, the possibility emerges of using a powerful tool: complex-variable functions. For example, the possibility emerges for directly measuring a derivative of the spectral density of a stationary stochastic process as an analytical function of the Laplace variable [14].

The method of discrete experiment [15] becomes increasingly important in electrochemistry. A discrete wavelet transform is widely applied in analysis, storage, and reconstruction of determinate signals [16, 17]. At the same time, it is more convenient to analyze stationary stochastic processes using one-side continuous wavelet transforms [18], which include the Haar transform [19], the Laplace transform [20–22], and wavelet transforms that utilize the Laguerre polynomials [23, 24]. It is essential that applying a wavelet transform to an analysis of statistical fluctuations and noises makes an inverse wavelet transform redundant, i.e. there is no need to reconstruct the original after the compression procedure. For this reason, many constraints drop out, including the requirement that the first several moments of a wavelet transform should vanish. Curiously enough, the Morlet wavelets [24], which have already turned classic, have a nontrivial initial moment.

The main aim of this work is to design an algorithm for estimating an operational spectral density, defined here as a quadrupled Laplace transform of a correlation function for a stationary discrete-time stochastic process.

To reach the goal to be sought we have introduced and utilized the Meixner wavelets. Earlier, wavelets based on Meixner polynomials were neither studied nor used.

THE NOTATION

The Meixner polynomials [25–29] are discrete orthogonal polynomials [30] first studied by Meixner in his classic work published in 1934 [31]. The orthogonality condition defines Meixner polynomials to within an accuracy of a constant factor. That is why there exist different forms of their representation in the literature. As a rule it is assumed [26, 29] that a Meixner polynomial of the nth order at the origin takes on the value equal to $n!$. We base this work on [29], where Meixner polynomials are denoted as $m_n^{(\gamma,\mu)}(x)$. Here, n is the polynomial order, x is an independent discrete variable ($x = 0, 1, \ldots$), and γ and μ are some parameters. At $\gamma = 1$, Meixner polynomials represent a discrete version of well-known continuous Laguerre polynomials [32]. Below, Meixner polynomials at $\gamma = 1$ are denoted as $m_n(x, \mu)$; by definition,

$$m_n(x,\mu) = (\mu^{n/2}/n!)m_n^{(\gamma=1,\mu)}(x). \qquad (1)$$

The first three Meixner polynomials are:

$$m_0(x, \mu) = 1,$$

$$m_1(x, \mu) = \{1 - (1-\mu)(x+1)\}/\mu^{1/2},$$

and

$$m_2(x, \mu) = \{1 - 2(1-\mu)(x+1) + (1-\mu)^2(x+2)(x+1)/2\}/\mu.$$

In the general case,

$$m_n(x, \mu) = \{1 + \ldots + (-1)^n(1-\mu)^n(x+n)\ldots(x+1)/n!\}/\mu^{n/2}.$$

A system of Meixner polynomials $m_n(x, \mu)$ is orthogonal, with the weight μ^x. The norm of Meixner polynomials $m_n(x, \mu)$ is independent of their order n:

$$\sum_0^{\infty} \delta x \mu^x m_n(x, \mu) m_r(x, \mu) = \delta_{nr}/(1-\mu), \quad (2)$$

where δ_{nr} is the Kronecker delta. In (2) and further on we use designations of definite sums [33], which are very similar to definite integrals in their properties. For example, by definition,

$$\sum_a^b \delta x(\ldots) = \sum_{x=a}^{b-1} (\ldots),$$

where a and b are positive integers. Note that a Meixner polynomial of order n is orthogonal with the same weight μ^x relative to any polynomial $P_r(x)$ whose order r is less than or equal to $n - 1$:

$$\sum_0^{\infty} \delta x \mu^x m_n(x, \mu) P_r(x) = 0. \quad (3)$$

Let $y(t)$ be a stochastic process with a finite intersection property, which occurs at fixed time instants $t = xt_0$, where t_0 is the digitization interval. For the correlation function $k(t)$ of a stochastic process $y(t)$ we select the standard relationship

$$k(t) = \langle y(0) y(t) \rangle, \quad (4)$$

where angle brackets correspond to the averaging over the ensemble of realizations. A discrete Laplace transform $K(p)$ for the correlation function $k(t)$ is

$$K(p) = t_0 \left\{ \sum_0^{\infty} \delta x \exp(-xpt_0) k(xt_0) - k(0)/2 \right\}, \quad (5)$$

where p is a Laplace variable ($p > 0$). The factor t_0 (digitization interval) is added into the right-hand side of (5) to provide for a correct transition to continuous time at $t_0 \longrightarrow 0$. It will be recalled that, by definition, the spectral density of a stochastic process is equal to $4K(p)$ in the Laplace domain.

MEIXNER WAVELETS

We define the Meixner wavelets $Y_n(p)$ for a stochastic process $y(t)$ by means of the equation

$$Y_n(p) = t_0 \sum_0^{\infty} \delta x m_n(x, \mu) \exp(-pt) y(t), \quad (6)$$

where $\mu = \exp(-2pt_0)$ and $t = xt_0$. The convergence of infinite sum (6) is ensured by exponential factor $\exp(-pt)$. It is owing to precisely this exponential factor that we can view a realization of a finite stochastic process as an infinite realization, hence the infinite upper integration bound in (6).The quantity p simultaneously acts as a scaling variable in the Meixner wavelet transform.

The dispersion of a Meixner wavelet is related to the operational spectral density $4K(p)$ through a very simple relation (see Appendix):

$$2\langle Y_n(p) Y_n(p) \rangle = 4K(p) t_0/\{1 - \exp(-2pt_0)\}. \quad (7)$$

DISCUSSION

First of all note that the right-hand side of (7) is independent of the Meixner polynomial order n. Hence, the dispersion of any Meixner wavelet has the same value. It follows that a multichannel algorithm may be constructed, by finding a discrete Laplace transform for the correlation function of a stationary discrete stochastic process (operational stochastic density) in the form

$$K(p) = \langle Y_n(p) Y_n(p) \rangle \{1 - \exp(-2pt_0)\}/(2t_0). \quad (8)$$

An analysis of the derivation of (7) and (8) shows that these relations remain valid for complex values of the Laplace variable, provided of course that the real part of the variable is positive.

CONCLUSIONS

The main result of this work, equation (8), suggests as follows.

(1) Meixner wavelets allow one to make spectral analysis of stationary discrete stochastic processes in the Laplace domain.

(2) A discrete Laplace transform for the correlation function of a stationary discrete-time stochastic process may be found by using a Meixner wavelet of any order.

(3) Based on the Meixner wavelet transform one can perform a multichannel analysis of a stationary discrete stochastic process when a multichannel analyzer has a single input, to which the process is supplied, and several fundamentally different channels for information processing. Each information channel is measuring the same quantity, namely, a discrete Laplace transform for the correlation function of the input stochastic process (operational stochastic density).

(4) The algorithm for a multichannel derivation of the spectral density of a stochastic process with the aid of a family of Meixner wavelets is unique. Such an algorithm does not exists within a Fourier analysis and is unknown in the framework of discrete wavelets [16, 34].

(5) Among the information wavelet technologies intended for analyzing stationary discrete-time stochastic processes, the Meixner wavelet transform is unique in that its dispersion has a clear physical meaning—within an a universal factor it equals the operational spectral density of the process under analysis.

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APPENDIX

According to the definition of a Meixner wavelet (6), its dispersion is the product of two infinite series

$$\langle Y_n^2(p)\rangle = \left\langle t_0^2 \sum_0^\infty \delta x_1 m_n(x_1, \mu)\exp(-pt_1)y(t_1) \times \sum_0^\infty \delta x_2 m_n(x_2, \mu)\exp(-pt_2)y(t_2)\right\rangle,$$

where $\mu = \exp(-2pt_0)$, $t_1 = x_1t_0$, and $t_2 = x_2t_0$. This product may be transformed into a sum of two infinite series:

$$\langle Y_n^2(p)\rangle = \Sigma(1) + \Sigma(2),$$

where

$$\Sigma(1) = \left\langle t_0^2 \sum_0^\infty \delta x_1 m_n(x_1, \mu)\exp(-pt_1)y(t_1) \times \sum_{x_1}^\infty \delta x_2 m_n(x_2, \mu)\exp(-pt_2)y(t_2)\right\rangle,$$

$$\Sigma(2) = \left\langle t_0^2 \sum_0^\infty \delta x_1 m_n(x_1, \mu)\exp(-pt_1)y(t_1) \times \sum_0^{x_1} \delta x_2 m_n(x_2, \mu)\exp(-pt_2)y(t_2)\right\rangle.$$

Let us first transform the expression for $\Sigma(1)$. As $y(t)$ is a stochastic process with a finite intersection property, averaging the quantity $y(t_1)y(t_2)$ in $\Sigma(1)$ leads to correlation function $k(t_2 - t_1)$ for the process under consideration:

$$\langle y(t_1)y(t_2)\rangle = k(t_2 - t_1).$$

Substituting $x_4 = x_2 - x_1$ for interior variable x_2, then

$$\Sigma(1) = t_0^2 \sum_0^\infty \delta x_1 \sum_0^\infty \delta x_4 m_n(x_1, \mu)\mathrm{e}^{-2pt_1}k(t_4) \times m_n(x_1 + x_4, \mu)e^{-pt_4}. \quad \text{(A1)}$$

According to the Newton series formula [33], a discrete Meixner polynomial $m_n(x_1 + x_4, \mu)$ may be written as

$$m_n(x_1 + x_4, \mu) = m_n(x_1, \mu) + P_{n-1}(x_1),$$

$$P_{n-1}(x_1) = x_4\Delta m_n(x_1, \mu) + x_4(x_4 - 1)\Delta^2 m_n(x_1, \mu)/2 + \ldots, \quad \text{(A2)}$$

where Δ and Δ^2 are operators of the first and second discrete derivatives. The polynomial $P_{n-1}(x_1)$ is a polynomial of order $(n - 1)$ relative to the variable x_1. Using this circumstance and the orthogonality of Meixner polynomials in the form of (3) and substituting (A2) into (A1), then

$$\Sigma(1) = t_0 \sum_0^\infty \delta x_1 m_n^2(x_1, \mu)e^{-2pt_1} t_0 \sum_0^\infty \delta x_4 k(t_4)e^{pt_4}. \quad \text{(A3)}$$

Using now (2) and (5), instead of (A3) we obtain

$$\Sigma(1) = \{K(p) + t_0k(0)/2\}t_0/\{1 - \exp(-2pt_0)\}. \quad \text{(A4)}$$

Similarly we can do unto the sum $\Sigma(2)$. Multiplying the two infinite series that constitute $\Sigma(2)$ and altering the summation order, then

$$\Sigma(2) = \left\langle t_0^2 \sum_0^\infty \delta x_2 \sum_{x_2+1}^\infty \delta x_1 m_n(x_1, \mu)m_n(x_2, \mu) \times y(t_1)y(t_2)e^{-pt_1}e^{-pt_2}\right\rangle.$$

Replacing the summation variable x_1 by $x_3 = x_1 - x_2$ and taking into account that $\langle y(t_1)y(t_2)\rangle = k(t_3)$, where $t_3 = x_3t_0$, then

$$\Sigma(2) = t_0^2 \sum_0^\infty \delta x_2 \sum_1^\infty \delta x_3 m_n(x_1, \mu) \times m_n(x_2 + x_3, \mu)k(t_3)e^{-pt_3}e^{-2pt_2}$$

or

$$\Sigma(2) = t_0^2 \sum_0^\infty \delta x_2 \sum_0^\infty \delta x_3 m_n(x_1, \mu) \times m_n(x_2 + x_3, \mu)k(t_3)e^{-pt_3}e^{-2pt_2} - t_0^2 \sum_0^\infty \delta x_2 m_n^2(x_1, \mu)e^{-2pt_2}k(0). \quad \text{(A5)}$$

Note that the first term in (A5) coincides with (A1). The second term in (A5), as follows from (2), is equal to $-k(0)t_0^2/\{1 - \exp(-2pt_0)\}$. Therefore, (A5) converts into

$$\Sigma(2) = \{K(p) - t_0k(0)/2\}t_0/\{1 - \exp(-2pt_0)\}. \quad \text{(A6)}$$

Finally, the dispersion of a Meixner transform for a stationary stochastic process takes the form

$$\langle Y_n^2(p)\rangle = K(p)2t_0/\{1 - \exp(-2pt_0)\}, \quad \text{(A7)}$$

which fully corresponds to equation (7).

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