International Journal of Game Theory (1990) 19: 101–106

Finite Approximations to a Zero-Sum Game With Incomplete Information

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Abstract: In this paper, we investigate a scheme for approximating a two-person zero-sum game G of incomplete information by means of a natural system G_{mn} of its finite subgames. The main question is: For large m and n, is an optimal strategy for G_{mn} necessarily an ϵ-optimal strategy for G?

Introduction

To formalize our idea of approximating a two-person zero-sum game of incomplete information by its subgames, we introduce what we shall call a game structure. A game structure is a system of the form $(\Omega, U, \tilde{F}_m, \tilde{G}_n)_{m,n=1}^{\infty}$. Here $\Omega = (\Omega, \mathscr{B}, P)$ is a probability space, $U = (U_{ij} : i = 1,\ldots,M; j = 1,\ldots,N)$ is a matrix of random variables on Ω (the payoff matrix), and $\tilde{F}_m$ and $\tilde{G}_n$ are sub-σ-fields of the σ-field $\mathscr{B}$ such that $\tilde{F}_{m+1} \supseteq \tilde{F}_m$ and $\tilde{G}_{n+1} \supseteq \tilde{G}_n$. We put $\tilde{F} = \tilde{F}_\infty$ = the σ-field generated by $\bigcup_m \tilde{F}_m$, $\tilde{G} = \tilde{G}_\infty$ = the σ-field generated by $\bigcup_n \tilde{G}_n$.

For $m,n = 1,2,\ldots,\infty$, let G_{mn} be the two-person, zero-sum game in which a strategy for player I is an $\tilde{F}_m$-measurable $\alpha: \Omega \dashrightarrow S^M$, and a strategy for player II is a $\tilde{G}_n$-measurable $\beta: \Omega \dashrightarrow S^N$. (Here S^M is the simplex $\{ x \in R^M : \sum_i x_i = 1, x_i \geq 0 \}$.) If player I plays α and player II plays β, then the payoff to I is $\Gamma(\alpha,\beta) = E(\sum_{ij} U_{ij}\, \alpha_i \beta_j)$. Thus in the game G_{mn}, $\tilde{F}_m$ and $\tilde{G}_n$ embody the information available to I and II, respectively. If $\tilde{F}_m$ and $\tilde{G}_n$ are finite, then G_{mn} is a finite approximation to the game $G = G_{\infty\infty}$.

By standard minimax theorems, each game G_{mn} has saddle point. Let V_{mn} denote the value of the game G_{mn} to player I, and let $V = V_{\infty\infty}$.

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0020–7276/90/1/101–106 $2.50

If $\tilde{F}_m$ and $\tilde{G}_n$ are finite, then the game G_{mn} is, at least in principle, solvable by finite methods. The question we shall study is: To what extent is an optimal strategy for G_{mn} a useful substitute for an optimal strategy for G? An ideal result along these lines would be

(1) Fix $\epsilon > 0$. Suppose that, for $m,n = 1,2,...$, α^{mn} is an optimal strategy for I in G_{mn}. Then, for all sufficiently large m and n, α^{mn} is an ϵ-optimal strategy for I in G.

As we shall see, (1) is, alas, in general false. The best we can do is a weaker version of (1) (Theorem 1), and a special case of (1) (Theorem 2). We shall state these theorems presently. For a strategy α for player I in the game G, let $\mathrm{Val}_n(\alpha) = \inf_\beta \Gamma(\alpha,\beta)$, where β ranges over $\tilde{G}_n$-measurable strategies for II. (Thus if α is $\tilde{F}_m$-measurable, then $\mathrm{Val}_n(\alpha)$ is the value to I of the strategy α in the game G_{mn}.) We shall write $\mathrm{Val}_G(\alpha)$ for $\mathrm{Val}_\infty(\alpha)$.

Theorem 1: For $m,n = 1,2,...$, suppose that α^{mn} is an optimal strategy for player I in G_{mn}, and that $\tilde{F}_m$ and $\tilde{G}_n$ are finite σ-fields. Then $\lim_{m\to\infty} \lim_{n\to\infty} \mathrm{Val}_G(\alpha^{mn}) = V$. Moreover, this convergence is uniform in the choices α^{mn} of optimal strategies, i.e., $\lim_{m\to\infty} \lim_{n\to\infty} \inf_{\alpha\in A(m,n)} \mathrm{Val}_G(\alpha) = V$, where $A(m,n)$ is the set of strategies optimal for player I in G_{mn}.

Theorem 2 says that, under an additional hypothesis, (1) does hold. This hypothesis, which we shall call (M), is a version of the "continuity of information" assumption first used in [Milgrom-Weber]. (M) says roughly that the joint probability on $\tilde{F}$ and $\tilde{G}$ is absolutely continuous with respect to the product probability on $\tilde{F} \times \tilde{G}$. A precise statement of (M) will be found in Sec. 2.

Theorem 2: Assume (M) holds. If, for $m,n = 1,2,...$, α^{mn} is an optimal strategy for I in G_{mn}, then $\lim_{\substack{m\to\infty \\ n\to\infty}} \mathrm{Val}_G(\alpha^{mn}) = V$, uniformly in the choices α^{mn} of optimal strategies.

Results

We first present an example which shows that assertion (1) of the introduction does not hold in general.

Example: A game structure in which (1) fails.

Let Ω be the interval [0,1] with Lebesgue measure, $M=N=2$, and the payoff $U_{ij} = \begin{cases} 1 \text{ if } i = j \\ -1 \text{ if } i \neq j \end{cases}$ for $i,j = 1,2$ (indepenent of ω). For $m = 1,2,..$, let $\tilde{F}_m = \tilde{G}_m$ = the σ-field on [0,1) generated by the partition $\{[(k\text{-}1)/2^m, k/2^m) : k = 1,2,..,2^m\}$. Thus $\tilde{F} = \tilde{G}$ = the Borel σ-field on Ω. It is easy to see that, for all m and n, $V_{mn} = 0$, and in G_{mn} the players have the optimal strategies $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1/2$, for all $\omega \in \Omega$.

For finite $m > 1$, consider the game $G_{m,m\text{-}1}$. The strategy $\underline{\alpha}^{m,m\text{-}1}$ given by

$$\underline{\alpha}_1^{m,n\text{-}1} = \begin{cases} 1 \text{ if } \omega \in [(k\text{-}1)/2^m, k/2^m) \\ 0 \text{ otherwise} \end{cases}, \; k \text{ odd}$$

$$\underline{\alpha}_2^{m,m\text{-}1} = 1 - \underline{\alpha}_1^{m,m\text{-}1}$$

is easily seen to be optimal for player I in the game $G_{m,m\text{-}1}$, i.e., $\text{Val}_{m\text{-}1}(\underline{\alpha}^{m,m\text{-}1}) = 0$. On the other hand, $\text{Val}_G(\underline{\alpha}^{m,m\text{-}1}) = -1$; $\underline{\alpha}^{m,m\text{-}1}$ is a very poor strategy for player I in G. Thus in any system $(\alpha^{mn} : m,n = 1,2,...,)$ of optimal strategies for player I in which $\alpha^{m,m\text{-}1} = \underline{\alpha}^{m,m\text{-}1}$ for all $m > 1$, $\lim_{\substack{m \to \infty \\ n \to \infty}} \text{Val}_G(\alpha^{mn}) < V$.

We shall next prove Theorems 1 and 2. We first require a series of lemmas. Our first lemma is a special case of Theorem 2 of [Blackwell-Dubins].

Lemma 2.1: Let (X_k) be a uniformly bounded sequence of random variables, and suppose that $X_k \to X_\infty$ a.s. as $k \to \infty$. Then $E(X_k \mid \tilde{G}_k) \to E(X_\infty \mid \tilde{G})$ a.s. as $k \to \infty$.

Our second lemma computes $\text{Val}_n(\alpha)$.

Lemma 2.2: Fix a strategy α for I in G, and fix $n \in \{1,2,...,\infty\}$. Define the random variable ξ by : ξ = the $j \in \{1,2,...,N\}$ which minimizes $E(\sum_i U_{ij}\,\alpha_i \mid \tilde{G}_n)$. In case of a tie, for definiteness, take the least such j. Then, for all strategies β for II in $G_{\infty n}$,

(i) $E(\sum_i U_{i\xi}\,\alpha_i) \leq \Gamma(\alpha,\beta)$, so

(ii) $\text{Val}_n(\alpha) = E(\min_j E(\sum_i U_{ij}\,\alpha_i \mid \tilde{G}_n))$.

Proof: (i) immediately implies (ii), so we prove (i). Let β be a strategy for II in $G_{\infty n}$, that is, a $\tilde{G}_n$-measurable $\beta : \Omega \to S^N$. Then, since $\sum_j \beta_j = 1$, we have

$$E(\sum_i U_{i\xi}\,\alpha_i \mid \tilde{G}_n) \leq \sum_j \beta_j E(\sum_i U_{ij}\,\alpha_i \mid \tilde{G}_n) = E(\sum_{ij} U_{ij}\,\alpha_i\,\beta_j \mid \tilde{G}_n) \text{ a.s.}$$

since β is $\tilde{G}_n$-measurable. Taking expected values on both sides yields (i).

Lemma 2.3: Let (α^k) be a sequence of strategies for I in G, and suppose that $\alpha^k \to \alpha$ a.s. as $k \to \infty$. Then

(i) for fixed $n = 1,2,\ldots,\infty$, $\mathrm{Val}_n(\alpha^k) \to \mathrm{Val}_n(\alpha)$ as $k \to \infty$, and
(ii) $\mathrm{Val}_k(\alpha^k) \to \mathrm{Val}_G(\alpha)$ as $k \to \infty$.

Proof: First note that applying (ii) in a system where $\tilde{G}_n = \tilde{G}_{n+1} = \ldots = \tilde{G}_\infty$ yields (i), so (i) is a special case of (ii). To prove (ii), let $X_k = \sum_i U_{ij}\,\alpha_i^k$ in lemma 2.1; then we have $E(\sum_i U_{ij}\,\alpha_i^k \mid \tilde{G}_k) \to E(\sum_i U_{ij}\,\alpha_i \mid \tilde{G})$ a.s. as $k \to \infty$, so by dominated convergence $E(\min_j E(\sum_i U_{ij}\,\alpha_i^k \mid \tilde{G}_k)) \to E(\min_j E(\sum_i U_{ij}\,\alpha_i \mid \tilde{G}))$. By lemma 2.2(ii), we are done.

Lemma 2.4: (i) For $m,n = 1,2,\ldots,\infty$, $\lim_{m\to\infty} V_{mn} = V_{\infty n}$ and $\lim_{n\to\infty} V_{mn} = V_{m\infty}$.
(ii) $\lim_{m\to\infty,\, n\to\infty} V_{mn} = V$.

Proof: Fix n, and let α be an optimal strategy for I in $G_{\infty n}$. Now for $m = 1,2,\ldots$, put $\alpha^m = E(\alpha \mid \tilde{F}_m)$. Thus α is a legal, though likely not optimal, strategy for I in G_{mn}. We have

$$\mathrm{Val}_n(\alpha^m) \le V_{mn} \le V_{\infty n}.$$

By lemma 2.1, $\alpha^m \to \alpha$ a.s. By lemma 2.3(i), $\lim_{m\to\infty} \mathrm{Val}_n(\alpha^m) = \mathrm{Val}_n(\alpha) = V_{\infty n}$. By the inequality directly above, we infer $\lim_{m\to\infty} V_{mn} = V_{\infty n}$. By symmetry, we also have $\lim_{n\to\infty} V_{mn} = V_{m\infty}$. for all m. This proves (i). Finally, it is easy to see that $V_{m\infty} \le V_{mn} \le V_{\infty n}$. Claim (ii) now follows by letting $m, n \to \infty$.

We are now ready to prove Theorem 1.

Proof of Theorem 1: Suppose that α^{mn} is an optimal strategy for player I in G_{mn}, for $m,n = 1,2,\ldots$. We claim that, for $m = 1,2,\ldots$, $\lim_{n\to\infty} \mathrm{Val}_G(\alpha^{mn}) = V_{m\infty}$. To prove this, fix m. We shall show that every subsequence of the sequence $\mathrm{Val}_G(\alpha^m)$ has in turn a subsequence which converges to $V_{m\infty}$. Indeed, since each α^{mn} is $\tilde{F}_m$-measurable, $\tilde{F}_m$ being a finite σ-field, by the Bolzano-Weierstrass theorem every subsequence of α^m has a (pointwise) convergent subsequence; thus we may assume that $\alpha^{mn} \to \alpha^m$ as $n \to \infty$. By lemma 2.3(ii), then, $\mathrm{Val}_n(\alpha^{mn}) \to \mathrm{Val}_G(\alpha^m)$ as $n \to \infty$. By hypothesis, $\mathrm{Val}_n(\alpha^{mn}) = V_{mn}$, so in fact $V_{mn} \to \mathrm{Val}_G(\alpha^m)$ as $n \to \infty$. Thus by lemma 2.4(i), $\mathrm{Val}_G(\alpha^m) = V_{m\infty}$. On the other hand, since $\alpha^{mn} \to \alpha^m$ as $n \to \infty$, by 2.3(i) we also have $\mathrm{Val}_G(\alpha^{mn}) \to \mathrm{Val}_G(\alpha^m) = V_{m\infty}$. This proves our claim.

Now by another use of lemma 2.4(i), $\lim_{m \to \infty} \lim_{n \to \infty} \mathrm{Val}_G(\alpha^{mn}) = V$. To prove the "moreover" clause in Theorem 1, let (ϵ_{mn}) be a sequence of numbers which converges to 0 as $m,n \to \infty$. For all m,n, note that there exists $\alpha^{mn} \in A(m,n)$ such that $\mathrm{Val}_G(\alpha^{mn}) - \epsilon_{mn} \leq \inf_{\alpha \in A(m,n)} \mathrm{Val}_G(\alpha) \leq \mathrm{Val}_G(\alpha^{mn})$. The "morevoer" clause follows at once.

We now consider Theorem 2. We must first discuss hypothesis (M).

$\tilde{F} \times \tilde{G}$ is the σ-field on $\Omega \times \Omega$ generated by sets of the form $S \times T$, where $S \in \tilde{F}$ and $T \in \tilde{G}$. Let Q and R be the probability measures on $(\Omega \times \Omega, \tilde{F} \times \tilde{G})$ defined by

$$Q(A) = P(\{\omega : (\omega,\omega) \in A\}) \quad \text{and}$$

$$R(A) = \iint_{\{(\omega,\eta) \in A\}} 1 \, P(d\omega)\, P(d\eta) \qquad \text{for } A \in \tilde{F} \times \tilde{G}.$$

We now state assumption (M).

(M) Q is absolutely continuous with respect to R, that is, for all $A \in \tilde{F} \times \tilde{G}$, if $R(A)=0$, then $Q(A)=0$.

Assumption (M) is a version of a hypothesis introduced in [Milgrom-Weber]. It is easy to see that (M) is satisfied either if $\tilde{F}$ and $\tilde{G}$ are indepedent (in which case $Q=R$), or if either $\tilde{F}$ or $\tilde{G}$ is atomic.

Lemma 2.5: Suppose (M) is satisfied. Then if (X_k) is a uniformly bounded sequence of $\tilde{F}$-measurable random variables which converges weakly to X_∞, and if Z is any bounded random variable, then

i) $E(X_k Z \mid \tilde{G}) \to E(X_\infty Z \mid \tilde{G})$ a.s. and

ii) $E(X_k Z \mid \tilde{G}_k) \to E(X_\infty Z \mid \tilde{G})$ a.s as $k \to \infty$.

Proof: First note that, since $E(E(X_k Z \mid \tilde{G}) \mid \tilde{G}_k) = E(X_k Z \mid \tilde{G}_k)$, by lemma 2.1,(i) implies (ii). Next, note that we may assume without loss of generality that Z is measurable in $\tilde{F} \vee \tilde{G}$ (the σ-field generated by $\tilde{F} \cup \tilde{G}$). This is because $E(X_k Z \mid \tilde{G}) = E(X_k \cdot E(Z \mid \tilde{F} \vee \tilde{G}) \mid \tilde{G})$, so we may replace Z by $E(Z \mid \tilde{F} \vee \tilde{G})$ if necessary. We shall therefore prove (i), assuming that Z is $\tilde{F} \vee \tilde{G}$-measurable.

Since Z is $\tilde{F} \vee \tilde{G}$-measurable, there exist a bounded $\tilde{F}$-measurable random variable $\hat{X}$, a bounded $\tilde{G}$-measurable random variable $\hat{Y}$, and a bounded, Borel measurable function $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ such that $Z = f(\hat{X},\hat{Y})$.

By (M) and the Radon-Nikodym theorem, there exists a bounded function $g: \Omega \times \Omega \to \mathbf{R}$ such that, for all $A \in \tilde{F} \times \tilde{G}$, $P(\{\omega : (\omega,\omega) \in A\}) = \iint_A g(\omega,\eta)P(d\omega)P(d\eta)$. It follows by standard methods that, for any vector $\mathbf{X}$ of $\tilde{F}$-measurable random variables, any vector $\mathbf{Y}$ of $\tilde{G}$-measurable random variables, and any Borel-measurable $h : \mathbf{R}^p \to \mathbf{R}$, we have

$$(*)\quad E(h(\mathbf{X},\mathbf{Y}) \mid \tilde{G})\,(\eta) = \int_\Omega h(\mathbf{X}(\omega), \mathbf{Y}(\eta))\, g(\omega,\eta)\, P(d\omega) \quad \text{a.s. } [\eta].$$

Now by (*) we have

$$E(X_k\, Z \mid \tilde{G})\,(\eta) = \int_\Omega X_k(\omega)\, f(\hat{X}(\omega), \hat{Y}(\eta))\, g(\psi,\eta)\, P(d\omega) \text{ a.s. } [\eta].$$

Since, by assumption, $X_k \dashrightarrow X_\infty$ weakly, we have $E(X_k\, Z \mid \tilde{G})\,(\eta) \dashrightarrow E(X_\infty Z \mid \tilde{G})\,(\eta)$ a.s., as desired.

In exact analogy to lemma 2.3, we have

Lemma 2.6: Assume that (*M*) holds. If (α^k) is a sequence of strategies for I in G which converges weakly to a strategy α, then

(i) for fixed $n = 1,2,\ldots,\infty$, $\mathrm{Val}_n(\alpha^k) \dashrightarrow \mathrm{Val}_n(\alpha)$, and

(ii) $\mathrm{Val}_k(\alpha^k) \dashrightarrow \mathrm{Val}_G(\alpha)$ as $k \dashrightarrow \infty$.

Proof of Theorem 2: Suppose that α^{mn} is an optimal strategy for player I in G_{mn}, for all finite m and n. We shall prove that every sequence (m_k,n_k) of pairs of integers such that $m_k \dashrightarrow \infty$ and $n_k \dashrightarrow \infty$ has a subsequence (m'_k,n'_k) such that $\mathrm{Val}_G(\alpha^{m'_k,n'_k}) \dashrightarrow V$ as $k \dashrightarrow \infty$. To conserve notation, let us write α^k for $\alpha^{m'_k,n'_k}$ and V_k for $V_{m'_k,n'_k}$. By weak compactness, we may choose the sequence (α^k) to converge weakly to a strategy α as $k \dashrightarrow \infty$. By lemma 2.6(ii), $\mathrm{Val}_{n'_k}(\alpha^k) \dashrightarrow \mathrm{Val}_G(\alpha)$ as $k \dashrightarrow \infty$. By assumption, $\mathrm{Val}_{n'_k}(\alpha^k) = V_k$, and by lemma 2.4(ii), $V_k \dashrightarrow V$; thus $\mathrm{Val}_G(\alpha) = V$.

On the other hand, since (α^k) converges weakly to α, by lemma 2.6(i), $\mathrm{Val}_G(\alpha^k) \dashrightarrow \mathrm{Val}_G(\alpha) = V$ as $k \dashrightarrow \infty$. Uniformity follows just as in the proof of Theroem 1. This completes the proof of Theorem 2.

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Received August 1985
Revised version June 1988
Final version November 1989