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The translation theorem*

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Summary. We state and prove the Translation Theorem. Then we apply the Translation Theorem to Soare's Extension Theorem, weakening slightly the hypothesis to yield a theorem we call the Modified Extension Theorem. We use this theorem to reprove several of the known results about orbits in the lattice of recursively enumerable sets. It is hoped that these proofs are easier to understand than the old proofs.

0 Introduction

In this paper, we will reconsider some of the known results about orbits in the lattice of recursively enumerable sets. For example, Soare showed that the maximal sets [7] form an orbit in this lattice. The proof of this and other results about orbits are similar in that they all construct an uniformly recursive enumeration satisfying several complex automorphism conditions and then apply Soare's Extension Theorem to these enumerations to yield the desired automorphism. Here we will focus on the recursiveness of these enumerations rather than the complex conditions they must satisfy or the Extension Theorem itself.

We show that it is enough to construct uniformly $\mathbf{0}''$-recursive enumerations satisfying these complex conditions rather than uniformly recursive enumerations. $\{X_{n,s}\}_{n,s<\omega}$ is an *uniformly* $\mathbf{0}''$*-recursive enumeration* if there is a function h such that $h \leq_T \mathbf{0}''$ and for all n, $X_{n,s} = W_{h(n),s}$. We do this by applying a new theorem, the Translation Theorem, to translate these uniformly $\mathbf{0}''$-recursive enumerations into uniformly recursive enumerations.

In Sect. 1, we state the Translation Theorem; the proof can be found in Sect. 2. In Sect. 1, we use the Translation Theorem to prove a slightly different version

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of the Extension Theorem, the Modified Extension Theorem. The hypothesis of the Modified Extension Theorem is weaker than the Extension Theorem in that uniformly $\mathbf{0}''$-recursive enumerations can be used rather than just uniformly recursive enumerations. However, the conclusion is weaker in the sense that it is not possible to use the Modified Extension Theorem to construct effective automorphisms. We do not consider this much of a weakness, since many of the applications of the Extension Theorem do not construct effective automorphisms and to prove the Extension Theorem using the new Δ_3-automorphism techniques we also must remove this possibility.

In the two remaining sections, we prove several results about orbits using uniformly $\mathbf{0}''$-recursive enumerations rather than uniformly recursive enumerations. In Sect. 3, we reprove Soare's result [7] that the maximal sets form an orbit. In Sect. 4, we reprove Maass's result [6] on the orbits of hyperhypersimple sets. We hope that these new proofs are easier to understand. We assume that the reader is familiar with the construction of automorphisms of the lattice of recursively enumerable sets and the use of the Extension Theorem. For the unfamiliar reader, we suggest [8, XV.4].

1 The statement of the translation theorem and the modified extension theorem

Before we can state the Translation Theorem and the Modified Extension Theorem, we need the following definitions. Only the first three definitions and Definition 1.6 are non-standard in the sense that they either do not appear in [8] or they are slightly different from the similar definition in [8]; otherwise our notation is standard.

Definition 1.1. $\{X_n\}_{n<\omega}$ is an *uniformly recursive collection* of r.e. sets if there is a recursive function h and for all n, $X_n = W_{h(n)}$. $\{X_n\}_{n<\omega}$ is an *uniformly $\mathbf{0}''$-recursive collection* of r.e. sets if there is a function h such that $h \leq_T \mathbf{0}''$ and for all n, $X_n = W_{h(n)}$. $\{X_{n,s}\}_{n,s<\omega}$, is an *uniformly $\mathbf{0}''$-recursive enumeration* if there is a function h such that $h \leq_T \mathbf{0}''$ and for all n, $X_{n,s} = W_{h(n),s}$.

Definition 1.2. For any e, if we are given uniformly recursive enumerations $\{X_{n,s}\}_{n\leq e,s<\omega}$ and $\{Y_{n,s}\}_{n\leq e,s<\omega}$ of r.e. sets $\{X_n\}_{n\leq e}$ and $\{Y_n\}_{n\leq e}$, define the *full e-state of x at stage s*, $\nu(e,x,s)$ with respect to (w.r.t.) $\{X_{n,s}\}_{n,s<\omega}$ and $\{Y_{n,s}\}_{n,s<\omega}$ to be the triple

$$\nu(e,x,s) = \langle e, \sigma(e,x,s), \tau(e,x,s)\rangle$$

where

$$\sigma(e,x,s) = \{i : i \leq e \wedge x \in X_{i,s}\}$$

and

$$\tau(e,x,s) = \{i : i \leq e \wedge x \in Y_{i,s}\}\,.$$

Definition 1.3. Given any collection of r.e. sets $\{X_n\}_{n<\omega}$ and $\{Y_n\}_{n<\omega}$, define the *final e-state of x*, $\nu(e,x)$ with respect to $\{X_n\}_{n<\omega}$ and $\{Y_n\}_{n<\omega}$ to be the triple

$$\nu(e,x) = \langle e, \sigma(e,x), \tau(e,x)\rangle$$

where

$$\sigma(e,x) = \{i : i \leq e \wedge x \in X_i\}$$

and

$$\tau(e,x) = \{i : i \leq e \wedge x \in Y_i\}\,.$$

Definition 1.4. Given recursive enumerations $\{X_s\}_{s<\omega}$ and $\{Y_s\}_{s<\omega}$ of X and Y, we define

(i) $X\backslash X = \{z : (\exists s)(z \in X_s - Y_s)\}$,

(ii) $X \searrow Y = (X\backslash Y) \cap Y$.

Definition 1.5. Given states $\nu = \langle e, \sigma, \tau\rangle$ and $\nu' = \langle e', \sigma', \tau'\rangle$, we define

(i) ν is an *initial segment* of ν' ($\nu \preceq \nu'$) iff $e \leq e'$, $\sigma = \sigma' \cap \{0, 1, \ldots, e\}$, and $\tau = \tau' \cap \{0, 1, \ldots, e\}$.

(ii) The *length of* ν, $|\nu|$, is e.

(iii) $\nu = \nu' \upharpoonright e$ iff $\nu \preceq \nu'$ and $|\nu| = e$.

(iv) ν *covers* ν' ($\nu \geq \nu'$) iff $e = e'$, $\sigma \supseteq \sigma'$ and $\tau \subseteq \tau'$.

Definition 1.6. Assume $\{T_s\}_{s<\omega}$ is a uniformly recursive enumeration of T, an infinite r.e. set. For any e, if we are given uniformly recursive enumerations $\{X_{n,s}\}_{n\leq e,s<\omega}$ and $\{Y_{n,s}\}_{n\leq e,s<\omega}$ of r.e. sets $\{X_n\}_{n\leq e}$ and $\{Y_n\}_{n\leq e}$. For each full e-state ν, define the r.e. set

$$D^T_\nu = \{x : \exists t \text{ such that } x \in T_t - T_{t-1} \wedge \nu = \nu(e, x, t) \text{ w.r.t. } \{X_{n,s}\}_{n\leq e,s<\omega} \text{ and } \{Y_{n,s}\}_{n\leq e,s<\omega}\}$$

If $x \in D^T_\nu$, we say that ν is the *entry e-state* of x w.r.t. $\{X_{n,s}\}_{n\leq e,s<\omega}$ and $\{Y_{n,s}\}_{n\leq e,s<\omega}$ into T. We say that D^T_ν is measured w.r.t. $\{X_{n,s}\}_{n\leq e,s<\omega}$ and $\{Y_{n,s}\}_{n\leq e,s<\omega}$.

Now we have all the definitions needed to state the Translation Theorem and Modified Extension Theorem. First an quick word about some of our notation. There are two kinds of hats: angled hats (^) and curved hats (⌒). The curves hats appear in the Translation Theorem while the angled hats in the Modified Extension Theorem. This notation seems natural since the sets $\widehat{X}$ and $\hat{X}$ play similar roles in the corresponding theorems.

The Translation Theorem. *Assume* $\{T^\dagger_s\}_{s<\omega}$, $\{\widehat{T}^\dagger_s\}_{s<\omega}$, $\{U^\dagger_{n,s}\}_{n,s<\omega}$, $\{\widehat{V}^\dagger_{n,s}\}_{n,s<\omega}$, $\{\widehat{U}^\dagger_{n,s}\}_{n,s<\omega}$, *and* $\{V^\dagger_{n,s}\}_{n,s<\omega}$ *are uniformly* $\mathbf{0}''$*-recursive enumerations of the infinite r.e. sets* $T^\dagger$ *and* $\widehat{T}^\dagger$, *and the uniformly* $\mathbf{0}''$*-recursive collection of r.e. sets* $\{U^\dagger_n\}_{n<\omega}$, $\{\widehat{V}^\dagger_n\}_{n<\omega}$, $\{\widehat{U}^\dagger_n\}_{n<\omega}$, *and* $\{V^\dagger_n\}_{n<\omega}$ *satisfying the following Conditions:*

$$\forall n[\widehat{T}^\dagger \searrow \widehat{U}^\dagger_n = T^\dagger \searrow \widehat{V}^\dagger_n = \emptyset]\,. \tag{1.7}$$

$$(\forall\nu)[D^{\widehat{T}^\dagger}_\nu \textit{ is infinite} \Rightarrow (\exists\nu' \geq \nu)[D^{T^\dagger}_{\nu'} \textit{ is infinite}]], \textit{ and} \tag{1.8}$$

$$(\forall\nu)[D^{T^\dagger}_\nu \textit{ is infinite} \Rightarrow (\exists\nu' \leq \nu)[D^{\widehat{T}^\dagger}_{\nu'} \textit{ is infinite}]]\,. \tag{1.9}$$

where for all e-states ν, $D^{T^\dagger}_\nu$ *is measured w.r.t.* $\{U^\dagger_{n,s}\}_{n\leq e,s<\omega}$ *and* $\{\widehat{V}^\dagger_{n,s}\}_{n\leq e,s<\omega}$ *and* $D^{\widehat{T}^\dagger}_\nu$ *is measured w.r.t.* $\{\widehat{U}^\dagger_{n,s}\}_{n\leq e,s<\omega}$ *and* $\{V^\dagger_{n,s}\}_{n\leq e,s<\omega}$. *Then there is a collection of uniformly r.e. sets* $\{U_n\}_{n<\omega}$, $\{\widehat{V}_n\}_{n<\omega}$, $\{\widehat{U}_n\}_{n<\omega}$, *and* $\{V_n\}_{n<\omega}$, *and uniformly recursive enumerations,* $\{T_s\}_{s<\omega}$, $\{\widehat{T}_s\}_{s<\omega}$, $\{U_{n,s}\}_{n,s<\omega}$, $\{\widehat{V}_{n,s}\}_{n,s<\omega}$, $\{\widehat{U}_{n,s}\}_{n,s<\omega}$, *and $\{V_{n,s}\}_{n,s<\omega}$, of these sets satisfying the following Conditions:*

$$T_{s+1} = T_s^\dagger \text{ and } \hat{T}_{s+1} = \hat{T}_s^\dagger , \tag{1.10}$$

$$\forall n[\hat{T} \searrow \hat{U}_n = T \searrow \hat{V}_n = \emptyset] , \tag{1.11}$$

$$\begin{gathered} \text{for all } n \text{ there is an } e_n \text{ such that } U_n^\dagger =^* U_{e_n} , \\ \hat{V}_n^\dagger \cup \bar{T} =^* \hat{V}_{e_n} \cap \bar{T},\ V_n^\dagger =^* V_{e_n},\ \text{and } \hat{U}_n^\dagger \cap \bar{\hat{T}} =^* \hat{U}_{e_n} \cap \bar{\hat{T}} , \end{gathered} \tag{1.12}$$

$$\begin{gathered} \text{for all } e, \text{ either } U_e \backslash T =^* \hat{V}_e \backslash T =^* V_e \backslash \hat{T} =^* \hat{U}_e \backslash \hat{T} =^* \emptyset \\ (\text{hence by Condition } 1.11,\ \hat{V}_e =^* \hat{U}_e =^* \emptyset), \text{ or} \\ \text{there is an } n \text{ such that } U_n^\dagger =^* U_e,\ \hat{V}_n^\dagger \cap \bar{T} =^* \hat{V}_e \cap \bar{T} , \\ V_n^\dagger =^* V_e,\ \text{and } \hat{U}_n^\dagger \cap \bar{\hat{T}} =^* \hat{U}_e \cap \bar{\hat{T}} , \end{gathered} \tag{1.13}$$

$$(\forall \nu)[D_\nu^{\hat{T}} \text{ is infinite} \Rightarrow (\exists \nu' \geq \nu)[D_{\nu'}^T \text{ is infinite}]], \text{ and} \tag{1.14}$$

$$(\forall \nu)[D_\nu^T \text{ is infinite} \Rightarrow (\exists \nu' \leq \nu)[D_{\nu'}^{\hat{T}} \text{ is infinite}]] \tag{1.15}$$

where for all e-states ν, D_ν^T is measured w.r.t. $\{U_{n,s}\}_{n\leq e,s<\omega}$ and $\{\hat{V}_{n,s}\}_{n\leq e,s<\omega}$ and $D_\nu^{\hat{T}}$ is measured w.r.t. $\{\hat{U}_{n,s}\}_{n\leq e,s<\omega}$ and $\{V_{n,s}\}_{n\leq e,s<\omega}$.

The Modified Extension Theorem. *Assume $\{T_s\}_{s<\omega}$, $\{\hat{T}_s\}_{s<\omega}$, $\{U_{n,s}\}_{n,s<\omega}$, $\{\hat{V}_{n,s}\}_{n,s<\omega}$, $\{\hat{U}_{n,s}\}_{n,s<\omega}$, and $\{V_{n,s}\}_{n,s<\omega}$ are uniformly $\mathbf{0}''$-recursive enumerations of the infinite r.e. sets T and $\hat{T}$ and the uniformly $\mathbf{0}''$-recursive collection of r.e. sets $\{U_n\}_{n<\omega}$, $\{\hat{V}_n\}_{n<\omega}$, $\{\hat{U}_n\}_{n<\omega}$, and $\{V_n\}_{n<\omega}$ satisfying the following Conditions:*

$$\forall n[T \searrow \hat{U}_n = \hat{T} \searrow \hat{V}_n = \emptyset] , \tag{1.16}$$

$$(\forall \nu)[D_\nu^{\hat{T}} \text{ is infinite} \Rightarrow (\exists \nu' \geq \nu)[D_{\nu'}^T \text{ is infinite}]], \text{ and} \tag{1.17}$$

$$(\forall \nu)[D_\nu^T \text{ is infinite} \Rightarrow (\exists \nu' \leq \nu)[D_{\nu'}^{\hat{T}} \text{ is infinite}]] , \tag{1.18}$$

where for all e-states ν, D_ν^T is measured w.r.t. $\{U_{n,s}\}_{n\leq e,s<\omega}$ and $\{\hat{V}_{n,s}\}_{n\leq e,s<\omega}$ and $D_\nu^{\hat{T}}$ is measured w.r.t. $\{\hat{U}_{n,s}\}_{n\leq e,s<\omega}$ and $\{V_{n,s}\}_{n\leq e,s<\omega}$. Then there is an uniformly $\mathbf{0}''$-recursive collection of r.e. sets $\{\hat{U}_n\}_{n\in\omega}$ and $\{\hat{V}_n\}_{n\in\omega}$ such that

$$\hat{U}_n \cap \bar{\hat{T}} =^* \hat{U}_n \cap \bar{\hat{T}},\ \ \hat{V}_n \cap \bar{T} =^* \hat{V}_n \cap \bar{T}, \text{ and} \tag{1.19}$$

$$\begin{gathered} \exists^\infty x \in T \text{ with final e-state } \nu \text{ w.r.t. to } \{U_n\}_{n<\omega} \text{ and } \{\hat{V}_n\}_{n<\omega} \\ \text{iff} \\ \exists^\infty \hat{x} \in \hat{T} \text{ with final e-state } \nu \text{ w.r.t. to } \{\hat{U}_n\}_{n<\omega} \text{ and } \{V_n\}_{n<\omega} . \end{gathered} \tag{1.20}$$

The statement of Soare's Extension Theorem is the same as the statement of the Modified Extension Theorem except the first two occurrences of "uniformly $\mathbf{0}''$-recursive" are replaced with "uniformly recursive". When one uses the proof in [7] or [8], one can add to the statement of the Extension Theorem that $\{\hat{U}_n\}_{n\in\omega}$ and $\{\hat{V}_n\}_{n\in\omega}$ are uniformly recursive collections of r.e. sets and hence the automorphism constructed is effective. However, this cannot be added, if one wishes to use the "tree of strategies" proof (see [1] or [2]). Since the array of sets constructed in the Modified Extension Theorem is an uniformly $\mathbf{0}''$-recursive collection of r.e. sets, the automorphism produced is an Δ_3-automorphism.

The Modified Extension Theorem follows fairly easily from the Translation Theorem and the Extension Theorem. Assume $\{T^\dagger_s\}_{s<\omega}$, $\{\hat{T}^\dagger_s\}_{s<\omega}$, $\{U^\dagger_{n,s}\}_{n,s<\omega}$, $\{\hat{V}^\dagger_{n,s}\}_{n,s<\omega}$, $\{\hat{U}^\dagger_{n,s}\}_{n,s<\omega}$, and $\{V^\dagger_{n,s}\}_{n,s<\omega}$ are uniformly $\mathbf{0}''$-recursive enumerations of the infinite r.e. sets $T^\dagger$ and $\hat{T}^\dagger$, and the uniformly $\mathbf{0}''$-recursive collection of r.e. sets $\{U^\dagger_n\}_{n<\omega}$, $\{\hat{V}^\dagger_n\}_{n<\omega}$, $\{\hat{U}^\dagger_n\}_{n<\omega}$, and $\{V^\dagger_n\}_{n<\omega}$ satisfying the hypothesis of the Modified Extension Theorem and hence the Translation Theorem. Apply the Translation Theorem to get uniformly r.e. sets $\{U_n\}_{n<\omega}$, $\{\hat{V}_n\}_{n<\omega}$, $\{\hat{U}_n\}_{n<\omega}$, and $\{V_n\}_{n<\omega}$, and uniformly recursive enumerations, $\{T_s\}_{s<\omega}$, $\{\hat{T}_s\}_{s<\omega}$, $\{U_{n,s}\}_{n,s<\omega}$, $\{\hat{V}_{n,s}\}_{n,s<\omega}$, $\{\hat{U}_{n,s}\}_{n,s<\omega}$, and $\{V_{n,s}\}_{n,s<\omega}$. Apply the Extension Theorem to the uniformly recursive enumeration of this uniformly recursive collection of r.e. sets to get the r.e. sets $\{\hat{U}_n\}_{n\in\omega}$ and $\{\hat{V}_n\}_{n\in\omega}$. For the conclusion of the Modified Extension Theorem restrict the above collection to $\{\hat{U}_{e_n}\}_{n\in\omega}$ and $\{\hat{V}_{e_n}\}_{n\in\omega}$. Using the Translation Theorem it is easy to see that this restricted collection satisfies the conclusion of the Modified Extension Theorem.

2 The proof of the translation theorem

This proof is very similar to the proofs of the extension theorems that can be found in [1] or [2]. We will build a "Δ_3-branching" tree Tr and construct the desired sets by using this tree. We will define, by induction, $Tr \subseteq \omega^{<\omega}$. The construction of the desired sets will be viewed as two giant pinball machines, M and $\hat{M}$, laid out on top of the tree, Tr. Unless noted, everything for the (angled) hatted side is the dual. Let Greek letters α, β, δ, γ, and δ range over $\omega^{<\omega}$. Let $\alpha^- \subset \alpha$ ($\neq \lambda$, the empty node) be such that for all $\beta \subset \alpha$, $\beta \subseteq \alpha^-$ and $\alpha \cap \beta$ be the least such that for all γ, if $\gamma \subseteq \beta$ and $\gamma \subseteq \alpha$, then $\gamma \subseteq \alpha \cap \beta$.

As we define Tr, we also define a mechanism for determining f_s, the approximation to the true path f (defined formally below) at stage s. (As usual , we will ensure that $f = \liminf_s f_s$.) Briefly, we will use the tree to provide us with indices for the sets $\{U^\dagger_n\}_{n<\omega}$, $\{\hat{V}^\dagger_n\}_{n<\omega}$, $\{\hat{U}^\dagger_n\}_{n<\omega}$, and $\{V^\dagger_n\}_{n<\omega}$ and all the entry states.

Each node $\alpha \in Tr$ will be given four r.e. sets $U^\dagger_\alpha$, $V^\dagger_\alpha$, $\hat{U}^\dagger_\alpha$ and $\hat{V}^\dagger_\alpha$ (α will be given the indices for these sets, more below). If $\alpha \subset f$ then we will ensure that $U^\dagger_\alpha = U^\dagger_{|\alpha|}$, $\hat{V}^\dagger_\alpha = \hat{V}^\dagger_{|\alpha|}$, $\hat{U}^\dagger_\alpha = \hat{U}^\dagger_{|\alpha|}$, $V^\dagger_\alpha = V^\dagger_{|\alpha|}$. Each node α will build four r.e. sets U_α, V_α, $\hat{U}_\alpha$ and $\hat{V}_\alpha$. If $\alpha \subset f$ then we will ensure that $U_\alpha =^* U^\dagger_\alpha$, $\hat{V}_\alpha \cap \bar{T} =^* \hat{V}^\dagger_\alpha \cap \bar{T}$, $\hat{U}_\alpha \cap \bar{\hat{T}} =^* U^\dagger_\alpha \cap \bar{\hat{T}}$, $V_\alpha =^* V^\dagger_\alpha$. If $\alpha \not\subset f$ then we will ensure that $U_\alpha \backslash T =^* \hat{V}_\alpha \backslash T =^* V_\alpha \backslash \hat{T} =^* \hat{U}_\alpha \backslash \hat{T} =^* \emptyset$. This will allow us to meet Conditions 1.12 and 1.13. (We will assume all eight sets associated with λ are all empty.)

To get the desired uniformly recursive enumerations, we will take some recursive function g from ω into Tr such that g is one to one, onto, and if $g(e) = \alpha$ then for all $\beta \subset \alpha$, there is a $j < e$ such that $g(j) = \beta$ (the existence of such a g is guaranteed by the Recursion Theorem). The enumeration will be the following: $U_{e,s} = U_{g(e),s}$,

$\widehat{V}_{e,s} = \widehat{V}_{g(e),s}$, $\widehat{U}_{e,s} = \widehat{U}_{g(e),s}$, and $V_{e,s} = V_{g(e),s}$. Since the sets we construct off the true path are finite outside T and $\hat{T}$ this enumeration will meet Conditions 1.12 and 1.13 (more details later).

Since the tree and the pinball machines will be interwoven with each other, we need a little general information about how the pinball machines will look and act before we can define *Tr* and the pinball machines. The surfaces of the two pinball machines will be the same, but M will only use balls (integers) from ω and $\hat{M}$ will only use balls ($\hat{x}$) from $\hat{\omega}$ (almost everything on the hatted side will wear hats).

The surfaces of the machines will be broken up into similar units, the α-*unit* on M and the $\hat{\alpha}$-*unit* on $\hat{M}$, for all $\alpha \in Tr$. The α-*unit* has one gate, G_α. When a ball, x, first arrives at the α-unit it is placed above G_α. When x passes by G_α, we say x has been *processed* by G_α. G_α will either hold x forever, use f_s to determine which β-unit x will enter next, where β is one of α's immediate successors or if for some s, $f_s <_L \alpha$ (defined formally below), x will be permanently removed from the α-unit.

We will consider the sets $T^\dagger$, $\{U_\alpha^\dagger\}_{\alpha\in Tr}$, and $\{\widehat{V}_\alpha^\dagger\}_{\alpha\in Tr}$ as subsets of ω and the sets $T^\dagger$, $\{\widehat{U}_\alpha^\dagger\}_{\alpha\in Tr}$, $\{V_\alpha^\dagger\}_{\alpha\in Tr}$ as subsets of $\hat{\omega}$. We will build T, $\{U_\alpha\}_{\alpha\in Tr}$, and $\{\widehat{V}_\alpha\}_{\alpha\in Tr}$ as subsets of ω and $\hat{T}$, $\{\widehat{U}_\alpha\}_{\alpha\in Tr}$, and $\{V_\alpha\}_{\alpha\in Tr}$, as subsets of $\hat{\omega}$. If $x \in T_s^\dagger$, then at stage $s+1$ we will remove x from the surface of M and place $x \in T_{s+1}$ (hence we will meet Condition 1.10).

To define $Tr \subseteq \omega^{<\omega}$, M, and $\hat{M}$ we will proceed as follows: First $\lambda \in Tr$. (λ is the empty string.) Now, given $\alpha \in Tr$, we must construct all the immediate successors of α in *Tr*. As we proceed, we will also define a mechanism for determining f_s. First we need the following definitions:

Definition 2.1. A set of e-states $\mathfrak{E}$ is an α-*entry set* if $|\alpha| = e - 1$.

Definition 2.2. Let $\alpha \in Tr$ and $e = |\alpha| + 1$. The α-entry set $\mathfrak{E}$ is *valid for* α, e_1 *and* e_2 if for all e-states ν, $\nu \in \mathfrak{E}$ iff the set $D_\nu^{T^\dagger}$ is infinite, where $D_\nu^{T^\dagger}$ is measured with respect to the enumeration $\{X_{i,s}\}_{i\le e}$ and $\{Y_{i,s}\}_{i\le e}$, where $X_{i,s} = U^\dagger_{\beta,s}$, if $i < e$ and $\alpha \restriction i = \beta$, $X_{e,s} = W_{e_1,s}$, $Y_{i,s} = \widehat{V}^\dagger_{\beta,s}$, if $i < e$ and $\alpha \restriction \beta$, and $Y_{e,s} = W_{e_2}, s$.

There are only finitely many α-entry sets. $\hat{\mathfrak{E}}$ is an $\hat{\alpha}$-*entry set* and $\hat{\mathfrak{E}}$ is *valid for* α, e_3 *and* e_4 are defined in the same manner. $\mathfrak{E}$ will always denote an α-entry set and $\hat{\mathfrak{E}}$ an $\hat{\alpha}$-entry set.

Each node, $\beta \notin \lambda$, in *Tr* will be given, in addition to the indices for four r.e. sets, a β^--entry set and a $\hat{\beta}^-$-entry set, $\mathfrak{E}_\beta$ and $\hat{\mathfrak{E}}_\beta$. If $\beta = \lambda$, let $\mathfrak{E}_\lambda = \hat{\mathfrak{E}}_\lambda = \{\langle 0, \emptyset, \emptyset\rangle\}$.

If X is a set of states, let $X \restriction e = \{\nu \restriction e : \nu \in X\}$. Let $\{\mathfrak{E}_i\}$ be a recursive indexing of all entry sets. Let r and n be recursive functions such that for all $i \neq j \le n(\alpha)$, $\mathfrak{E}_{r(\alpha,i)} \neq \mathfrak{E}_{r(\alpha,j)}$ and $\{\mathfrak{E}_{r(\alpha,0)}, \mathfrak{E}_{r(\alpha,1)}, \ldots, \mathfrak{E}_{r(\alpha,n(\alpha))}\}$ is the set of all α-entry sets.

Defining α's immediate successors: Assume $\alpha \in Tr$ and that $\mathfrak{E}_\alpha$ and $\hat{\mathfrak{E}}_\alpha$ are defined. Let $\beta = \alpha\char`^\langle e_1, e_2, e_3, e_4, i, j, k\rangle$. If

$$i \le n(\alpha) \text{ and if } \nu \in \mathfrak{E}_{r(\alpha,i)}, \text{ then } \nu \restriction |\alpha| \in \mathfrak{E}_\alpha, \text{ and} \tag{2.3}$$

$$j \le \hat{n}(\alpha) \text{ and if } \hat{\nu} \in \hat{\mathfrak{E}}_{\hat{r}(\alpha,j)}, \text{ then } \hat{\nu} \restriction |\alpha| \in \hat{\mathfrak{E}}_\alpha\,, \tag{2.4}$$

then let $\beta \in Tr$, $\mathfrak{E}_\beta = \mathfrak{E}_{r(\alpha,i)}$, $\hat{\mathfrak{E}}_\beta = \hat{\mathfrak{E}}_{\hat{r}(\alpha,j)}$, $U^\dagger_{\beta,s} = W_{e_1,s}$, $\hat{V}^\dagger_{\beta,s} = W_{e_2,s}$, $\hat{U}^\dagger_{\beta,s} = W_{e_3,s}$, and $V^\dagger_{\beta,s} = W_{e_4,s}$. (We will later use k to help us find the approximation to the true path at stage s.)

Before we continue we need the following definition and lemma. Cof is the index set such that $x \in \mathrm{Cof}$ iff $\bar{W}_x$ is finite. It is well known that Cof is Σ_3-complete (see [8]) and in fact it is very easy to show the following lemma (the reader who wishes to see the missing proof is directed to [1]) or [1]:

Lemma 2.5. *If $A \in \Sigma_3$ then there is a recursive function g such that*

$$x \in A \Leftrightarrow \exists! k[W_{g(x,k)} = \omega];$$

and

$$x \notin A \Leftrightarrow \forall k[W_{g(x,k)} =^* \emptyset].$$

A mechanism for determining f_s. Given an $\alpha^\frown\langle e_1, e_2, e_3, e_4, i, j, k\rangle \in Tr$, let $n = |\alpha| + 1$. Determining whether for all s, $U^\dagger_{n,s} = W_{e_1,s}$ and for all $q < e_1$, there exists an s such that $U^\dagger_{n,s} \neq W_{q,s}$, is recursive in $\mathbf{0}''(\Delta_3)$. (First find an e such that for all s, $U^\dagger_{n,s} = W_{e,s}$, since $\{U^\dagger_{n,s}\}_{n,s<\omega}$ is a uniformly $\mathbf{0}''$-recursive enumeration such an e can be found using $\mathbf{0}''$. Then ask, using $\mathbf{0}''$, whether for all s, $W_{e,s} = W_{e_1,s}$ and for all $q < e_1$, there exists an s such that $W_{e,s} \neq W_{q,s}$). In fact, for all $\alpha^\frown\langle e_1, e_2, e_3, e_4, i, j, k\rangle \in Tr$, this can be done uniformly in $\mathbf{0}''$ for any e_i.

Since for any $\gamma = \alpha^\frown\langle e_1, e_2, e_3, e_4, i, j, k\rangle \in Tr$, for all $\beta \subseteq \gamma$ the enumerations of $U^\dagger_\beta$, $V^\dagger_\beta$, $\hat{U}^\dagger_\beta$ and $\hat{V}^\dagger_\beta$ are fixed, for any i, determining whether for all $\nu \in \mathfrak{E}_{r(\alpha,i)}$, $D^{T^\dagger}_\nu$ is infinite (measured with respect to given enumerations of $U^\dagger_\beta$ and $\hat{V}^\dagger_\beta$, for $\beta \subseteq \gamma$) can be done uniformly in $\mathbf{0}''$. Therefore determining for all α, whether $\mathfrak{E}_{r(\alpha,i)}$ is valid for α, e_1, and e_2 can be done uniformly in $\mathbf{0}''$.

Let R be the set such that $\langle \alpha, e_1, e_2, e_3, e_4, i, j\rangle \in R$ if and only if

$$\forall s(U^\dagger_{m,s} = W_{e_1,s}) \wedge \forall q < e_1 \exists s(U^\dagger_{m,s} \neq W_{q,s}),$$
$$\forall s(\hat{V}^\dagger_{m,s} = W_{e_2,s}) \wedge \forall q < e_2 \exists s(\hat{V}^\dagger_{m,s} \neq W_{q,s}),$$
$$\forall s(\hat{U}^\dagger_{m,s} = W_{e_3,s}) \wedge \forall q < e_3 \exists s(\hat{U}^\dagger_{m,s} \neq W_{q,s}),$$
$$\forall s(V^\dagger_{m,s} = W_{e_4,s}) \wedge \forall q < e_4 \exists s(V^\dagger_{m,s} \neq W_{q,s}),$$
$$i \leq n(\alpha) \text{ and } \mathfrak{E}_{r(\alpha,i)} \text{ is valid for } \alpha,\ e_1, \text{ and } e_2, \text{ and}$$
$$j \leq \hat{n}(\alpha) \text{ and } \hat{\mathfrak{E}}_{\hat{r}(\alpha,j)} \text{ is valid for } \alpha,\ e_3, \text{ and } e_4$$

(where $m = |\alpha|+1$). The R is Δ_3. For all α there exists at most one $\langle e_1, e_2, e_3, e_4, i, j\rangle$ such that $\langle \alpha, e_1, e_2, e_3, e_4, i, j\rangle \in R$. Let $\beta = \alpha^\frown\langle e_1, e_2, e_3, e_4, i, j, k\rangle \in Tr$. Now by Lemma 2.5 there is a recursive function g such that:

(i) $\exists! k[W_{g(\alpha,e_1,e_2,e_3,e_4,i,j,k)} = \omega]$ iff $\langle \alpha, e_1, e_2, e_3, e_4, i, j\rangle \in R$

(ii) $\forall k[W_{g(\alpha,e_1,e_2,e_3,e_4,i,j,k)} =^* \emptyset]$ iff $\langle \alpha, e_1, e_2, e_3, e_4, i, j\rangle \notin R$.

Let $C_\beta = W_{g(\alpha,e_1,e_2,e_3,e_4,i,j,k)}$.

Definition 2.6. (The true path) Let f be a branch in Tr such that $\lambda \subset f$ and if $\alpha \subseteq f$ and there is a unique immediate successor β of α in Tr such that $C_\beta = \omega$ then $\beta \subseteq f$. f is called the *true path*.

Claim 2.7. *f is an infinite branch.*

Proof. Let $\alpha \in Tr$ such that $\alpha \subseteq f$. Let $n = |\alpha| + 1$. Now there exists e_i such that for all s, $U^\dagger_{n,s} = W_{e_1,s}$, $\hat{V}^\dagger_{n,s} = W_{e_2,s}$, $\hat{U}^\dagger_{n,s} = W_{e_3,s}$, $V^\dagger_{n,s} = W_{e_4,s}$, for all $q < e_1$ there exists s such that $U^\dagger_{n,s} \neq W_{q,s}$, for all $q < e_2$ there exists s such that $\hat{V}^\dagger_{n,s} \neq W_{q,s}$, for all $q < e_3$ there exists s such that $\hat{U}^\dagger_{n,s} \neq W_{q,s}$, and for all $q < e_4$ there exists s such that $V^\dagger_{n,s} \neq W_{q,s}$. There must exist $i \leq n(\alpha)$ and $j \leq \hat{n}(\alpha)$, such that $\mathfrak{E}_{r(\alpha,i)}$ is valid for α, e_1 and e_2 and $\hat{\mathfrak{E}}_{\hat{r}(\alpha,j)}$ is valid for α, e_3 and e_4. Since Conditions (2.3) and (2.4) hold for i and j, we have that for all k, $\alpha\hat{}\langle e_1, e_2, e_3, e_4, i, j, k\rangle \in Tr$. Therefore there exists a unique k such that for $\beta = \alpha\hat{}\langle e_1, e_2, e_3, e_4, i, j, k\rangle \in Tr$, $C_\beta = \omega$. Thus $\beta \subseteq f$. $\square$

Hence if $\beta = \alpha\hat{}\langle e_1, e_2, e_3, e_4, i, j, k\rangle \subset f$, then for all s, $U^\dagger_{n,s} = U^\dagger_{\beta,s}$, $\hat{V}^\dagger_{n,s} = \hat{V}^\dagger_{\beta,s}$, $\hat{U}^\dagger_{n,s} = \hat{U}^\dagger_{\beta,s}$, $V^\dagger_{n,s} = V^\dagger_{\beta,s}$, $\mathfrak{E}_\beta$ is valid for α, e_1 and e_2 and $\hat{\mathfrak{E}}_\beta$ is valid for α, e_3 and e_4. C_β is called the "chip set" of β and is used to determine the approximation to the true path, f_s, at stage s. During the course of the construction we will ensure that $f = \liminf f_s$ measured with respect to $<_L$ (defined below). From now on we will restrict the range of the lower case Greek letters α, β, γ, and δ to Tr.

Definition 2.8. Let $\alpha, \beta \in Tr$.

(i) α is to the *left* of $\beta\,(\alpha <_L \beta)$ if

$$\exists\gamma \in Tr[\gamma\hat{}\langle e_1, e_2, e_3, e_4, i, j, k\rangle \subseteq \alpha \wedge \gamma\hat{}\langle e'_1, e'_2, e'_3, e'_4, i', j', k'\rangle \subseteq \beta \\ \wedge \langle e_1, e_2, e_3, e_4, i, j, k\rangle < \langle e'_1, e'_2, e'_3, e'_4, i', j', k'\rangle]$$

(ii) $\alpha \leq \beta$ if $\alpha <_L \beta$ or $\alpha \subseteq \beta$ (to the left or above).

(iii) $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$.

(iv) $\alpha \leq^* \beta$ if $\alpha <_L \beta$ or $\beta \subseteq \alpha$ (to the left or below).

(v) $\alpha <^* \beta$ if $\alpha <_L \beta$ or $\beta \subsetneq \alpha$.

(vi) Let h be an infinite branch in $[Tr]$ we say $h <_L \alpha\,(\alpha \leq h$, $h <_L \alpha$, or $h \leq^* \alpha)$ if there exists a $\beta \subset h$ such that $\beta <_L \alpha\,(\alpha \leq \beta$, $\beta <_L \alpha$, or $\beta \leq^* \alpha)$.

We will now consider the action of the balls in our pinball machines. We say *x is in the α-unit at stage s* if x is above gate G_α at stage s. The *α-region* E_α of M will be the collection of β-units for $\alpha \subseteq \beta$. We say $x \in E_{\alpha,s}$ iff for some $\beta \supseteq \alpha$, x is in the β-unit at stage s. We say $x \in E_{\alpha,\infty}$ iff there is some $\beta \supseteq \alpha$ and some s such that for all $t \geq s$, x is in the β-unit at stage t. We define $\hat{E}_\alpha$, $\hat{E}_{\alpha,s}$, and $\hat{E}_{\alpha,\infty}$ in a similar manner.

We will associate with the balls a function, $\alpha(x, s)$, into Tr such that $\alpha(x, s) = \alpha$ iff x is at the α-unit at stage s. Hence if $x \in T_s$ then x is not in the pinball machine and thus $\alpha(x, s) \uparrow$. $\alpha(x, s)$ will be partially determined by f_s. If $\alpha(x, s) = \alpha$ and $f_s <_L \alpha$, we will set $\alpha(x, s + 1) = f_s \cap \alpha = \beta$ and we will place x above G_β (assuming we have not removed x from M at stage $s + 1$). In addition, if $x \leq s$, we will *β-reject* x for all β such that $f_s <_L \beta$. We will only allow x to move into a β-unit if $\beta \leq^* \alpha(x, s)$ and x is not β-rejected. Hence, we will be able to ensure that $f_s \not<_L \alpha(x, s + 1)$. Also we will only allow x to move to the α-unit during or shortly after stages where $\alpha \subseteq f_s$.

We will meet the following requirements, F_α, $\hat{F}_\alpha$, Q_α, $\hat{Q}_\alpha$, $D_{\alpha,\nu}$ and $\hat{D}_{\alpha,\nu}$, for all α.

$$F_\alpha: \quad \begin{gathered} \text{If } \alpha <_L f\text{, then } E_{\alpha,\infty} =^* \emptyset \text{ and } U_\alpha \backslash T =^* \hat{V}_\alpha \backslash T =^* \emptyset\,. \\ \text{If } \alpha \subset f\text{, then } E_{\alpha,\infty} =^* \bar{T}\,, \\ \text{If } f <_L \alpha\text{, then } E_{\alpha,\infty} = \emptyset \text{ and } U_\alpha \backslash T =^* \hat{V}_\alpha \backslash T =^* \emptyset\,. \end{gathered}$$

$$Q_\alpha: \quad \begin{gathered} \text{If } x \in U_{\beta,s+1} - U_{\beta,s} \text{ then either } \alpha(x,s) \supseteq \beta \text{ or } x \in T_s\,. \\ \text{If } x \in \hat{V}_{\beta,s+1} - \hat{V}_{\beta,s} \text{ then } \alpha(x,s) \supseteq \beta\,. \\ \text{If } x \in T_s\text{, then for all } \beta\text{, } x \in U^\dagger_{\beta,s} \text{ iff } x \in U_{\beta,s+1}\,. \end{gathered}$$

$$\text{If } \alpha(x,s) = \alpha \text{ then for all } \beta \subseteq \alpha,\ x \in U_{\beta,s} \text{ iff } x \in U^\dagger_{\beta,s} \text{ and } x \in \hat{V}_{\beta,s} \text{ iff } x \in \hat{V}^\dagger_{\beta,s}\,.$$

$$D_{\alpha,\nu}: (\alpha \subset f) \quad \begin{gathered} \text{If } D^T_\nu \text{ is infinite then } \nu \in \mathfrak{E}_\alpha\,. \\ \text{If } \nu \in \mathfrak{E}_\alpha \text{ and } \nu = \langle |\alpha|, \sigma, \tau \rangle \text{ then for all } \beta \supseteq \alpha\,, \\ D^T_{\nu^*} \text{ is infinite, where } \nu^* \text{ is the } |\beta|\text{-state } \langle |\beta|, \sigma, \tau \rangle\,. \end{gathered}$$

(In the requirement $D_{\alpha,\nu}$, D^T_ν and $D^T_{\nu^*}$ are measured with respect to $\{U_{\gamma,s}\}_{\gamma\subseteq\beta,s<\omega}$ and $\{\hat{V}_{\gamma,s}\}_{\gamma\subseteq\beta,s<\omega}$.)

First we will show this will be enough to meet Conditions (1.10) through (1.15) of the Translation Theorem. Recall that g is a recursive function from ω into Tr such that g is one to one, onto, and if $g(e) = \alpha$ then for all $\beta \subset \alpha$, there is a $j < e$ such that $g(j) = \beta$. The enumeration will be the following: $U_{e,s} = U_{g(e),s}$, $\hat{V}_{e,s} = \hat{V}_{g(e),s}$, $\hat{U}_{e,s} = \hat{U}_{g(e),s}$, and $V_{e,s} = V_{g(e),s}$.

Recall that if $x \in T^\dagger_s$, then at stage $s+1$ we will remove x from the surface of M and place $x \in T_{s+1}$ and hence $\alpha(x, s+1)\uparrow$. Therefore $T_{s+1} = T^\dagger_s$ and by Q_α, we have for all n, $T \searrow \hat{V}_n = \emptyset$.

By F_α and Q_α and their duals, if $g(e) \not\subset f$, then $U_e \backslash T =^* \hat{V}_e \backslash T =^* V_e \backslash T =^* \hat{U}_e \backslash T =^* \emptyset$ and if $g(e) \subset f$ then for $n = |g(e)|$, $U^\dagger_n =^* U_e$, $\hat{V}^\dagger_n \cap \bar{T} =^* \hat{V}_e \cap \bar{T}$, $V^\dagger_n =^* V_e$, and $\hat{U}^\dagger_n \cap \bar{\hat{T}} =^* U_e \cap \bar{\hat{T}}$. Since f is infinite, (1.12) and (1.13) are met.

Let $\nu = \langle e, \sigma, \tau \rangle$ be an e-state. Let α be the greatest substring of f such that for some $i \le e$, $g(i) = \alpha$. To meet (1.14) and (1.15), we need to translate the e-state ν measured with respect to $\{U_{e',s}\}_{e'\le e,s<\omega}$ and $\{\hat{V}_{e',s}\}_{e'\le e,s<\omega}$ to an $|\alpha|$-state $t(\nu)$ measured with respect to $\{U_{\beta,s}\}_{\beta\subseteq\alpha,s<\omega}$ and $\{\hat{V}_{\alpha,s}\}_{\beta\subseteq\alpha,s<\omega}$. Let $t(\nu) = \langle |\alpha|, u(\sigma), u(\tau) \rangle$ where $u(\varrho) = \{|\beta| : \beta \subseteq \alpha \wedge g(j) = \beta \wedge j \in \varrho\}$ (this is a well-defined $|\alpha|$-state). Since we will meet the requirement F_α,

$$\text{(2.9)} \quad \begin{gathered} D^T_\nu \text{ is infinite (measured with } \{U_{e',s}\}_{e'\le e,s<\omega} \text{ and } \{\hat{V}_{e',s}\}_{e'\le e,s<\omega}) \\ \text{iff} \\ D^T_{t(\nu)} \text{ is infinite (measured with respect to } \{U_{\beta,s}\}_{\beta\subseteq\alpha,s<\omega} \text{ and } \{\hat{V}_{\beta,s}\}_{\beta\subseteq\alpha,s<\omega})\,. \end{gathered}$$

Assume D^T_ν is infinite (measured as above). Then $D^T_{t(\nu)}$ is infinite (measured as above). Since we met $D_{\alpha,t(\nu)}$, $t(\nu) \in \mathfrak{E}_\alpha$. Thus $D^{T^\dagger}_{t(\nu)}$ is infinite (measured w.r.t. the given enumerations of $U^\dagger_n$ and $\hat{V}^\dagger_n$). By the hypothesis of the Translation Theorem, there exists a $|\alpha|$-state $\nu' \leq t(\nu)$ such that $D^{\hat{T}\dagger}_{\nu'}$ is infinite (measured w.r.t. the given enumerations of $\hat{U}^\dagger_n$ and $V^\dagger_n$). Since $\alpha \subset f$, $\nu' \in \mathfrak{E}_\alpha$. Now using the inverse of the above procedure we can untranslate this $|\alpha|$-state ν' into a $g^{-1}(\alpha)$-state $\hat{\nu} = \hat{t}^{-1}(\nu') = \langle g^{-1}(\alpha), \hat{\sigma}, \hat{\tau}\rangle$. Let $\nu^* = \langle e, \hat{\sigma}, \hat{\tau}\rangle$ (since $g^{-1}(\alpha) \leq e$, this is a well-defined state). Since $\nu' \leq t(\nu)$, $\nu^* \leq \nu$. Since we meet $\hat{D}_{\alpha,\hat{\nu}}$ and (2.9), the set $D^{\hat{T}}_{\nu*}$ is infinite (measured w.r.t. $\{\hat{U}_{\gamma,s}\}_{\gamma\subseteq\alpha,s<\omega}$ and $\{V_{\gamma,s}\}_{\gamma\subseteq\alpha,s<\omega}$). Hence (1.15) is met. Similar reasoning, shows that (1.14) is met.

We will now explore the action the α-unit will take to meet the above requirements. The behavior of the α-unit depends on α's mode, $m(\alpha, s)$. α has three modes. If $m(\alpha, s) = \mathit{off}$, then there are no balls in the α-region and until α changes its mode no balls will be placed in the α-region. If $m(\alpha, s) = \mathit{on}$ then the α-unit will maintain the balls (in terms of the natural action to meet Q_α) that are in the α-unit. If $m(\alpha, s) = a$ (active), then, in addition, to maintaining the balls in the α-unit, α will actively seek out certain balls to ensure $D_{\alpha,\nu}$ is met (α will *pull* balls it knows will enter T and place them in the proper state). f_s will play a role in determine $m(\alpha, s)$. If $f_s <_L \alpha$, then we will ensure that $m(\alpha, s + 1) = \mathit{off}$. Otherwise, we will only allow α's mode to change when $\alpha \subseteq f_s$.

When α's mode is active, the α-unit will try to verify the states in $\mathfrak{E}_\alpha$ and $\hat{\mathfrak{E}}_\alpha$ are actually the valid entry states. If x enters T at stage s from the α-unit, we will *mark* the entry state of x on $\mathfrak{E}_\alpha$. Only after all the states in $\mathfrak{E}_\alpha$ and $\hat{\mathfrak{E}}_\alpha$ are marked, will we allow α to change its mode (assuming $\alpha <_L f_s$ or $\alpha \subset f_s$). If all the states in $\mathfrak{E}_\alpha$ and $\hat{\mathfrak{E}}_\alpha$ are marked, we say that $\mathfrak{E}_\alpha$, and $\hat{\mathfrak{E}}_\alpha$ are *completely marked.* (Generally marks will be used to witness the occurrence of certain events.)

During the construction we also will use a function $p(\alpha, s)$. $p(\alpha, s)$ will be used in two ways. First it will be used as a priority ordering (the lower the number the higher priority). If $m(\alpha, s) = a$, $m(\beta, s) = a$, and they are both actively seeking the same ball (to help met $D_{\alpha,\nu}$ or $D_{\beta,\nu}$) then $p(-, s)$ will be used to determine which unit will receive the ball. $p(-, s)$ will also be used as a restraint. Unless $m(\alpha, s) = a$ and α "pulls" a ball into the α-unit, we will only allow balls less than $p(\alpha, s)$ to enter the α-unit. We will be careful to ensure that $p(-, s)$ is a non-decreasing function. The following notation will be helpful.

Definition 2.10. (i) The *α-state of x at stage s*, $\nu_\alpha(x, s)$, is the $|\alpha|$-state, $\langle|\alpha|, \sigma, \tau\rangle$, where $\sigma = \{|\beta| : x \in U^\dagger_{\beta,s} \wedge \beta \subseteq \alpha\}$ and $\tau = \{|\beta| : x \in \hat{V}^\dagger_{\beta,s} \wedge \beta \subseteq \alpha\}$.

(ii) The *$\hat{\alpha}$-state of $\hat{x}$ at stage s*, $\hat{\nu}_{\hat{\alpha}}(\hat{x}, s)$, is the $\hat{\alpha}$-state, $\langle|\hat{\alpha}|, \sigma, \tau\rangle$, where $\sigma = \{|\beta| : x \in \hat{U}^\dagger_{\beta,s} \wedge \beta \subseteq \hat{\alpha}\}$ and $\tau = \{|\beta| : x \in V^\dagger_{\beta,s} \wedge \beta \subseteq \hat{\alpha}\}$. (We will always use $\hat{\nu}_\alpha(\hat{x}, s)$ as shorthand for $\hat{\nu}_{\hat{\alpha}}(\hat{x}, s)$.)

The construction

The steps for $\hat{M}$ are the dual of those presented.

Stage $s = 0$. Let $T_0 = \hat{T}_0 = U_{\alpha,0} = V_{\alpha,0} = \hat{U}_{\alpha,0} = \hat{V}_{\alpha,0} = \emptyset$. Let $f_0 = \lambda$. Let $\alpha(x, 0) = \lambda$, $m(\alpha, 0) = \mathit{off}$, and $p(\alpha, 0) = -1$, for all $\alpha \in \mathit{Tr}$ and for all x.

Stage $s + 1$. Unless otherwise explicitly defined all parameters remain the same from stage s to stage $s + 1$. Recall that if $\beta = \alpha\hat{\ }\langle e_1, e_2, e_3, e_4, i, j, k\rangle \in \mathit{Tr}$ then $U^\dagger_{\beta,s+1} = W_{e_1,s+1}$, $\hat{V}^\dagger_{\beta,s+1} = W_{e_2,s+1}$, $\hat{U}^\dagger_{\beta,s+1} = W_{e_3,s+1}$, and $V^\dagger_{\beta,s+1} = W_{e_4,s+1}$.

Step (1) (Enumeration into T.) If $x \in T^\dagger_s - T^\dagger_{s-1}$, then remove x from the surface of M, place $x \in T_{s+1}$, let $\alpha(x, s+1) \uparrow$, and if $\alpha = \alpha(x, s)$ and for all $t \leq s$, $\alpha \leq^* \alpha(x, t)$ (i.e. x has never been below α) mark $\nu_\alpha(x, s)$ on $\mathfrak{E}_\alpha$ (if $\nu_\alpha(x, s)$ appears on $\mathfrak{E}_\alpha$).

Step (2) (Pulling to meet $D_{\alpha,\nu}$.) Let $x \in T^\dagger_{s+1} - T^\dagger_s$. If there exists an α such that

(2.1) $m(\alpha, s) = \alpha$,

(2.2) $f_s \not<_L \alpha$, and $x \geq |\alpha|$,

(2.3) for all $t \leq s$, $\alpha \leq^* \alpha(x, t)$, and x is not α-rejected,

(2.4) if $m(\alpha(x, s), s) = a$ then $p(\alpha, s) < p(\alpha(x, s), s)$ (so $\alpha \mathrel{\not\mid} \alpha(x, s)$),

(2.5) for all β with $\alpha \cap \alpha(x, s) \subseteq \beta \subset \alpha$ if $m(\beta, s) = a$ then $p(\alpha, s) < p(\beta, s)$,

(2.6) $\nu_\alpha(x, s + 1) \in \mathfrak{E}_\alpha$ and $\nu_\alpha(x, s + 1)$ is unmarked,

then let α be such that if γ also satisfies (2.1) through (2.6) then either $\alpha <_L \gamma$ or $p(\alpha, s) < p(\gamma, s)$. Place x above G_α and let $\alpha(x, s + 1) = \alpha$. We say α *pulled x away from* $\alpha(x, s)$. (At the next stage x will enter T and $\nu_\alpha(x, s + 1) \in \mathfrak{E}_\alpha$ will become marked.)

Step (3) (Removing balls from units to the right of the true path.) If $x \leq s + 1$, then for all β such that $f_s <_L \beta$, x is β-rejected (x is permanently removed from the β-unit). If $x \leq s + 1$, $x \notin T_{s+1}$, x has not been pulled away from $\alpha(x, s)$, and $f_s <_L \alpha(x, s)$, then let $\alpha(x, s + 1) = \alpha(x, s) \cap f_s$ and place x above $G_{\alpha(x,s+1)}$.

Step (4) (Movement on the pinball machine due to f_s.) Suppose $\beta \subseteq f_s$ ($\beta \neq \lambda$), $x \notin T_{s+1}$, $|\beta| \leq x \leq p(\beta, s)$, x is not β-rejected, $\beta^- = \alpha(x, s)$, and x has not been pulled from $\alpha(x, s)$. Let $\alpha(x, s + 1) = \beta$ and place x above $G_{\alpha(x,s+1)}$.

Step (5) (Enumeration into U_β and $\widehat{V}_\beta$; meeting Q_α.) For all α and for all $x \leq s+1$, such that $\alpha = \alpha(x, s + 1)$ (if $\alpha(x, s + 1)$ was not defined by the above steps then let $\alpha(x, s + 1) = \alpha(x, s))$ then for all $\beta \subseteq \alpha$, $x \in U_{\beta,s}$ iff $x \in U^\dagger_{\beta,s}$, and $x \in \widehat{V}_{\beta,s}$ iff $x \in \widehat{V}^\dagger_{\beta,s}$. For all $x \leq s + 1$, if $x \in T_s$, then for all β, if $x \in U^\dagger_{\beta,s}$ then $x \in U_{\beta,s+1}$.

(Clearly this meets Q_α.)

The next two steps do not have duals.

Step (6) (Action by f_s and changing α's mode.)

Substep (6.1) (Turning off the α-units to the right of the true path.) If $f_s <_L \alpha$ and $m(\alpha, s) = on$ or a, then let $m(\alpha, s + 1) = off$ and $p(\alpha, s + 1) = s + 1$.

Substep (6.2) (Changing from *on* or *off* to *active*.) If $\alpha \subseteq f_s$ and $m(\alpha, s) = off$ or *on*, let $m(\alpha, s + 1) = a$, $p(\alpha, s + 1) = s + 1$, and clear all the markers.

Substep (6.3) (Changing from *active* to *on*.) If $\alpha \subseteq f_s$, $m(\alpha, s) = a$, and $\mathfrak{E}_\alpha$ and $\hat{\mathfrak{E}}_\alpha$ are fully marked, do the following: Let $m(\alpha, s + 1) = on$ and $p(\alpha, s + 1) = s + 1$.

Step (7) (Determining f_{s+1}.) We will define $\gamma_{i,s+1}$ by induction, for $i \leq s + 1$. Let $\gamma_{0,s+1} = \gamma$. If there is a stage $t \leq s$ such that $\gamma_{i,s+1} \subseteq f_t$, let t be the greatest such stage, otherwise let $t = 0$. If there is an immediate successor β of $\gamma_{i,s+1}$ in *Tr* such that $C_{\beta,s+1} \neq C_{\beta,t}$, then let $\gamma_{i+1,s+1}$ be the $<_L$-least such β. Otherwise let $\gamma_{i+1,s+1} = \gamma_{i,s+1}$ (and therefore $f_{s+1} = \gamma_{i,s+1}$). Let $f_{s+1} = \gamma_{s+1,s+1}$.

The verification

The lemmas are dual for $\hat{M}$.

As we noted before, the true path f is an infinite branch in Tr. It is a straight forward argument to prove that $f = \liminf f_s$ (measured with respect to $<_L$). (For a very similar proof the reader is directed to [1] or [2].) Clearly Step 5 meets Q_α. Hence if $\alpha(x,s) = \alpha$, then $\nu_\alpha(x,s) = \nu$ iff $\nu(|\alpha|, x, s) = \nu$ measured with respect to $\{U_{\beta,s}\}_{\beta\subseteq\alpha}$ and $\{\hat{V}_{\beta,s}\}_{\beta\subseteq\alpha}$.

Some Easy Facts about $\alpha(-,-)$ *and* $m(-,-)$. For all x and for all s, we have $x \geq |\alpha(x,s)|$; $f_s \not<_L \alpha(x,s+1)$; if $f_s <_L \alpha$ and $s \geq x$ then x is α-rejected (unable to enter the α-unit) at stage $s+1$; and if x is α-rejected at stage s then for all $t > s$, $\alpha(x,t) < \alpha$ (see Steps 2.2, 2.3, 3, and 4). For all x and s, either $\alpha(x,s+1) \leq^* \alpha(x,s)$; or $\alpha(x,s+1) \subset \alpha(x,s)$, x is $\alpha(x,s)$-rejected at stage $s+1$ and $f_s <_L \alpha(x,s)$ (see Step 2.3, 3, and 4). x is in the α-unit at stage s iff $\alpha = \alpha(x,s)$ (by definition of $\alpha(x,s)$ and Step 1). If $m(\alpha, s+1) = \mathit{off}$ then for all x, $\alpha \not\subseteq \alpha(x,s+1)$ and either $m(\alpha,s) = \mathit{off}$ or $f_s <_L \alpha$ (see Steps 2.1, 3, 4, and 6.1). We will use these facts without too much reference in the next three lemmas.

Lemma 2.11. *For all $k \geq -1$, all $\alpha \in Tr$, if there is a stage s such that $p(\alpha,s) = k$, then there exists a stage $t > s$ such that either*

(i) *$p(\alpha,t) = t \neq k$, or*

(ii) *no new balls enter the α-region after stage t, no new balls enter the $\hat{\alpha}$-region after stage t and for all $t_1 \geq t$, $m(\alpha,t_1) = m(\alpha,t)$.*

Proof. By induction on k. $p(\alpha,s)$ is a non-decreasing function in s (see Substeps 6.1, 6.2 and 6.3). If for all s, $p(\alpha,s) = -1$ then for all s, $m(\alpha,s) = \mathit{off}$ and $E_{\alpha,s} = \hat{E}_{\alpha,s} = \emptyset$ (see Substeps 6.1, 6.2 and 6.3). For all $\alpha \in Tr$ there does not exist a stage s such that $p(\alpha,s) = 0$ (see Stage 0 and Step 6). Hence the lemma holds for $k = -1, 0$.

By induction assume the lemma holds for $k' < k \neq -1, 0$. For all $\alpha \in Tr$, if $m(\alpha,s) \neq m(\alpha,s+1)$ then $p(\alpha,s) \neq p(\alpha,s+1) = s+1$ (see Step 6). For all $k \geq 1$, for all $\alpha \in Tr$, there exists a stage s such that $p(\alpha,s) = k$ iff $p(\alpha,k) = k$ and if $p(\alpha,k) = k$ then either $\alpha \subseteq f_k$ (see Substep 6.2 and 6.3) or $m(\alpha,k-1) \neq \mathit{off}$ and $f_k <_L \alpha$ (see Substep 6.1). If $m(\alpha,k-1) \neq \mathit{off}$ and $f_k <_L \alpha$, then $m(\alpha,k) = \mathit{off}$, $p(\alpha,k) = k$ and either there exists a stage $t > k$ such that $p(\alpha,t) > k$ or for all $s \geq k$, $m(\alpha,s) = \mathit{off}$, $p(\alpha,s) = k$, and $E_{\alpha,s} = \hat{E}_{\alpha,s} = \emptyset$. Hence it only remains to show the lemma for $\alpha \subseteq f_k$ and k. We will do this by reverse induction for $\alpha \subseteq f_k$.

But first we must note the following: If $\beta <^* f_k$ and $m(\beta,k) \neq \mathit{off}$ then there is stage $s_1 < k$ such that $\beta \subseteq f_{s_1}$ and for all s_2, if $s_1 \leq s_2 \leq k$, $f_{s_2} \not<_L \beta$. Hence by the induction hypothesis for k there is a stage $t_1 \geq k$ such that for all $\beta <^* f_k$ either

(a) $m(\beta,t_1) = \mathit{off}$ (hence if β turns on at some later stage s_1, $p(\beta,s_1) = s_1 > k$),

(b) $p(\beta,t_1) > k$, or

(c) no new balls enter the β-region after stage t_1, no new balls enter the $\hat{\beta}$-region after stage t_1, and for all $t_3 \geq t_1$, $m(\beta,t_1) = m(\beta,t_3)$.

Let $\alpha \subseteq f_k$. Assume that the lemma holds for all β such that $\alpha \subset \beta \subseteq f_k$. Assume for all $t \geq k$, $p(\alpha,t) = k$. Hence for all $t \geq k$, $m(\alpha,t) = m(\alpha,k) \neq \mathit{off}$ and $f_t \not<_L \alpha$ (see Step 6). By the induction hypothesis for α there is a stage $t_2 \geq t_1$ such that the above Conditions (a), (b) and (c) hold for all $\beta <^* \alpha$ and t_2 rather than t_1.

Assume $\alpha = \lambda$. Now for all t, $\lambda \subseteq f_t$. Therefore, we may assume $m(\lambda,k) = a$ (if $m(\lambda,k) = \mathit{on}$, then there exists a stage $t > k$ such that $m(\lambda,t) = a$; see

Step 6.2). For all $x > k$, if $x \notin T_t$ then $\alpha(x,t) = \lambda$ (see Step 4). Since $T^\dagger$ is infinite, there exists an $x > k$ and s_1 such that $x \in T^\dagger_{s_1} - T^\dagger_{s_1-1}$. Since we are assuming all eight sets associated with λ are empty, $\nu_\lambda(x,t) = \langle 0, \emptyset, \emptyset\rangle$. Hence at some later stage r both $\mathfrak{E}_\lambda = \{\langle 0,\emptyset,\emptyset\rangle\}$ and $\hat{\mathfrak{E}}_\lambda = \{\langle 0,\emptyset,\emptyset\rangle\}$ are completely marked. Therefore $m(\lambda, r+1) = on$ and $p(\lambda, r+1) = r+1 > k$. A contradiction.

Assume $\alpha \neq \lambda$. New balls may only enter the α-region through Step 2 or 4. Since for all $t > t_2$, $p(\alpha,t) = k$, after stage t_2 the action of Step 4 will be able to place only finitely many balls into the α-region. Hence we must show Step 2 can only put finitely many balls into the α-region. Let x be a ball which is placed in α-region after stage t_2 by Step 2. Say x enters the β-unit at stage t_3.

First we will show that $m(\alpha,k) = a$. Since $\alpha \not\subseteq \alpha(x,t_3-1)$, β must be active at stage t_3 (see Step 2.1). Therefore $p(\beta,t_3) = k_1 \geq k$ (this follows since β must satisfy either (a), (b) or (c) and the choice of t_2). Thus $\alpha \subseteq \beta \subseteq f_{k_1}$ and hence if $m(\alpha,k_1-1) = on$ then $p(\alpha,k_1) = k_1$ and $m(\alpha,k_1) = on$. So $m(\alpha,k) = a$.

Now since $m(\alpha,k) = a$, if Step 2 places a ball into the α-region they must place that ball into the α-unit (see 2.5). Since $\mathfrak{E}_\alpha$ is finite, Step 2 can only place finitely many balls into the α-unit (see Steps 1 and 2.6).

Therefore there exists a $t \geq t_2$ such that no new balls enter the α-region after stage t. Similar reasoning shows there is a t such that no new balls enter the $\hat{\alpha}$-region after stage t. $\square$

Lemma 2.12. (i) *If $f <_L \alpha$, then for all x there exists s_x such that for all $s \geq s_x$, $\alpha(x,s) < \alpha$ and for almost all x and for all s, $\alpha \not\subseteq \alpha(x,s)$. (Hence $E_{\alpha,\infty} = \emptyset$ and $U_\alpha \backslash T =^* \hat{V}_\alpha \backslash T =^* \emptyset$.)*

(ii) *If $\alpha <_L f$, then for almost all x and for all s, $\alpha <^* \alpha(x,s)$. (Hence $E_{\alpha,\infty} =^* \emptyset$ and $U_\alpha \backslash T =^* \hat{V}_\alpha \backslash T =^* \emptyset$.)*

(iii) *For all $x \notin T$, $\alpha(x) = \lim_{s\to\infty} \alpha(x,s)$ exists.*

Proof. We will use without reference many of the facts about $\alpha(-,-)$ mentioned earlier.

(i) Assume $f <_L \alpha$. Let $s_x \geq x$ be such that $f_{s_x} <_L \alpha$. We have that for all $s \geq s_x$, $\alpha(x,s) < \alpha$. Since there exists a $\beta \subseteq \alpha$, such that C_β is finite (otherwise α is on the true path), there exists an r such that for all $s \geq r$, $\alpha \not\subseteq f_s$ and $m(\alpha,s) = off$. Therefore, for all $x \geq r$, for all s, $\alpha \not\subseteq \alpha(x,s)$.

(ii) Assume $\alpha <_L f$. There exists a stage t such that for all $s \geq t$, $\alpha <_L f_s$ or $f_s \subset \alpha$. Hence for all $s \geq t$ and for all $\beta \leq^* \alpha$, $p(\beta,t) = p(\beta,s)$. Therefore by the above lemma, there exists a $t_1 \geq t$ such that no new balls enter the β-region after stage t_1 and hence $E_{\beta,\infty} =^* \emptyset$, for any $\beta \leq^* \alpha$. If $x > \max\{p(\beta,t) : \beta \leq^* \alpha\}$, then for all s, $\alpha <^* \alpha(x,s)$ (see Steps 3 and 4). Hence for almost all x, $\alpha <^* \alpha(x,s)$, for all s.

(iii) Given $x \notin T$ do the following: Let $\alpha \subset f$ such that $|\alpha| = x$. Let $t \geq x$ be such that for all $s \geq t$, $f_s \not<_L \alpha$. Hence for all $s \geq t$ and for $\beta <_L \alpha$, $p(\beta,t) = p(\beta,s)$. Now, there exists a $t_1 \geq t$ such that $\alpha \subset f_{t_1}$ and for all $\beta <_L \alpha$, no new balls enter the β-region after stage t_1. If $\alpha(x,t_1) <_L \alpha$, then for all $s \geq t_1$, $\alpha(x,s) = \alpha(x,t_1)$; otherwise since $t_1 \geq x$ and $\alpha \subseteq f_{t_1}$, for all β with $\alpha <_L \beta$, x is β-rejected at stage t_1 (see Step 3) and therefore for all $s > t_1$, $\alpha(x,s) \subseteq \alpha(x,s+1) \subseteq \alpha$ (see Steps 2.2, 2.3, 3 and 4). Hence $\lim_{s\to\infty} \alpha(x,s)$ exists. $\square$

Notation 2.13. Let $k_\lambda = 0$. For $\alpha \subset f$ with $\alpha \neq \lambda$, let $k_\alpha > k_{\alpha^-}$ be the least stage such that

$$\alpha \subseteq f_{k_\alpha},$$

$$\text{for all } t \geq k_\alpha,\ \ f_t \not<_L \alpha,$$

$$\text{for all } s,\ \text{for all } x \geq k_\alpha,\ \text{if } x \notin T_s \text{ then } \alpha(x,s) \not<_L \alpha, \text{ and}$$

$$\text{for all } s,\ \text{for all } \hat{x} \geq k_\alpha,\ \text{if } \hat{x} \notin \hat{T}_s \text{ then } \hat{\alpha}(\hat{x},s) \not<_L \alpha.$$

Such a k_α exists since $\alpha \subset f$ and the above lemma. If $x \geq k_\alpha$, then for all $\beta <_L \alpha$, x will not enter the β-unit. If $x \geq k_\alpha$ and $\alpha(x,s) \supseteq \alpha$, then after stage s such an x cannot leave the α-region unless x enters T and hence for all $t \geq s$, either $\alpha(x,t) \supseteq \alpha$ or $x \in T_t$. If $x \geq k_\alpha$ is placed in U_α or $\hat{V}_\alpha$ at stage s then $\alpha \subseteq \alpha(x,s)$ and x is always in the α-region after stage s. If $x \geq k_\alpha$ and x is not in the α-region at stage s then $\nu_{\alpha^-}(x,s) = \nu_\alpha(x,s)$.

Lemma 2.14. *Assume $\alpha \subset f$. Then*

(i) *for all $s \geq k_\alpha$, $m(\alpha,s) \neq$ off, and*

(ii) *for all $k \geq k_\alpha$ there exist s and t such that $t > s \geq k$, $m(\alpha,s) = a$, and $m(\alpha,t) =$ on. Hence $\lim_{s\to\infty} p(\alpha,s) = \infty$.*

Proof. (i) Since $\alpha \subseteq f_{k_\alpha}$ and for all $s \geq k_\alpha$, $f_s \not<_L \alpha$, $m(\alpha,k_\alpha) \neq$ *off* (see Step 6.2) and for all $s \geq k_\alpha$, $m(\alpha,s) \neq$ *off* (Step 6.1).

(ii) By induction on α. Since $\alpha \subset f$, there exists $s \geq k$ such that $m(\alpha,s) = a$ and $p(\alpha,s) \geq 0$ (see Step 9.2). If $\alpha = \lambda$, then there exists a $t > s$ such that (ii) holds (see the paragraph of the proof of Lemma 2.12 which begins "Assume $\alpha = \lambda$.").

Assume that $\alpha \neq \lambda$ and for all $t_3 \geq s$, $m(\alpha,t_3) = a$. Hence for all $t_3 \geq s$, $p(\alpha,t_3) = p(\alpha,s)$. By 2.12 and the induction hypothesis, there exists a stage $t_1 \geq s \geq k_\alpha$ such that no balls greater than t_1 enter the α-region or $\hat{\alpha}$-region after stage t_1 and for all $\beta \subset \alpha$, $p(\alpha,t_1) < p(\beta,t_1)$. By the choice of k_α and t_1, no balls greater than k_α enter the β-region or $\hat{\beta}$-region for any $\beta \leq^* \alpha$. Therefore for all $\beta \leq^* \alpha$, β cannot pull from any node γ after stage t_1.

We will show that $\mathfrak{C}_\alpha$, and $\hat{\mathfrak{C}}_\alpha$ are completely marked at stage t_1. Let $\alpha = \alpha^{-\frown}\langle e_1,e_2,e_3,e_4,i,j,k\rangle$. Let $\nu \in \mathfrak{C}_\alpha$. Then for all s, $U^\dagger_{n,s} = W_{e_1,s}$, $\hat{V}^\dagger_{n,s} = W_{e_2,s}$, $\hat{U}^\dagger_{n,s} = W_{e_3,s}$, $V^\dagger_{n,s} = W_{e_4,s}$, and $\mathfrak{C}_\alpha$ is valid for α, e_1 and e_2. Hence there must exist an $x \geq t_2$ and $s_1 \geq t_2$ such that $\nu_\alpha(x,s_1) = \nu$, $x \in T^\dagger_{s_1} - T^\dagger_{s_1-1}$, and for all $t_3 \leq s_1$, $\alpha <^* \alpha(x,t_3)$. Let s_1 be the least such stage (this determines the x). If ν were unmarked at stage t_1, Step 2 will place x into a β-unit at stage s_1 for some $\beta \leq^* \alpha$. Hence $\mathfrak{C}_\alpha$ is completely marked. We can show $\hat{\mathfrak{C}}_\alpha$ is completely marked in a similar fashion.

Since $\alpha \subset f$, there exists a $t \geq t_1$ such that $\alpha \subseteq f_t$, $m(\alpha,t) = a$ and $\mathfrak{C}_\alpha$ and $\hat{\mathfrak{C}}_\alpha$ are completely marked at stage t. Therefore $m(\alpha,t+1) =$ *on*; $p(\alpha,t+1) = t+1$; and $\mathfrak{C}_\alpha$, and $\hat{\mathfrak{C}}_\alpha$ are not marked at all at stage $t+1$ (see Step 6.3). $\square$

Lemma 2.15. *For all $\alpha \subset f$, let $\beta \subset f$ such that $\beta^- = \alpha$. Then for almost all $x \notin T$, $\beta \subseteq \alpha(x)$. (Hence $E_{\beta,\infty} =^* \bar{T}$ and the requirement F_α is met.)*

Proof. By induction on α. By induction hypothesis, we know that for almost all $x \notin T$, $\alpha \subseteq \alpha(x)$. We also know that for all γ, if $\beta <_L \gamma$ then for all x there exists a stage s such that for all $t \geq s$, $\alpha(x,t) <_L \gamma$ and if $\gamma <_L \beta$ then for all $x \geq k_\alpha$ and for all s, $\gamma <^* \alpha(x,s)$ (see Lemma 2.12). Hence for all $x \notin T$, if $x \geq k_\alpha$ then either $\alpha \supseteq \alpha(x)$ or $\beta \subseteq \alpha(x)$ (only Step 3 can move a ball upwards in the machine).

Assume there are infinitely many $x \notin T$ such that $\alpha = \alpha(x)$. Fix $x \geq k_\beta$ such that $x \notin T$ and $\alpha = \alpha(x)$. Let s_x be such that for all $s \geq s_x$, $\alpha(x,s) = \alpha$. Since $\beta \subset f$, x is not β-rejected (see Step 3) and there exists a $t_x \geq s_x$ such that $\beta \subset f_{t_x}$, and $x \leq p(\beta, t_x)$ (by the above lemma). Step 4 will set $\alpha(x, t_x + 1) = \beta$, a contradiction. $\square$

Lemma 2.16. *For all $\alpha \subset f$, $D_{\alpha,\nu}$ is met,*

Proof. By induction on α. Let $e = |\alpha|$. Assume $\nu = \langle |\alpha|, \sigma, \tau\rangle \in \mathfrak{E}_\alpha$. By Lemma 2.14, $\mathfrak{E}_\alpha$, and $\hat{\mathfrak{E}}_\alpha$ are completely marked and unmarked infinitely many times. We can mark ν (on $\mathfrak{E}_\alpha$) at stage s if some ball x, $x \in T_{s+1} - T_s$ and $\nu_\alpha(x,s) = \nu$ and for all $t \leq s+1$, $\alpha \leq^* \alpha(x,t)$. If $\beta \supseteq \alpha$ then the $|\beta|$-state of x at stage $s+1$ measured with respect to $\{U_{\gamma,s}\}_{\gamma \subseteq \beta, s<\omega}$ and $\{\hat{V}_{\gamma,s}\}_{\gamma\subseteq\beta,s<\omega}$ is $\langle |\beta|, \sigma, \tau\rangle$. Since $\lim_{s\to\infty} p(\alpha, s) = \infty$, there are infinitely many such balls. Hence if $\nu = \langle e, \sigma, \tau\rangle \in \mathfrak{E}_\alpha$ then $D^T_{\nu^*}$ is infinite, where ν^* is the $|\beta|$-state $\langle |\beta|, \sigma, \tau\rangle$.

Assume D^T_ν is infinite (measured with respect to $\{U_{\beta,s}\}_{\beta\subseteq\alpha,s<\omega}$ and $\{\hat{V}_{\beta,s}\}_{\beta\subseteq\alpha,s<\omega}$). If $\nu \restriction e-1 = \nu$, then, by induction, we are done. If $x \geq k_\alpha$ then if $\alpha(x,s) \supseteq \alpha$ then for all $t \geq s$ $\alpha(x,t) \supseteq \alpha$ (x can not move above α). Assume that $x \in D^T_\nu$, $x \in T_{s+1} - T_s$, and $x \geq k_\alpha$. By Q_α, $\alpha \subseteq \alpha(x,s)$, since $\nu \restriction e-1 \neq \nu$. Therefore $\nu_\alpha(x,s) = \nu$ and $x \in D^{T^\dagger}_\nu$ measured w.r.t. to $\{U^\dagger_{n,s}\}_{n\leq e,s<\omega}$ and $\{\hat{V}^\dagger_{n,s}\}_{n\leq e,s<\omega}$. Hence $D^{T^\dagger}_\nu$ is infinite, by Q_α, and ν must be in $\mathfrak{E}_\alpha$. $\square$

3 Maximal sets form an orbit

Let M_1 and M_2 be maximal sets. We show that M_1 and M_2 are automorphic in the lattice of recursively enumerable sets. This is a result of Soare (see [7] or [8]) but our proof is different.

Since M_i is maximal we know that either $W_e \cup M_i =^* \omega$ or $W_e \subseteq^* M_i$ and furthermore deciding whether $W_e \cup M_i =^* \omega$ or $W_e \subseteq^* M_i$ can be done recursively in $\mathbf{0}''$. This and the fact that maximal sets are simple will be the only facts that we will use about maximal sets. As always we will consider $\hat{\omega}$ as a copy of ω; integers from $\hat{\omega}$ will always wear hats; M_1 as a subset of ω; and M_2 as a subset of $\hat{\omega}$.

Since we are using the Modified Extension Theorem it is enough to find uniformly $\mathbf{0}''$-enumerations $\{M_{1,s}\}_{s<\omega}$, $\{M_{2,s}\}_{s<\omega}$, $\{U_{n,s}\}_{n,s<\omega}$, $\{\hat{V}_{n,s}\}_{n,s<\omega}$, $\{\hat{U}_{n,s}\}_{n,s<\omega}$, and $\{V_{n,s}\}_{n,s<\omega}$ of the (hopefully) uniformly $\mathbf{0}''$-recursive collection of r.e. sets M_1, M_2, $\{U_n\}_{n<\omega}$, $\{\hat{V}_n\}_{n<\omega}$, $\{\hat{U}_n\}_{n<\omega}$, and $\{V_n\}_{n<\omega}$ satisfying the following Conditions:

$$\forall n[M_1 \searrow \hat{U}_n = M_2 \searrow \hat{V}_n = \emptyset]\,, \tag{3.1}$$

$$(\forall \nu)[D^{M_2}_\nu \text{ is infinite} \Rightarrow (\exists \nu' \geq \nu)[D^{M_1}_{\nu'} \text{ is infinite}]], \text{ and} \tag{3.2}$$

$$(\forall \nu)[D^{M_1}_\nu \text{ is infinite} \Rightarrow (\exists \nu' \leq \nu)[D^{M_2}_{\nu'} \text{ is infinite}]]\,, \tag{3.3}$$

$$\begin{gathered}\text{if } n = 2m \text{ then } U_n =^* W_m \text{ and } V_n = \emptyset \text{ and}\\ \text{if } n = 2m+1 \text{ then } V_n =^* W_m \text{ and } U_n = \emptyset\,,\end{gathered} \tag{3.4}$$

$$\begin{gathered}\exists^\infty x \in \overline{M_1} \text{ with final } e\text{-state } \nu \text{ w.r.t. to } \{U_n\}_{n<\omega} \text{ and } \{\hat{V}_n\}_{n<\omega}\\ \text{iff}\\ \exists^\infty \hat{x} \in \overline{M_2} \text{ with final } e\text{-state } \nu \text{ w.r.t. to } \{\hat{U}_n\}_{n<\omega} \text{ and } \{V_n\}_{n<\omega}\,.\end{gathered} \tag{3.5}$$

where for all e-states ν, $D_\nu^{M_1}$ is measured w.r.t. $\{U_{n,s}\}_{n\le e,s<\omega}$ and $\{\hat{V}_{n,s}\}_{n\le e,s<\omega}$ and $D_\nu^{M_2}$ is measured w.r.t. $\{\hat{U}_{n,s}\}_{n\le e,s<\omega}$ and $\{V_{n,s}\}_{n\le e,s<\omega}$. (In this section M_1 with play the role of T and M_2 that of $\hat{T}$.)

Before we construct this enumeration, we will show that this is enough to conclude that these sets are automorphic. First, by the Modified Extension Theorem, there is an uniformly $\mathbf{0}''$-recursive collection of r.e. sets $\{\hat{U}_n\}_{n\in\omega}$ and $\{\hat{V}_n\}_{n\in\omega}$ such that

$$\hat{U}_n \cap \overline{M_2} =^* \hat{U}_n \cap \overline{M_2},\ \ \hat{V}_n \cap \overline{M_1} =^* \hat{V}_n \cap \overline{M_1}, \tag{3.6}$$

and

$$\begin{gathered}\exists^\infty x \in M_1 \text{ with final } e\text{-state } \nu \text{ w.r.t. to } \{U_n\}_{n<\omega} \text{ and } \{\hat{V}_n\}_{n<\omega}\\ \text{iff}\\ \exists^\infty \hat{x} \in M_2 \text{ with final } e\text{-state } \nu \text{ w.r.t. to } \{\hat{U}_n\}_{n<\omega} \text{ and } \{V_n\}_{n<\omega}.\end{gathered} \tag{3.7}$$

From (3.5), (3.6), and (3.7), we have that

$$\begin{gathered}\exists^\infty x \in \omega \text{ with final } e\text{-state } \nu \text{ w.r.t. to } \{U_n\}_{n<\omega} \text{ and } \{\hat{V}_n\}_{n<\omega}\\ \text{iff}\\ \exists^\infty \hat{x} \in \hat{\omega} \text{ with final } e\text{-state } \nu \text{ w.r.t. to } \{\hat{U}_n\}_{n<\omega} \text{ and } \{V_n\}_{n<\omega}.\end{gathered} \tag{3.8}$$

By (3.4), it is easy to see

$$\begin{gathered}\exists^\infty x \in \omega \text{ with final } e\text{-state } \nu \text{ w.r.t. to } \{W_e\}_{e<\omega} \text{ and } \{\hat{V}_{2e+1}\}_{e<\omega}\\ \text{iff}\\ \exists^\infty \hat{x} \in \hat{\omega} \text{ with final } e\text{-state } \nu \text{ w.r.t. to } \{\hat{U}_{2e}\}_{e<\omega} \text{ and } \{W_e\}_{e<\omega},\end{gathered} \tag{3.9}$$

and hence $\Phi(W_e) = \hat{U}_{2e}$ and $\Phi^{-1}(W_e) = \hat{V}_{2e+1}$ defines an automorphism of the lattice of the recursively enumerable sets modulo the finite sets such that $\Phi(M_1) =^* M_2$. Φ can be easily converted into an automorphism Ψ of the lattice of the recursively enumerable sets such that $\Psi(M_1) = M_2$ (see [8, XV.2.7]).

We will now focus on meeting (3.1) through (3.5). We will just pick any enumeration of M_1 and M_2. To meet (3.1), we will not enumerate integers into $\hat{U}_n$ ($\hat{V}_n$) once they have entered M_2 (M_1). Since we will meet (3.4), we can let $\hat{U}_{2e+1} = \hat{V}_{2e} = \emptyset$.

A first (failed) attempt to meet (3.5) might go as follows: if $U_{2e} \cup M_1 =^* \omega$ then let $\hat{U}_{2e} = \omega$, otherwise let $\hat{U}_{2e} = \emptyset$, and if $V_{2e+1} \cup M_2 =^* \omega$ then let $\hat{V}_{2e+1} = \omega$, otherwise let $\hat{V}_{2e+1} = \emptyset$ (without choosing any enumeration of these sets). Since M_1 and M_2 are both maximal, this will meet (3.5) but as we will see this fails to meet the entry Conditions (3.2) and (3.3). Assume that $W_0 \cup M_1 = \omega$ and we have the bad luck to enumerate U_0, $\hat{V}_0$, $\hat{U}_0$, and V_0 such that when we only consider 0-states $D_\nu^{M_1}$ is infinite (measured w.r.t. the bad enumeration of U_0 and $\hat{V}_0$) iff $\nu \in \{\langle 0,\emptyset,\emptyset\rangle, \langle 0,\{0\},\emptyset\rangle\}$ and $D_\nu^{M_2}$ is infinite iff $\nu \in \{\langle 0,\{0\},\emptyset\rangle\}$ (measured w.r.t. to the enumeration of $\hat{U}_0$ and V_0). Hence (3.3) is not met if $\nu = \langle 0,\emptyset,\emptyset\rangle$. We must ensure that our entry states cohere; this will be done by carefully controlling the enumerations of the desired sets.

We will do this by induction on $e \in \omega \cup \{-1\}$. Assume that we have enumerations $\{U_{n,s}\}_{n\le e,s<\omega}$, $\{\hat{V}_{n,s}\}_{n\le e,s<\omega}$, $\{\hat{U}_{n,s}\}_{n\le e,s<\omega}$, and $\{V_{n,s}\}_{n\le e,s<\omega}$ such that

Conditions (3.1) through (3.5) are satisfied when restricted to e-states and $n \le e$. Furthermore assume that for all $n \le e$, we have sets $\mathfrak{E}_n$ and $\mathfrak{R}_n$ of n-states such that

(3.10) $\quad \nu \in \mathfrak{E}_n$ iff $D_\nu^{M_1}$ is infinite iff $D_\nu^{M_2}$ is infinite, and

$$\begin{aligned} &\nu \in \mathfrak{R}_n \text{ iff } \exists^\infty x \in \overline{M_1}, \nu(n,x) = \nu \text{ iff } \exists^\infty \hat{x} \in \overline{M_2}, \hat{\nu}(n,\hat{x}) = \nu \text{ iff} \\ &\text{for all } x \in \overline{M_1}, \text{ if there exists a stage } s \text{ such that } \nu(n,x,s) = \nu, \\ (3.11) \quad &\qquad \text{then } \nu(n,x) = \nu \\ &\text{for all } \hat{x} \in \overline{M_2}, \text{ if there exists a stage } s \text{ such that } \hat{\nu}(n,\hat{x},s) = \nu, \\ &\qquad \text{then } \hat{\nu}(n,\hat{x}) = \nu \end{aligned}$$

(where $D_\nu^{M_1}$ and $\nu(n,x,s)$ are measured w.r.t. $\{U_{n,s}\}_{n \le e, s<\omega}$ and $\{\widehat{V}_{n,s}\}_{n \le e, s<\omega}$, $\nu(n,x)$ w.r.t. $\{U_n\}_{n \le e}$ and $\{\widehat{V}_n\}_{n \le e}$, $D_\nu^{M_2}$ and $\hat{\nu}(n,\hat{x},s)$ w.r.t. $\{\widehat{U}_{n,s}\}_{n \le e, s<\omega}$ and $\{V_{n,s}\}_{n \le e, s<\omega}$, and $\hat{\nu}(n,\hat{x})$ w.r.t. $\{U_n\}_{n \le e}$ and $\{\widehat{V}_n\}_{n \le e}$). If $n = -1$, let $\mathfrak{E}_{-1} = \mathfrak{R}_{-1} = \{\langle -1, \emptyset, \emptyset \rangle\}$. Given this we will define the enumeration of U_{e+1}, $\widehat{V}_{e+1}$, $\widehat{U}_{e+1}$, and V_{e+1} as follows:

Assume that $e + 1 = 2m$. Hence we must ensure that $U_{e+1} =^* W_m$. For all s, let $\widehat{V}_{e+1,s} = V_{e+1,s} = \emptyset$. Let $\mathfrak{E}^*_{e+1} = \{\langle e+1, \sigma, \tau \rangle : \langle e, \sigma, \tau \rangle \in \mathfrak{E}_e\}$ and $\mathfrak{R}^*_{e+1} = \{\langle e+1, \sigma, \tau \rangle : \langle e, \sigma, \tau \rangle \in \mathfrak{R}_e\}$. There are two cases: either $W_m \cup M_1 =^* \omega$ or $W_m \subseteq^* M_1$. If $W_m \subseteq^* M_1$, we let $\mathfrak{E}_{e+1} = \mathfrak{E}^*_{e+1}$, $\mathfrak{R}_{e+1} = \mathfrak{R}^*_{e+1}$, $U_{e+1,s+1} = W_{m,s+1} \cap M_{1,s}$, and $\widehat{U}_{e+1,s+1} = \emptyset$. Assume $W_m \cup M_1 =^* \omega$. For all x, $\hat{x}$ and stages s, do the following: Assume $x \notin U_{e+1,s}$. We will add x to U_{e+1} at stage $s+1$ iff $x \in W_{m,s+1}$ and either $x \in M_{1,s}$ or $\nu(e+1,x,s) \in \mathfrak{R}^*_{e+1}$ and for all $\nu \in \mathfrak{E}^*_{e+1}$, $|D^{M_1}_{\nu,s+1}| \ge x$. Assume $\hat{x} \notin \widehat{U}_{e+1,s}$. We will add $\hat{x}$ to $\widehat{U}_{e+1}$ at stage $s+1$ iff $\hat{x} \notin M_{2,s}$, $\hat{\nu}(e+1,\hat{x},s) \in \mathfrak{R}^*_{e+1}$, and for all $\nu \in \mathfrak{E}^*_{e+1}$, $|D^{M_2}_{\nu,s+1}| \ge \hat{x}$. (Where $\nu(e+1,x,s)$ and $D_\nu^{M_1}$ are measured w.r.t. $\{U_{n,s}\}_{n \le e+1, s<\omega}$ and $\{\widehat{V}_{n,s}\}_{n \le e+1, s<\omega}$, and $\hat{\nu}(e+1,\hat{x},s)$ and $D_\nu^{M_2}$ are measured w.r.t. $\{\widehat{U}_{n,s}\}_{n \le e+1, s<\omega}$ and $\{V_{n,s}\}_{n \le e+1, s<\omega}$.) Let $\mathfrak{R}_{e+1} = \{\langle e+1, \sigma \cup \{e+1\}, \tau \rangle : \langle e, \sigma, \tau \rangle \in \mathfrak{R}_e\}$ and $\mathfrak{E}_{e+1} = \mathfrak{E}^*_{e+1} \cup \mathfrak{R}_{e+1}$.

By our enumeration if $\nu \in \mathfrak{E}^*_{e+1}$ then $D_\nu^{M_1}$ and $D_\nu^{M_2}$ are infinite. Since $\mathfrak{R}_e$ is the set of maximal e-states and M_1 and M_2 are maximal sets, $\mathfrak{R}_{e+1}$ is the set of maximal $(e+1)$-states and hence (3.11) holds. Since M_1 and M_2 are simple, if $\nu \in \mathfrak{R}_{e+1}$ then $D_\nu^{M_1}$ and $D_\nu^{M_2}$ are infinite. Since for an integer x to be raised into a maximal $(e+1)$-state, x must be in a maximal e-state, (3.10) holds for $\mathfrak{E}_{e+1}$. From (3.10) and (3.11) it is easy to see that the rest of the induction hypothesis holds. The case where $e+1$ is odd is done in a similar fashion. Hence the enumeration of $\{U_n\}_{n<\omega}$, $\{\widehat{V}_n\}_{n<\omega}$, $\{\widehat{U}_n\}_{n<\omega}$, and $\{V_n\}_{n<\omega}$ constructed in this manner will satisfy Conditions (3.1) through (3.5). Conditions (3.10) and (3.11) are *exactly* the special properties of maximal sets which allow us to conclude that all maximal sets are automorphic.

However, there is still one remaining problem. Why is this enumeration an uniformly $\mathbf{0}''$-enumeration? It should be clear that there are functions g_0, g_1, g_2, and g_3 recursive in $\mathbf{0}''$ such that for all e and s, $U_{e,s} = W_{g_0(e,s)}$, $\widehat{V}_{e,s} = W_{g_1(e,s)}$, $\widehat{U}_{e,s} = W_{g_2(e,s)}$, and $V_{e,s} = W_{g_3(e,s)}$. We need functions g_0, g_1, g_2, and g_3 recursive in $\mathbf{0}''$ such that for all e and s, $U_{e,s} = W_{g_0(e),s}$, $\widehat{V}_{e,s} = W_{g_1(e),s}$, $\widehat{U}_{e,s} = W_{g_2(e),s}$, and

$V_{e,s} = W_{g_3(e),s}$. To find such a function we must do the above construction on a tree and use the Recursion Theorem as follows:

Let $Tr = 2^{<\omega}$. At $\alpha \in Tr$, we will construct r.e. sets U_α, $\widehat{V}_\alpha$, $\widehat{U}_\alpha$, and V_α and an enumeration of these sets (we build U_α and its enumeration in a similar manner to the way we built U_{e+1} and its enumeration). The details of this construction are as follows: We will do this by induction on $\alpha \in Tr$. If $\alpha = \lambda$, let $\mathfrak{E}_\alpha = \mathfrak{R}_\alpha = \{\langle -1, \emptyset, \emptyset \rangle\}$ and for all s, $U_{\alpha,s} = \widehat{V}_{\alpha,s} = \widehat{U}_{\alpha,s} = V_{\alpha,s} = \emptyset$. Assume that we have enumerations $\{U_{\beta,s}\}_{\beta\subset\alpha,s<\omega}$, $\{\widehat{V}_{\beta,s}\}_{\beta\subset\alpha,s<\omega}$, $\{\widehat{U}_{\beta,s}\}_{\beta\subset\alpha,s<\omega}$, and $\{V_{\beta,s}\}_{\beta\subset\alpha,s<\omega}$, and sets $\mathfrak{E}_\beta$ and $\mathfrak{R}_\beta$ of $|\beta|$-states. Assume that $|\alpha| - 1 = 2m$. We will ensure that $U_\alpha =^* W_m$. For all s, let $\widehat{V}_{\alpha,s} = V_{\alpha,s} = \emptyset$. Let $\mathfrak{E}^*_\alpha = \{\langle |\alpha|, \sigma, \tau \rangle : \langle e, \sigma, \tau \rangle \in \mathfrak{E}_{\alpha^-}\}$ and $\mathfrak{R}^*_\alpha = \{\langle |\alpha|, \sigma, \tau \rangle : \langle e, \sigma, \tau \rangle \in \mathfrak{R}_{\alpha^-}\}$. There are two cases: either $\alpha = \alpha^{-}\hat{\ }0$ or $\alpha = \alpha^{-}\hat{\ }1$ (this will be used to code whether $W_m \cup M_1 =^* \omega$ or $W_m \subseteq^* M_1$). If $\alpha = \alpha^{-}\hat{\ }0$, we let $\mathfrak{E}_\alpha = \mathfrak{E}^*_\alpha$, $\mathfrak{R}_\alpha = \mathfrak{R}^*_\alpha$, $U_{\alpha,s+1} = W_{m,s+1} \cap M_{1,s}$, and $\widehat{U}_{\alpha,s+1} = \emptyset$. Assume $\alpha = \alpha^{-}\hat{\ }1$. For all x, $\hat{x}$, and stages s, do the following: Assume $x \notin U_{\alpha,s}$. We will add x to U_α at stage $s+1$ iff $x \in W_{m,s+1}$ and either $x \in M_{1,s}$ or $\nu(|\alpha|, x, s) \in \mathfrak{R}^*_\alpha$ and for all $\nu \in \mathfrak{E}^*_\alpha$, $|D^{M_1}_{\nu,s+1}| \geq x$. Assume $\hat{x} \notin \widehat{U}_{\alpha,s}$. We will add $\hat{x}$ to $\widehat{U}_\alpha$ at stage $s+1$ iff $\hat{x} \notin M_{2,s}$, $\hat{\nu}(|\alpha|, \hat{x}, s) \in \mathfrak{R}^*_\alpha$, and for all $\nu \in \mathfrak{E}^*_\alpha$, $|D^{M_2}_{\nu,s+1}| \geq \hat{x}$. (Where $D^{M_1}_\nu$ and $\nu(|\alpha|, x, s)$ are measured w.r.t. $\{U_{\beta,s}\}_{\beta\subseteq\alpha,s<\omega}$ and $\{\widehat{V}_{\beta,s}\}_{\beta\subseteq\alpha,s<\omega}$, and $D^{M_2}_\nu$ and $\hat{\nu}(|\alpha|, \hat{x}, s)$ are measured w.r.t. to $\{\widehat{U}_{\beta,s}\}_{\beta\subseteq\alpha,s<\omega}$ and $\{V_{\beta,s}\}_{\beta\subseteq\alpha,s<\omega}$.) Let $\mathfrak{R}_\alpha = \{\langle |\alpha|, \sigma \cup \{|\alpha|\}, \tau \rangle : \langle e, \sigma, \tau \rangle \in \mathfrak{R}_\alpha\}$ and $\mathfrak{E}_\alpha = \mathfrak{E}^*_\alpha \cup \mathfrak{R}_\alpha$.

By the Recursion Theorem there are recursive functions h_0, h_1, h_2, and h_3 from Tr into ω such that $U_{\alpha,s} = W_{h_0(\alpha),s}$, $\widehat{V}_{\alpha,s} = W_{h_1(\alpha),s}$, $\widehat{U}_{\alpha,s} = W_{h_2(\alpha),s}$, and $V_{\alpha,s} = W_{h_3(\alpha),s}$. Using $\mathbf{0}''$ choose an infinite branch f through Tr as follows: $\lambda \subseteq f$, if $\alpha \subseteq f$ and $|\alpha| = 2m$ then $\alpha\hat{\ }1 \subseteq f$ iff $W_m \cup M_1 =^* \omega$, and if $\alpha \subseteq f$ and $|\alpha| = 2m + 1$ then $\alpha\hat{\ }1 \subseteq f$ iff $W_m \cup M_2 =^* \omega$. If $\alpha \subset f$ and $|\alpha| = e + 1$ then $U_{e,s} = W_{h_0(\alpha),s}$, $\widehat{V}_{e,s} = W_{h_1(\alpha),s}$, $\widehat{U}_{e,s} = W_{h_2(\alpha),s}$, and $V_{e,s} = W_{h_3(\alpha),s}$. Hence we have found an uniformly $\mathbf{0}''$-enumeration of $\{M_{1,s}\}_{s<\omega}$, $\{M_{2,s}\}_{s<\omega}$, $\{U_{n,s}\}_{n,s<\omega}$, $\{\widehat{V}_{n,s}\}_{n,s<\omega}$, $\{\widehat{U}_{n,s}\}_{n,s<\omega}$, and $\{V_{n,s}\}_{n,s<\omega}$ satisfying Conditions (3.1) through (3.5). Therefore M_1 and M_2 are automorphic sets.

4 Orbits of hyperhypersimple sets

Let H_1 and H_2 be hyperhypersimple sets. Fix some enumeration of H_1 and H_2. Recall from [8], that $\mathscr{L}^*(H)$ is the lattice of r.e. supersets of H modulo the finite sets. We say that Ψ is a Σ_3-isomorphism from $\mathscr{L}^*(H_1)$ to $\mathscr{L}^*(H_2)$ iff Ψ is an isomorphism from $\mathscr{L}^*(H_1)$ to $\mathscr{L}^*(H_2)$ and there is a total Σ_3-function h such that $\Psi(W_e \cup H_1) =^* (W_{h(e)} \cup H_2)$. Assume that Ψ is a Σ_3-isomorphisms from $\mathscr{L}^*(H_1)$ to $\mathscr{L}^*(H_2)$. Hence

$$\begin{array}{ll} & \exists^\infty x \in \overline{H_1} \text{ with final } e\text{-state } \nu \text{ w.r.t. to } \{W_n\}_{n<\omega} \text{ and } \{W_{h^{-1}(n)}\}_{n<\omega} \\ (4.1) & \text{iff} \\ & \exists^\infty \hat{x} \in \overline{H_2} \text{ with final } e\text{-state } \nu \text{ w.r.t. to } \{W_{h(n)}\}_{n<\omega} \text{ and } \{W_n\}_{n<\omega} \end{array}$$

(as above we will consider $\hat{\omega}$ as a copy of ω; integers from $\hat{\omega}$ will always wear hats; H_1 as a subset of ω; and H_2 as a subset of $\hat{\omega}$). Maass [6] showed that H_1 and H_2

are automorphic sets. We will now, using the above format, provide a new proof of this result.

Before we continue with this proof we will quickly review some needed facts about hyperhypersimple sets and Σ_3-functions. If h is total and a Σ_3-function, then h is recursive in $\mathbf{0}''$. By Lachlan [5] (see (8, X.2.8]), we know that for all e there is a least n_e such that

$$W_e \cap W_{n_e} \subseteq H_1 \quad \text{and} \quad W_e \cup W_{n_e} \cup H_1 = \omega\,, \tag{4.2}$$

and similarly for H_2 for all e there is a least $\hat{n}_e$ such that

$$W_e \cap W_{\hat{n}_e} \subseteq H_2 \quad \text{and} \quad W_e \cup W_{\hat{n}_e} \cup H_2 = \omega \tag{4.3}$$

(in this case think of W_e and $W_{\hat{n}_e}$ as subsets of $\hat{\omega}$). In addition, we will make the further assumption that for all e,

$$W_e \cap W_{n_e} \backslash H_1 = \emptyset \quad \text{and} \quad W_e \cap W_{\hat{n}_e} \backslash H_2 = \emptyset \tag{4.4}$$

for the above enumeration of H_1 and H_2. Furthermore the functions $g(e) = n_e$ and $\hat{g}(e) = \hat{n}_e$ are recursive in $\mathbf{0}''$.

We are using the Modified Extension Theorem to help construct the desired automorphism. As above, it is enough to find uniformly $\mathbf{0}''$-enumerations $\{H_{1,s}\}_{s<\omega}$, $\{H_{2,s}\}_{s<\omega}$, $\{U_{n,s}\}_{n,s<\omega}$, $\{\widehat{V}_{n,s}\}_{n,s<\omega}$, $\{\widehat{U}_{n,s}\}_{n,s<\omega}$, and $\{V_{n,s}\}_{n,s<\omega}$ of the uniformly $\mathbf{0}''$-recursive collection of r.e. sets H_1, H_2, $\{U_n\}_{n<\omega}$, $\{\widehat{V}_n\}_{n<\omega}$, $\{\widehat{U}_n\}_{n<\omega}$, and $\{V_n\}_{n<\omega}$ satisfying Conditions (3.1), (3.2), (3.3) and (3.4) and the following Condition:

$$\begin{aligned}
&\text{if } n = 4m \text{ then } U_n =^* W_m \text{ and } V_n = \emptyset\,,\\
&\text{if } n = 4m+1 \text{ then } U_n \cap \overline{H_1} =^* W_{g(m)} \cap \overline{H_1} \text{ and } V_n = \emptyset\,,\\
&\text{if } n = 4m+2 \text{ then } V_n =^* W_m \text{ and } U_n = \emptyset\,,\\
&\text{if } n = 4m+3 \text{ then } V_n \cap \overline{H_2} =^* W_{\hat{g}(m)} \cap \overline{H_2} \text{ and } U_n = \emptyset\,.
\end{aligned} \tag{4.5}$$

(where H_1 will play the role of M_1 and H_2 will play the role of M_1 in the Conditions from Sect. 3).

We will focus on meeting these Conditions. We will use the above enumerations of H_1 and H_2. To meet (3.1), we will not enumerate integers into $\widehat{U}_n$ ($\widehat{V}_n$) once they have entered H_2 (H_1). Since we will meet (4.5), we can let $\widehat{V}_{4e} = \widehat{V}_{4e+1} = \widehat{U}_{4e+2} = \widehat{V}_{4e+3} = \emptyset$.

A first (again failed) attempt to meet (3.5) might go as follows: let $U_{4e} =^* W_e$, $\widehat{U}_{4e} =^* W_{h(e)}$, $U_{4e+1} =^* W_{g(e)}$, $\widehat{U}_{4e+1} =^* W_{\hat{g}(h(e))}$, $V_{4e+2} =^* W_e$, $\widehat{V}_{4e+2} =^* W_{h^{-1}(e)}$, $V_{4e+2} =^* W_{\hat{g}(e)}$, and $\widehat{V}_{4e+2} =^* W_{g(h^{-1}(e))}$ (without choosing any enumeration of these sets). By Conditions (4.1), (4.2), and (4.3), for all e, $W_{h(g(e))} \cap \overline{H_2} =^* W_{\hat{g}(h(e))} \cap \overline{H_2}$ and $W_{h^{-1}(\hat{g}(e))} \cap \overline{H_1} =^* W_{g(h^{-1}(e))} \cap \overline{H_1}$ and therefore Condition (3.5) holds. But this may not meet the entry Conditions (3.2) and (3.3) (again we can produce an example as above). Hence, in addition to constructing these sets, we also must construct the enumerations of these sets.

We will do this by induction on $e \in \omega \cup \{-1\}$. Assume that we have enumerations $\{U_{n,s}\}_{n\leq 2e,s<\omega}$, $\{\widehat{V}_{n,s}\}_{n\leq 2e,s<\omega}$, $\{\widehat{U}_{n,s}\}_{n\leq 2e,s<\omega}$, and $\{V_{n,s}\}_{n\leq 2e,s<\omega}$, such that Conditions (3.1), (3.2), (3.3), (3.5), and (4.5) are satisfied when restricted to n and

n-states where $n \leq 2e$. Furthermore assume that for all $n \leq e$, we have sets $\mathfrak{C}_n$ and $\mathfrak{R}_n$ of $2n$-states such that $\mathfrak{R}_n \subseteq \mathfrak{C}_n$ and

$$\nu \in \mathfrak{C}_n \text{ iff } D_\nu^{H_1} \text{ is infinite iff } D_\nu^{H_2} \text{ is infinite,} \tag{4.6}$$

$$\text{if } \exists^\infty x \in \overline{H_1}, \nu(2n, x) = \nu \text{ or } \exists^\infty \hat{x} \in \overline{H_2}, \hat{\nu}(2n, \hat{x}) = \nu \text{ then } \nu \in \mathfrak{R}_n, \text{ and} \tag{4.7}$$

$$\begin{gathered} \text{if } \nu \in \mathfrak{R}_n \text{ then} \\ \text{for all } x \in \overline{H_1}, \text{ if there exists a stage } s \text{ such that } \nu(2n, x, s) = \nu, \\ \text{then } \nu(2n, x) = \nu, \text{ and} \\ \text{for all } \hat{x} \in \overline{H_2}, \text{ if there exists a stage } s \text{ such that } \hat{\nu}(2n, \hat{x}, s) = \nu, \\ \text{then } \hat{\nu}(2n, \hat{x}) = \nu \end{gathered} \tag{4.8}$$

(where $D_\nu^{H_1}$ and $\nu(2n, x, s)$ are measured w.r.t. $\{U_{n,s}\}_{n \leq 2e, s<\omega}$ and $\{\hat{V}_{n,s}\}_{n \leq 2e, s<\omega}$, $\nu(2n, x)$ w.r.t. $\{U_n\}_{n \leq 2e}$ and $\{\hat{V}_n\}_{n \leq 2e}$, $D_\nu^{H_2}$ and $\hat{\nu}(2n, \hat{x}, s)$ w.r.t. $\{\hat{U}_{n,s}\}_{n \leq 2e, s<\omega}$ and $\{V_{n,s}\}_{n \leq 2e, s<\omega}$, and $\hat{\nu}(2n, \hat{x})$ w.r.t. $\{U_n\}_{n \leq 2e}$ and $\{\hat{V}_n\}_{n \leq 2e}$). If $n = -1$, let $\mathfrak{C}_{-1} = \mathfrak{R}_{-1} = \{\langle -1, \emptyset, \emptyset \rangle\}$.

Since Conditions (4.7) and (4.8) together are weaker than Condition (3.11), we will have more difficulties constructing our enumeration. These two Conditions are weaker than Condition (3.11), since we cannot tell using $\mathbf{0}''$ whether $W_e \cap \overline{H_1} =^* \emptyset$ or not (that would imply that H_1 and H_2 are semi-low$_2$). Given this we will define the enumeration of U_{e+1}, $\hat{V}_{e+1}$, $\hat{U}_{e+1}$, V_{e+1}, U_{e+2}, $\hat{V}_{e+2}$, $\hat{U}_{e+2}$, and V_{e+2} as follows:

Until otherwise noted $`\nu(2e+2, x, s)$ and $`D_\nu^{H_1}$ are $\nu(2e+2, x, s)$ and $D_\nu^{H_1}$ measured w.r.t. $\{X_{n,s}\}_{n \leq 2e+1, s<\omega}$ and $\{Y_{n,s}\}_{n \leq 2e+1, s<\omega}$, where for $n \leq 2e$, $X_{n,s} = U_{n,s}$ and $Y_{n,s} = \hat{V}_{n,s}$, $X_{2e+1,s} = W_{m,s}$, $X_{2e+2,s} = W_{g(m),s}$, $Y_{2e+1,s} = \emptyset$, and $Y_{2e+2,s} = \emptyset$; $`\nu(2e+2, x)$ is $\nu(2e+2, x)$ measured w.r.t. $\{X_n\}_{n \leq 2e+1}$ and $\{Y_n\}_{n \leq 2e+1}$, where for $n \leq 2e$, $X_n = U_n$ and $Y_n = \hat{V}_n$, $X_{2e+1} = W_m$, $X_{2e+2} = W_{g(m)}$, $Y_{2e+1} = \emptyset$, and $Y_{2e+2} = \emptyset$; $`\hat{\nu}(2e+2, \hat{x}, s)$ and $`D_\nu^{H_2}$ are $\nu(2e+2, \hat{x}, s)$ and $D_\nu^{H_2}$ measured w.r.t. $\{X_{n,s}\}_{n \leq 2e+1, s<\omega}$ and $\{Y_{n,s}\}_{n \leq 2e+1, s<\omega}$, where for $n \leq 2e$, $X_{n,s} = \hat{U}_{n,s}$ and $Y_{n,s} = V_{n,s}$, $X_{2e+1,s} = W_{h(m),s}$, $X_{2e+2,s} = W_{\hat{g}(h(m)),s}$, $Y_{2e+1,s} = \emptyset$, and $Y_{2e+2,s} = \emptyset$; $`\hat{\nu}(2e+2, x)$ is $\nu(2e+2, x)$ measured w.r.t. $\{X_n\}_{n \leq 2e+1}$ and $\{Y_n\}_{n \leq 2e+1}$, where for $n \leq 2e$, $X_n = \hat{U}_n$ and $Y_n = V_n$, $X_{2e+1} = W_{h(m)}$, $X_{2e+2} = W_{\hat{g}(h(m))}$, $Y_{2e+1} = \emptyset$, and $Y_{2e+2} = \emptyset$; $D_\nu^{H_1}$ and $\nu(2e+2, x, s)$ are measured w.r.t. $\{U_{n,s}\}_{n \leq 2e+2, s<\omega}$ and $\{\hat{V}_{n,s}\}_{n \leq 2e+2, s<\omega}$; $\nu(2e+2, x)$ w.r.t. $\{U_n\}_{n \leq 2e+2}$ and $\{\hat{V}_n\}_{n \leq 2e+2}$; $D_\nu^{H_2}$ and $\hat{\nu}(2e+2, \hat{x}, s)$ w.r.t. $\{\hat{U}_{n,s}\}_{n \leq 2e+2, s<\omega}$ and $\{V_{n,s}\}_{n \leq 2e+2, s<\omega}$; and $\hat{\nu}(2e+2, \hat{x})$ w.r.t. $\{U_n\}_{n \leq 2e+1}$ and $\{\hat{V}_n\}_{n \leq 2e+1}$.

Assume that $e+1 = 4m$. We will ensure that $U_{e+1} =^* W_m$, $\hat{U}_{e+1} \cap \overline{H_2} =^* W_{h(m)} \cap \overline{H_2}$, $U_{e+2} \cap \overline{H_1} =^* W_{g(m)} \cap \overline{H_1}$ and $U_{4e+2} \cap \overline{H_2} =^* W_{\hat{g}(h(m))} \cap \overline{H_2}$. For all s, let $\hat{V}_{e+1,s} = V_{e+1,s} = \hat{V}_{e+2,s} = V_{e+2,s} = \emptyset$. Using $\mathbf{0}''$, let $\mathfrak{R}_{e+1}$ be the set of $(2e+2)$-states such that $\nu = \langle (2e+2), \sigma, \tau \rangle \in \mathfrak{R}_{e+1}$ iff $\nu \restriction 2e \in \mathfrak{R}_e$, $`D_\nu^{H_1}$ is infinite, $`D_\nu^{H_2}$ is infinite, and either $2e+1 \in \sigma$ or $2e+2 \in \sigma$ (by Condition (4.4), $\{2e+1, 2e+2\} \not\subseteq \sigma$). Let $\mathfrak{C}_{e+1}^* = \{\langle 2e+2, \sigma, \tau \rangle : \langle 2e, \sigma, \tau \rangle \in \mathfrak{C}_e\}$ and $\mathfrak{C}_{e+1} = \mathfrak{C}_{e+1}^* \cup \mathfrak{R}_{e+1}$. For all x, $\hat{x}$ and stages s, do the following. Assume $x \notin U_{e+1,s} \cup U_{e+2,s}$. We will add x to U_{e+1} at stage $s+1$ iff $x \in W_{m,s+1}$ and either $x \in H_{1,s}$ or $`\nu(2e+2, x, s+1) = \nu^* \in \mathfrak{R}_{e+1}$, and either for all $\nu \in \mathfrak{C}_{e+1}^*$,

$|D^{H_1}_{\nu,s}| \geq x$ or $x \in H_{1,s+1} - H_{1,s}$ and for all $\nu \in \mathfrak{E}^*_{e+1}$, $|D^{H_1}_{\nu,s}| \geq |D^{H_1}_{\nu^*,s}|$. Add x to U_{e+2} at stage $s+1$ iff $x \in W_{\hat{g}(h(m)),s+1}$, ${}^{\backprime}\hat{\nu}(2e+2,x,s+1) = \nu^* \in \mathfrak{R}_{e+1}$, and either for all $\nu \in \mathfrak{E}^*_{e+1}$, $|D^{H_1}_{\nu,s}| \geq x$ or $x \in H_{1,s+1} - H_{1,s}$ and for all $\nu \in \mathfrak{E}^*_{e+1}$, $|D^{H_1}_{\nu,s}| \geq |D^{H_1}_{\nu^*,s}|$. Assume $\hat{x} \notin \hat{U}_{e+1,s} \cup \hat{U}_{e+2,s}$. Add $\hat{x}$ to $\hat{U}_{e+1}$ at stage $s+1$ iff $\hat{x} \in W_{h(m),s+1}$, ${}^{\backprime}\hat{\nu}(2e+2,\hat{x},s+1) = \nu^* \in \mathfrak{R}_{e+1}$, and either for all $\nu \in \mathfrak{E}^*_{e+1}$, $|D^{H_2}_{\nu,s}| \geq \hat{x}$ or $\hat{x} \in H_{2,s+1} - H_{2,s}$ and for all $\nu \in \mathfrak{E}^*_{e+1}$, $|D^{H_2}_{\nu,s}| \geq |D^{H_2}_{\nu^*,s}|$. Add $\hat{x}$ to $\hat{U}_{e+2}$ at stage $s+1$ iff $\hat{x} \in W_{\hat{g}(h(m)),s+1}$, ${}^{\backprime}\nu(2e+2,\hat{x},s+1) = \nu^* \in \mathfrak{R}_{e+1}$, and either for all $\nu \in \mathfrak{E}^*_{e+1}$, $|D^{H_2}_{\nu,s}| \geq \hat{x}$ or $\hat{x} \in H_{2,s+1} - H_{2,s}$ and for all $\nu \in \mathfrak{E}^*_{e+1}$, $|D^{H_2}_{\nu,s}| \geq |D^{H_2}_{\nu^*,s}|$.

We will now show that this enumeration satisfies the desired properties. Clearly Condition (3.1) holds for $2e+1$ and $2e+2$. By Conditions (4.4) and (4.8), for all $x\,(\hat{x})$ if ${}^{\backprime}\nu(2e+2,x,s) = \nu \in \mathfrak{R}_{e+1}$ (${}^{\backprime}\hat{\nu}(2e+2,\hat{x},s) = \nu \in \mathfrak{R}_{e+1}$) then for all $t \geq s$, if $x \notin H_{1,t+1}$ ($\hat{x} \notin H_{2,t+1}$) then ${}^{\backprime}\nu(2e+2,x,t+1) = \nu$ (${}^{\backprime}\hat{\nu}(2e+2,\hat{x},s) = \nu$). Therefore Condition (4.8) holds for $\mathfrak{R}_{e+1}$. By induction on l, we can show for all $\nu \in \mathfrak{E}_{e+1}$, $|D^{H_1}_{\nu}| \geq l$ and $|D^{H_2}_{\nu}| \geq l$. Now using the fact that Condition (4.8) holds for $\mathfrak{R}_{e+1}$, it is clear that Condition (4.6) holds for $\mathfrak{E}_{e+1}$ and hence Conditions (3.2) and (3.3) hold for $(2e+2)$-states.

Assume $X = \{x : x \in \overline{H_1}$ and $\nu(2e+2,x) = \nu = \langle 2e+2,\sigma,\tau\rangle\}$ is infinite. We will show that $\exists^\infty \hat{x} \in \overline{H_2}, \hat{\nu}(2e+2,x) = \nu$, $\nu \in \mathfrak{R}_{e+1}$, and either $2e+1 \in \sigma$ or $2e+2 \in \sigma$. By the induction hypothesis, $\exists^\infty \hat{x} \in \overline{H_2}, \hat{\nu}(2e,x) = \nu \restriction 2e$ and $\nu \restriction 2e \in \mathfrak{R}_e$. There exists an infinite subset Y of X such that for all $x \in Y$, ${}^{\backprime}\nu(2e+2,x) = \nu^* = \langle 2e+2,\sigma^*,\tau^*\rangle$ where $\nu^* \restriction 2e = \nu \restriction 2e$. Since $\overline{H_1} \subseteq W_m \cup W_{h(m)}$, $2e+1 \in \sigma^*$ or $2e+2 \in \sigma^*$. By the choice of h, g, and $\hat{g}$, $\exists^\infty \hat{x} \in \overline{H_2}$, ${}^{\backprime}\hat{\nu}(2e+2,x) = \nu^*$. Since $H_1\,(H_2)$ is simple, ${}^{\backprime}D^{H_1}_{\nu^*}$ (${}^{\backprime}D^{H_2}_{\nu^*}$) is infinite. Therefore $\nu \in \mathfrak{R}_{e+1}$. Hence for all $x \notin H_1$, if ${}^{\backprime}\nu(2e+2,x) = \nu^*$ then $\nu(2e+2,x) = \nu^*$ and for all $\hat{x} \notin H_2$, if ${}^{\backprime}\hat{\nu}(2e+2,\hat{x}) = \nu^*$ then $\hat{\nu}(2e+2,\hat{x}) = \nu^*$. So $\nu^* = \nu$, $\exists^\infty \hat{x} \in \overline{H_2}, \hat{\nu}(2e+2,x) = \nu$, and either $2e+1 \in \sigma$ or $2e+2 \in \sigma$. Using similar reasoning we can show that Conditions (3.5) and (4.7) hold for $(2e+2)$-states and $U_{e+1} =^* W_m$, $\hat{U}_{e+1} \cap \overline{H_2} =^* W_{h(m)} \cap \overline{H_2}$, $U_{e+2} \cap \overline{H_1} =^* W_{g(m)} \cap \overline{H_1}$ and $\hat{U}_{4e+2} \cap \overline{H_2} =^* W_{\hat{g}(h(m))} \cap \overline{H_2}$.

The case where $e+1 = 4m+2$ is done in a similar fashion. Hence the enumeration of $\{U_n\}_{n<\omega}$, $\{\hat{V}_n\}_{n<\omega}$, $\{\hat{U}_n\}_{n<\omega}$, and $\{V_n\}_{n<\omega}$ constructed in this manner will satisfy Conditions (3.1), (3.2), (3.3), (3.5), and (4.5). As before to show that this enumeration is an uniformly $\mathbf{0}''$-enumeration we must translate the above construction to one done on a tree; a construction where we receive the needed information through the tree rather than using an oracle for $\mathbf{0}''$. We use the Recursion Theorem to find the indices for the r.e. sets constructed at each node and an oracle for $\mathbf{0}''$ to pick out a *correct* path through the tree and hence the indices witness the fact that our enumeration is an uniformly $\mathbf{0}''$-enumeration as desired. Other than defining a possible tree for this construction, we will not provide any details.

We will define a tree Tr by induction. First $\lambda \in Tr$. Assume that $\alpha \in Tr$, when $\alpha\hat{\ }\langle m_0, m_1, m_2, m_3, \mathfrak{R}\rangle$, where $|\alpha| = m_0$, $m_i \in \omega$, and $\mathfrak{R}$ is a set of $(2|\alpha|+2)$-states such that if $\alpha \neq \lambda$ and $\alpha = \beta\hat{\ }\langle m_0', m_1', m_2', m_3', \mathfrak{R}'\rangle$ then $\mathfrak{R} \restriction 2|\alpha| = \mathfrak{R}'$. When we translate the above inductive step to one done at some node of the tree, m_0 will play the role of m, m_1 will play the role of $h(m)$, m_2 will play the role of $g(m)$, m_3

will play the role of $\hat{g}(h(m))$, and $\mathfrak{R}$ the role of $\mathfrak{R}_{e+1}$. The rest of the details follow without much difficulty.

Hence we have found uniformly $\mathbf{0}''$-enumerations $\{H_{1,s}\}_{s<\omega}$, $\{H_{2,s}\}_{s<\omega}$, $\{U_{n,s}\}_{n,s<\omega}$, $\{\hat{V}_{n,s}\}_{n,s<\omega}$, $\{\hat{U}_{n,s}\}_{n,s<\omega}$, and $\{V_{n,s}\}_{n,s<\omega}$ satisfying Conditions (3.1), (3.2), (3.3), (3.5), and (4.5). Therefore H_1 and H_2 are automorphic sets.

5 Conclusion

We would like to point out that Downey and Stob's result [3] that the hemimaximal sets form an orbit and their work on orbits of Friedberg splittings of hyperhypersimple sets can also be recast in this format. This follows in a natural fashion after the proof in Sects. 3 and 4. Much of Downey and Stob's work [4] on e-splittings and e^*-splittings (see [4] for definitions) and orbits can also be recast in this format.

One of the aspects that all of the proofs in this paper have in common is that they all use a tree to provide information recursive in $\mathbf{0}''$. This is also similar to the Δ_3-automorphism techniques. In fact, one can combine the proof of Soare's Extension Theorem, the Extension Theorem, and the proof in Sect. 3 (or Sect. 4) to produce a *single* tree argument showing that the maximal sets form an orbit (or Maass's result on the orbits of hyperhypersimple sets). Such a proof may be shorter but we believe such a proof would hide the exact properties about maximal and hyperhypersimple sets which allowed us to prove these results. By proving these results in pieces, we believe that these properties are more obvious to the reader.

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