An Existence Theorem in the Calculus of Variations Based on Sobolev's Imbedding Theorems

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1. Introduction

This paper deals with the minimum problem concerning an integral of the form

$$I(x)=\int_{\Omega} f(t,x,x^{(1)},\ldots,x^{(m)},\ldots,x^{(l)})\,dt \tag{1.1}$$

where l is an arbitrary positive integer, where $x^{(m)}$ denotes the vector $x^{(m)}=(x_1^{(m)},\ldots,x_{\sigma_m}^{(m)})$ of the derivatives of x of order m taken in an arbitrary but fixed order, and where Ω is a bounded open domain in the n-dimensional Euclidean space E^n of points $t=(t_1,\ldots,t_n)$.

In certain basic aspects (*e.g.* in the use of convexity considerations and of the reflexitivity of the Sobolev spaces) the method of the present paper is the same as the one used in a recent paper by F. E. BROWDER [*3*][1], while in other aspects the treatment is different. This will be clear from the following outline of the existence proof given in the present paper:

The Sobolev space $W_p^l=W_p^l(\Omega)$ is a reflexive Banach space. Therefore, the closed ball $B_R\subset W_p^l$ with radius R and center 0 is weakly compact, *i.e.* compact in the relative topology of B_R induced by the weak topology of W_p^l. Consequently for the proof of the existence of an $x_0\in B_R$ minimizing $I(x)$ in B_R it will be sufficient to show that $I(x)$ is weakly lower semi-continuous (see [*10*]). To do this, we use the notation

$$f(x;y)=f(t,x^{(1)},\ldots,x^{(l-1)},y^{(l)}), \tag{1.2}$$

$$I(x;y)=\int_{\Omega} f\big(t,x(t);y(t)\big)\,dt \tag{1.3}$$

such that

$$I(x)=I(x;x). \tag{1.4}$$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

Under assumptions to be specified later the following statements α) and β) will be proved:

α) for fixed $x_0 \in B_R$, $I_0(y) = I(x_0; y)$ is weakly lower semi-continuous in y,

β) for fixed $y \in B_R$, $I(x; y)$ is weakly continuous in x.

As to the proof of α) a nearly obvious argument (given in detail in [*11*; theorem 4.1]) shows that it will be sufficient to prove: to each, y_0 in B_R there exists a bounded linear functional $l_0 = l_0(y)$ on W_p^l such that

$$I_0(y) - I_0(y_0) \geqq l_0(y - y_0), \qquad y, y_0 \in B_R. \tag{1.5}$$

The main assumption for the proof of (1.5) will be a convexity assumption on f with respect to the highest derivatives which implies a corresponding convexity of $I(x; y)$ with respect to y [2]. A linear functional l_0 satisfying (1.5) is then obtained as follows: if (for x fixed) Π denotes the product space of W_p^l with the real line consisting of couples $u = (y, r)$, then the set $\Pi_1 \subset \Pi$ above the graph $r = I_0(y)$, i.e. the set of points (y, r) with $r \geqq I_0(y)$ is convex and has therefore under rather general conditions a closed supporting hyperplane at the point $u_0 = (y_0, I_0(y_0))$ given by an equation of the form

$$h_0(u) = h_0(u_0) \tag{1.6}$$

where h_0 is a continuous linear functional on Π. It is then proved that the restriction of h_0 to W_p^l multiplied by a proper constant is an l_0 of the desired properties (section 3).

The proof of the statement β) above is given in section 4. It is based on the extended form (4.2) of the Friedrichs-Sobolev inequality (4.1). The latter is proved in Appendix A by applying SOBOLEV's imbedding theorems. The existence of a minimizing $x_0 \in B_R$ then follows easily by combining the results of sections 3 and 4 (section 5).

The inequality (4.1) is a generalization of the Friedrichs inequality given for the case $l = 1$ in an inner product space [*4*; p. 489]. (See also [*7*; section 5] for a more general case.) From the point of view taken in the present paper an essential feature of FRIEDRICH's inequality and its generalization (4.2) is that they immediately imply the weak continuity of the L_p norm of the derivatives of order $< l$ of elements $x \in B_R$. (See Remark 1 at the end of section 4.)

The compactness theorem known as RELLICH's lemma (Rellichscher Auswahlsatz) [*4*; p. 489] is a direct consequence of FRIEDRICH's original inequality. Correspondingly it is shown in Appendix *B* that inequality (4.2) implies immediately a compactness theorem which may be considered as a generalization of RELLICH's lemma. This theorem is not new. It is a special case of KONDRASEV's theorem on the complete continuity of the SOBOLEV imbedding operator [*13*; chapter I, § 11].

[2] Instead of a convexity assumption, a positivity assumption on the second differential (implying convexity) was used in [*11*]. The author is indebted to G. MINTY for pointing out in conversations that convexity would be sufficient. Moreover in that paper in which the case $l = 1$ was treated the assumption just mentioned was made not only with respect to the first derivatives of x but also with respect to x itself. This resulted in a theorem of unnecessarily narrow scope as was pointed by L. M. GRAVES [Math. Reviews **15**, 39 (1954)].

2. Preliminaries

Ω will always be a bounded open domain in E^n. Such an Ω is called a Sobolev domain if the Sobolev imbedding theorems [*13*; pp. 56, 57] hold. Sufficient conditions for this to be the case may be found in [*13*; p. 66] or [*12*; chapter IV].

By "derivative" of a function $x=x(t)$ defined in Ω we will always mean generalized derivative as defined in [*13*; p. 33]. The L_p spaces and the norm $\|x\|_p$ of an $x\in L_p$ are defined as usual. We always assume $p>1$.

For any positive integer l the Banach space $W_p^l=W_p^l(\Omega)$ is then defined as the space of all $x\in L_p(\Omega)$ which have derivatives in $L_p(\Omega)$ of order up to and including l, while the norm of x is defined by

$$\|x\|_{W_p^l}=(\|x\|_p^p+\|x\|_{l,p}^p)^{1/p}. \tag{2.1}$$

Here $\|x\|_{l,p}$ is defined as follows: let $\|x\|_E$ denote the Euclidean norm of the vector $x^{(l)}$ defined in the first paragraph of the introduction; then

$$\|x\|_{l,p}=\Big(\int_\Omega \|x^{(l)}\|_E^p\,dt\Big)^{1/p}. \tag{2.2}$$

Theorem 2.1. *The Banach space W_p^l is reflexive.*

Proof. If the norm $\|x\|_{l,p}$ were re-defined by replacing in its definition (2.2) the Euclidean norm $\|x\|_E$ by the p norm, *i.e.* by

$$\|x\|=\Big[\sum_{i=1}^{l_\sigma}(x_i^{(l)})^p\Big]^{1/p},$$

then the proof would be the same as the one given for theorem 2.1 in [*11*]. (*Cf.* also [*2*; p. 863] and [*3*] where this norm is used.) However a perusal of that proof makes clear how it should be modified to meet our case by using the following:

Lemma 2.1. *For any positive s let $\|y\|_{E_s}$ be the Euclidean norm of $y=(y_1,\ldots,y_s)$. Then the space of all vector functions $y(t)=(y,(t),\ldots,y_s(t))$ for which $\|y(t)\|_{E_s}\in L_p$ with norm*

$$\|y\|=\Big\{\int_\Omega \|y(t)\|_{E_s}^p\,dt\Big\}^{1/p} \tag{2.3}$$

is uniformly convex.

Proof. Since L_p is uniformly convex, it is easily seen that the proof given by M. M. Day for his theorem 3 in [*5*] is valid for the present case. (*Cf.* also the concluding remarks in [*5*].)

We now specify the assumptions on the integrand f of the integral (1.1). t and σ_m ($m=0, 1, \ldots, l$) are defined as in the introduction, and $p^{(m)}$ is a vector with components $p_1^{(m)}, \ldots, p_{\sigma_m}^m$. Then the function $f(t, p^{(0)}, \ldots, p^{(l)})$ is supposed to satisfy the following four conditions:

A) f satisfies a Hoelder (or Lipschitz) condition with respect to the variables $p^{(0)}, p^{(1)}, \ldots, p^{(l-1)}$.

B) f is convex with respect $p^{(l)}$.

C) For any couple x, y of elements of B_{2R}, $f(x; y)$ (see (1.2)) exists and is integrable over Ω.

D) To every couple x_0, y_0 in B_R there exists a neighborhood $U(y_0)$ of y_0 such that $I(x_0; y)$ is bounded from above for $y \in U(y_0)$.

Remark. D) is satisfied if to every couple x_0, y_0 of elements of B_R there exists a constant $C = C(x_0, y_0)$ and a neighborhood $U(y_0)$ of y_0 such that for each $t \in \Omega$

$$f(x_0, y) < C \, \| y^{(l)} \|_E^p \quad \text{for} \quad y \in U(y_0).$$

3. Proof of the lower semicontinuity of $I_0(y) = I(x_0; y)$

As already pointed out it is sufficient to prove that (1.5) holds for some bounded linear functional l_0. To prove the existence of such a functional we note first that the convexity assumption B) on f implies immediately the convexity of $I_0(y)$, and that assumption D) implies that $I_0(y)$ is locally bounded from above. It now follows from a well known theorem [*1*; Chapter II, § 5, proposition 2] that $I_0(y)$ is continuous in B_R. From this it is easily seen that the set $\Pi_1 \subset \Pi$ defined in the introduction contains interior points (namely all points (y_0, r_0) with $r_0 > I_0(y_0)$; see also [*9*]), and that the graph of I_0 belongs to the boundary of Π_1.

From these properties we conclude by a well known theorem [*1*; chapter II, § 3, proposition 3] that every point $u_0 = (y_0, I_0(y_0))$ of the graph of I_0 admits a closed supporting hyperplane H_0. Therefore there exists a continuous linear functional $h_0 = h_0(u)$ on Π such that

$$h_0(u) = h_0(u_0) \tag{3.1}$$

is an equation for H_0 while

$$h_0(u) \geqq h_0(u_0) \quad \text{for} \quad u \in \Pi_1. \tag{3.2}$$

Now let e_0 be an arbitrary element of Π which is not an element of W_p^l. Then every $u \in \Pi$ has the unique representation

$$u = r\, e_0 + y, \qquad r \text{ real}, \; y \in W_p^l. \tag{3.3}$$

Thus

$$h_0(u) = r\, h_0(e_0) + h_0(y). \tag{3.4}$$

Since the points of the graph of I_0 are characterized by $r = I_0(y)$, we see from (3.4) that

$$h_0(u) = I_0(y)\, h_0(e_0) + h_0(y)$$

for u on the graph. But since the graph belongs to Π_1 we see from (3.2) that

$$I_0(y)\, h_0(e_0) + h_0(y) \geqq I_0(y_0)\, h(e_0) + h(y_0)$$

or

$$\big(I_0(y) - I_0(y_0)\big)\, h_0(e_0) \geqq -h_0(y - y_0). \tag{3.5}$$

If now

$$h_0(e_0) \neq 0, \tag{3.6}$$

then (3.5) implies (1.5) if we take for l_0 the restriction of the linear functional $-h_0/h_0(e_0)$ to W_p^l.

Suppose now $h_0(e_0)=0$. Then e_0 is an element of the hyperspace H given by $h_0(u)=0$. Since e_0 by definition $\notin W_p^l$, the two hyperspaces H and W_p^l of Π are not identical, and W_p^l contains an element u_1 which is not in H, *i.e.* for which $h_0(u_1)\neq 0$. We now set $e_1=e_0+u_1$. Then $h_0(e_1)=h_0(e_0)+h_0(u_1)=h_0(u_1)\neq 0$, *i.e.* (3.6) is satisfied with e_0 replaced by e_1. On the other hand, $e_1\notin W_p^l$ since otherwise $e_0=e_1-u_1$ would be in W_p^l.

4. The weak continuity of $I(x;y)$ with respect to x

The goal of this section is the proof of

Theorem 4.1. *For fixed $y_0\in B_R$, $I(x;y_0)$ is weakly continuous in x in the weak topology of B_R. The continuity is uniform as y_0 varies over B_R.*

The proof will be based on

Theorem 4.2. *Let Ω be a Sobolev domain. Then there exists a positive constant $M=M(\Omega)$ with the following property: if for any positive δ, $I_\delta\subset\Omega$ denotes a cube of sidelength δ, and if for any positive integer N, $\Omega_N=\Omega_N(\delta)$ denotes the union of N such cubes $I_\delta', \ldots, I_\delta^N$ with disjoint interiors, then for $m=0, 1, \ldots, l-1$ there exist for $\delta<1$ an Ω_N and bounded linear functionals $\lambda^j_{\beta_1\ldots\beta_n}$ on W_p^l $(j=1, \ldots, N)$, $\sum_i \beta_i=k$ where k varies from 0 to $l-m-1$ such that the following Friedrichs-Sobolev inequality holds*:

$$\int_{\Omega_N}\left|\frac{\partial^m x}{\partial t_1^{m_1}\ldots\partial t_n^{m_n}}\right|^p dt$$

$$\leq M^p\,\delta^{(l-m)p}\int_{\Omega_N}\left[\sum_{\sum\beta_i=l-m}\left(\frac{\partial^l x}{\partial t_1^{m_1+\beta_1}\ldots\partial t_n^{m_n+\beta_n}}\right)^2\right]^{p/2} dt+ \tag{4.1}$$

$$+\delta^{-mp}\sum_{j=1}^{N}\sum_{k=0}^{l-m-1}\left[\sum_{\sum\beta_i=k}(\lambda^j_{\beta_1}\ldots\beta_n(x))^2\right]^{p/2} \qquad \left(\sum m_i=m\right).$$

The proof will be given in Appendix A.

Our next goal is the proof of

Theorem 4.3. *Let η be a given positive number. Then* (*with the notations used in theorem* 4.2) *there exist a positive $\delta<1$* (*which may be chosen arbitrarily small*) *and bounded linear functionals $\lambda^j_{\beta_1\ldots\beta_n}$ on W_p^l such that for $x\in B_{2R}\subset W_p^l$*

$$\int_{\Omega}\left|\frac{\partial^m x}{\partial t_1^{m_1}\ldots\partial t_n^{m_n}}\right|^p dt$$

$$\leq M^p\,\delta^{(l-m)p}\int_{\Omega}\left[\sum_{\sum\beta_i=l-m}\left(\frac{\partial^l x}{\partial t_1^{m_1+\beta_1}\ldots\partial t_n^{m_n+\beta_n}}\right)^2\right]^{p/2} dt+ \tag{4.2}$$

$$+\delta^{-mp}\sum_{j=1}^{N}\sum_{k=0}^{m-l-1}\left[\sum_{\sum\beta_i=k}(\lambda^j_{\beta_1\ldots\beta_n}(x))^2\right]^{p/2}+\eta\,.$$

Moreover, the integral at the right member of this inequality may be replaced by $(2R)^p$.

The proof is based on the next three lemmas.

Lemma 4.1. *There exists a* $q>p$ *and a constant* $C=C(q)$ *such that for each* $x\in W_p^l$

$$\|x_\nu^{(m)}\|_q \leqq C\,\|x\|_{W_p^l} \tag{4.3}$$

for $m=0,1,\ldots,l-1$ *and* $\nu=1,\ldots,\sigma_m$. *(For the notation see the first paragraph of the introduction.)*

Proof. Suppose first that $m\geqq l-n/p$. Then according to a result by SOBOLEV [*13*, Theorem, p. 69], (4.3) is true for every positive $q<np/(n-(l-m)p)$. Since the latter number is greater than p, the lemma is proved under the assumption made.

If however $m<l-n/p$, then by the same theorem $x_\nu^{(m)}$ is continuous, and moreover $|x_\nu^{(m)}|\leqq \text{const}\,\|x\|_{W_p^l}$. This inequality obviously implies (4.3) for every positive q since the measure of Ω is finite.

Lemma 4.2. *Let* $0<p<q$. *Let* α *be a positive number, and* $\mathfrak{F}=\{z\}$ *a family of elements of* $L_q(\Omega)$ *for which*

$$\int_\Omega |z|^q\,dt<\alpha; \tag{4.4}$$

then there corresponds to each positive η *a* ζ *such that*

$$\int_E |z|^p\,dt<\eta, \qquad z\in\mathfrak{F} \tag{4.5}$$

if, with $\mu(E)$ *denoting the measure of the set* E,

$$\mu(E)<\zeta, \qquad E\in\Omega. \tag{4.6}$$

Proof. Setting $z_1=x^p$, one sees that it is sufficient to consider the case $p=1$, and a proof in this case is given in [*8*; p. 163].

Lemma 4.3. *With the notation of theorem* 4.2 *there corresponds to each positive* η *an* $\Omega_N=\Omega_N(\delta)$ *(with arbitrarily small* δ*) such that for each* $x\in B_R\subset W_p^l$

$$\int_{\Omega-\Omega_N} |x_\nu^{(m)}|^p\,dt<\eta \qquad (m=0,1,\ldots,l-1). \tag{4.7}$$

Proof. We first choose a q according to lemma 4.1. We then see from (4.3) that (4.4) is satisfied with $\mathfrak{F}=\{x_\nu^{(m)}\,|\,x\in B_R\}$ and with $\alpha=C^q R^q$. On the other hand since Ω is a bounded open set, there exists an $\Omega_N(\delta)$ with arbitrarily small δ such that (4.6) is true with $E=\Omega-\Omega_N$. Thus our assertion follows from lemma 4.2.

Proof of Theorem 4.3. (4.2) is now an immediate consequence of theorem 4.1 and lemma 4.3. The additional assertion of our theorem follows from the fact that integral at the right member of (4.2) is not greater than $\|x\|_{l,p}^p\leqq\|x\|_{W_p^l}$ as is seen from definitions (2.1) and (2.2).

We are now in a position to prove theorem 4.1. In obvious notation we have for $x, a, y_0 \in B_R$

$$I(x;y_0)-I(a;y_0)$$

$$(4.8)\qquad = \int_\Omega \sum_{m=0}^{l-1} [f(t, a^0, a^1, \dots, a^{(m-1)}, x^{(m)}, x^{(m+1)}, \dots, x^{(l-1)}, y_0^l) - $$
$$- f(t, a^0, a^1, \dots, a^{(m-1)}, a^{(m)}, x^{(m+1)}, \dots, x^{(l-1)}, y_0^l)]\, dt.$$

From the definition of the vector $x^{(m)}$ (see the first paragraph of the introduction) we see that the bracket in (4.8) can be written as a sum of differences in each of which only one component $a_\nu^{(m)}$ changes to $x_\nu^{(m)}$. From assumption A) we conclude therefore that the right member of (4.8) is majorized by a finite number, say σ, terms of the form

$$(4.9)\qquad L\int_\Omega \left|\frac{\partial^m (x-a)}{\partial t_1^{m_1} \dots \partial t_n^{m_n}}\right|^\alpha dt, \quad \sum m_i = m$$

where L and α are positive constants, the latter $\leqq 1$. Since $p>1$, it is easily seen from HOELDER's inequality that the expression (4.9) is not greater than

$$(4.10)\qquad \left\{L_1 \int_\Omega \left|\frac{\partial^m (x-a)}{\partial t_1^{m_1} \dots \partial t_n^{m_n}}\right|^p dt\right\}^{\alpha/p}, \quad L_1 = L^{p/\alpha} (\mu(\Omega))^{(p-\alpha)/\alpha}$$

where $\mu(\Omega)$ denotes the measure of Ω.

Now x and a are elements of B_R. Therefore $x-a \in B_{2R}$ and we may apply theorem 4.3 to the integral in (4.10) with x replaced by $x-a$. We see from (4.8) and (4.10) that with $\sigma_1 = \sigma^{p/\alpha}$

$$(4.11)\qquad \begin{aligned} |I(x;y_0)-I(a;y_0)|^{p/\alpha} &\leqq \sigma_1 L_1 \eta + \sigma_1 L_1 (M\delta^{l-m} 2R)^p + \\ &+ \sigma_1 L_1 \delta^{-mp} \sum_{j=1}^{N} \sum_{k=0}^{l-m-1} \Big[\sum_{\Sigma \beta_i = k} (\lambda^j_{\beta_1 \dots \beta_n}(x-a))^2\Big]^{p/2}. \end{aligned}$$

Now if ε is a given positive number, we first choose η such that the first term of the right member of (4.11) is less than $\varepsilon/3$, then δ such that the second term is also majorized by $\varepsilon/3$. After η and δ are fixed, the third term of the right member of (4.11) is a finite sum of powers of bounded linear functionals. It is therefore clear that there exists a weak neighborhood $U(a)$ such that this term is also not greater then $\varepsilon/3$ for $x \in U(a)$. This obviously proves theorem 4.1.

Remark 1. The argument at the end of the proof just given shows also that for $x \in B_R$ and for $m=0, 1, \dots, l-1$ the L_p norm of any derivative $x_\nu^{(m)}$ of x is continuous in the weak topology of B_R. Indeed, the absolute value of $\|x_\nu^{(m)}\|_p - \|a_\nu^m\|_p$ is not greater than $\|(x-a)_\nu^m\|_p$, and $\|(x-a)_\nu^{(m)}\|^p$ is the integral in (4.10), and therefore by theorem 4.3 is majorized by the right member of (4.11) with σ_1 and L_1 replaced by 1.

Remark 2. For x in B_R the norm $\|x\|_{l,p}$ (defined by (2.2)) is lower semicontinuous in the weak topology of B_R. Indeed, since a Banach space norm $\|x\|_{W_p^l}$ has this property [*10*; lemma 2.4], and on account of (2.1), our assertion follows from remark 1 (with $m=0$).

5. Proof of the existence of an $x_0 \in B_R$ minimizing the integral (1.1)

As pointed out in the introduction, it is sufficient to prove the weak lower semicontinuity of $I(x)$. Let ε be a given positive number. For x, a in B_R we see from (1.4) that

$$I(x)-I(a) \geqq I(a;x)-I(a;a)-|I(x;x)-I(a;x)|. \tag{5.1}$$

Now since $I(a;x)$ is weakly lower semicontinuous in x (section 3), there exists a weak neighborhood $V(a)$ such that the first difference of the right member of (5.1) is not smaller than $-\varepsilon$ for $x \in V(a)$, and by theorem 4.1 there exists a weak neighborhood $U(a)$ such that $|I(x;x)-I(a;x)|<\varepsilon$ for $x \in U(a)$. Thus we see from (5.1) that $I(x)-I(a) \geqq -2\varepsilon$ for $x \in V(a) \cap U(a)$.

Appendix A

Proof of theorem 4.2. We first consider the cube $I=I_1 \subset E^n$ of side length 1 with the origin as center and with sides parallel to the coordinate axes. The norm $\|x\|_{W_p^l(I)}$ of $x \in W_p^l(I)$ is then defined in section 2 with Ω replaced by I. With SOBOLEV we introduce a new norm $\|x\|^{0}_{W_p^l}$ in the following way: Let S^l be the vector space of all polynomials

$$P=\sum_{k=0}^{l-1} \sum_{\sum \alpha_i=k} a_{\alpha_1} \dots \alpha_n t_1^{\alpha_1} \dots t_n^{\alpha_n} \tag{A.1}$$

of degree less than l in the variables $t_1, \dots, t_n$. S^l may be considered a finite dimensional subspace of the linear space underlying $W_p^l(I)$. Therefore there exists a "supplementary" space L_p^l such that every $x \in W_p^l(I)$ has the unique representation

$$x=P+x_1, \quad P \in S^l, \quad x_1 \in L_p^l. \tag{A.2}$$

Then with the norm $\| \ \|_{l,p}$ defined by (2.2) and with $\|P\|_{S^l}$ given by

$$\|P\|_{S^l}=\left\{\sum_{k=0}^{l-1} \left[\sum_{\sum \alpha_i=k} (a_{\alpha_1} \dots \alpha_n)^2\right]^{p/2}\right\}^{1/p} \tag{A.3}$$

the new norm is defined by

$$\|x\|^{0}_{W_p^l}=\{\|P\|^p_{S^l}\|+x_1\|^p_{l,p}\}^{1/p} \tag{A.4}$$

which may also be written as

$$\|x\|^{0}_{W_p^l}=\{\|P\|^p_{S^l}+\|x\|^p_{l,p}\}^{1/p} \tag{A.5}$$

since by (A.2) the l^{th} derivatives of x agree with those of x_1, and therefore $\|x\|_{l,p}=\|x_1\|_{l,p}$.

Now I is clearly a Sobolev domain. Therefore the following result of SOBOLEV is valid [*13*; p. 72]:

Theorem A.1. *The norms* $\|x\|_{W_p^l(I)}$ *and* $\|x\|^{0}_{W_p^l(I)}$ *are equivalent*, i.e. *there exist two positive constants* m *and* M (*not depending on* x) *such that*

$$m\,\|x\|^{0}_{W_p^l(I)} \leqq \|x\|_{W_p^l(I)} \leqq M\,\|x\|^{0}_{W_p^l(I)}. \tag{A.6}$$

Since $\|x\|_p \leqq \|x\|_{W_p^l(I)}$ (see (2.1)), we conclude from (A.6) that $\|x\|_p \leqq M \|x\|^0_{W_p^l(I)}$, or written out

$$\int_I \|x\|^p\,dt \leqq M^p \left\{ \int_I \left[\sum_{\Sigma \alpha_i = l} \left(\frac{\partial^l x}{\partial t_1^{\alpha_1} \dots \partial t_n^{\alpha_n}} \right)^2 \right]^{p/2} dt + \sum_{k=0}^{l-1} \left[\sum_{\Sigma \alpha_i = k} (a_{\alpha_1 \dots \alpha_n}(x))^2 \right]^{p/2} \right\}, \quad x \in W_p^l(I). \tag{A.7}$$

Here we consider the $a_{\alpha_1 \dots \alpha_n}$ as bounded linear functionals on $W_p^l(I)$; this is possible since by (A.1) these coefficients are certainly bounded linear functionals on the finite dimensional space S^l which by the Hahn-Banach theorem may be extended to $W_p^l(I)$.

We now want to extend (A.7) to the derivatives $x_\nu^{(m)}$ of x ($m = 0, 1, \dots, l-1$; $\nu = 1, \dots, \sigma_m$). Since the $(l-m)^{\text{th}}$ derivatives of $x_\nu^{(m)}$ are l^{th} derivatives of x we see that $x_\nu^{(m)} \in W_p^{l-m}(I)$. Therefore (A.7) remains valid if we replace x by $x_\nu^{(m)}$ and l by $l-m$, and the $a_{\alpha_1 \dots \alpha_n}$ by appropriate linear functionals on $W_p^{l-m}(I)$. Thus for

$$x_\nu^{(m)} = \frac{\partial^m x}{\partial t_1^{m_1} \dots \partial t_n^{m_n}}, \qquad \sum_{\rho=1}^{n} m_\rho = m \tag{A.8}$$

we obtain

$$\int_I \left| \frac{\partial^m}{\partial t_1^{m_1} \dots \partial t_n^{m_n}} \right|^p dt \leqq M^p \int_I \left[\sum_{\Sigma \beta_i = l-m} \left(\frac{\partial^l x}{\partial t_1^{m_1+\beta_1} \dots \partial t_n^{m_n+\beta_n}} \right)^2 \right]^{p/2} dt + \sum_{k=0}^{l-m-1} \left[\sum_{\Sigma \beta_i = k} (L_{\beta_1 \dots \beta_n;\nu}(x))^2 \right]^{p/2} \tag{A.9}$$

where we have set

$$L_{\beta_1 \dots \beta_n}(x) = M\, a^{(m)}_{\beta_1 \dots \beta_n;\nu} \tag{A.10}$$

and where the $a^{(m)}_{\beta_1 \dots \beta_n;\nu}$ are obtained as follows: if $P \in S^l$ is the polynomial occurring in the representation (A.2) of x, then the $a^{(m)}_{\beta_1 \dots \beta_n;\nu}$ are the coefficients of the polynomial

$$P_{l-m;\nu} = \frac{\partial^m P}{\partial t_1^{m_1} \dots \partial t_n^{m_n}} \in S^{l-m}. \tag{A.11}$$

We claim that the $L_{\beta_1 \dots \beta_n;\nu}$ defined by (A.10) are linear bounded functionals on $W_p^l(I)$. Indeed the linearity is obvious. As to the boundedness, we know already that the $a^{(m)}_{\beta_1 \dots \beta_n;\nu}$ are linear and bounded on $W_p^{l-m}(I)$, *i.e.* that there exists a $C > 0$ such that

$$|a_{\beta_1 \dots \beta_n}(x_\nu^{(m)})| \leqq C\, \|x_\nu^{(m)}\|_{W_p^{l-m}} = C\{\|x_\nu^{(m)}\|_p + \|x_\nu^{(m)}\|_{l-m,p}\}. \tag{A.12}$$

Now the p^{th} power of $\|x_\nu^{(m)}\|_{l-m,p}$ is by definition precisely the integral at the right member of (A.9), and this integral clearly $\leqq \|x\|_{l,p} \leqq \|x\|_{W_p^l(I)}$. On the other hand $\|x_\nu^{(m)}\|_p \leqq$ Const. $\|x\|_{W_p^l(I)}$ which inequality follows from [*13*; theorem, p. 69] in the same way as the inequality (4.3). These inequalities together with (A.12) and (A.10) prove our assertion.

We now consider the cube $I_\delta \subset E^n$ of side length $\delta < 1$ with center $\vartheta^0 = (\vartheta_1^0 \dots \vartheta_n^0)$, and parallel to the cube I. Setting

(A.13) $$\vartheta_i = \vartheta_i^0 + \delta t, \quad \xi(\vartheta) = x(t) \quad \text{where} \quad \vartheta = (\vartheta_1 \dots \vartheta_n),$$

and taking into account that the Jacobian

(A.14) $$\frac{\partial(t_1, \dots, t_n)}{\partial(\vartheta_1, \dots, \vartheta_n)} = \delta^{-n}$$

and that

(A.15) $$\frac{\partial^m x}{\partial t_1^{m_1} \dots \partial t_n^{m_n}} = \delta^m \frac{\partial^m \xi}{\partial \vartheta_1^{m_1} \dots \partial \vartheta_n^{m_n}},$$

we see that (A.9) written in the new variables becomes

(A.16) $$\begin{aligned} &\int_{I_\delta} \left| \frac{\partial^m \xi}{\partial \vartheta_1^{m_1} \dots \partial \vartheta_n^{m_n}} \right|^p d\vartheta \\ &\leqq M^p \delta^{(l-m)p} \int_{I_\delta} \left[\sum_{\Sigma \beta_i = l-m} \left(\frac{\partial^l \xi}{\partial \vartheta_1^{m_1+\beta_1} \dots \partial \vartheta_n^{m_n+\beta_n}} \right)^2 \right]^{p/2} d\vartheta + \\ &+ \delta^{-mp} \sum_{k=0}^{l-m-1} \left[\sum_{\Sigma \beta_i = k} (\lambda_{\beta_1 \dots \beta_n}(\xi))^2 \right]^{p/2}, \end{aligned}$$

where

(A.17) $$\lambda_{\beta_1 \dots \beta_n}(\xi) = \delta^{n/p} L_{\beta_1 \dots \beta_n; \nu}(x).$$

We claim the $\lambda_{\beta \dots \beta_n}$ are bounded linear functionals on $W_p^l(I_\delta)$ with bounds independent of δ. Since we know that the $L_{\beta_1 \dots \beta_n}$ are bounded linear functionals on $W_p^l(I)$, we have to prove: if $L(x)$ is a bounded linear functional on $W_p^l(I)$, then

$$\lambda(\xi) = \delta^{n/p} L(x), \quad 0 < \delta < 1$$

is a bounded linear functional on $W_p^l(I_\delta)$ with a bound independent of δ.

We omit the obvious proof of the linearity. Let γ be a bound for L. Then

$$\begin{aligned} |\lambda(\xi)|^p &= \delta^n |L(x)|^p \leqq \delta^n \gamma^p \|x\|_{W_p^l(I)}^p \\ &= \delta^n \gamma^p \left\{ \int_I |x|^p dt + \int_I \left[\sum_{\Sigma \alpha_i = l} \left(\frac{\partial^l}{\partial t_1^{\alpha_1} \dots \partial t_n^{\alpha_n}} \right)^2 \right]^{p/2} dt \right\} \\ &= \delta^n \gamma^p \left\{ \delta^{-n} \int_{I_\delta} |\xi|^p d\vartheta + \delta^{-n} \delta^{lp} \int_{I_\delta} \left[\sum_{\Sigma \alpha_i = l} \left(\frac{\partial^l \xi}{\partial \vartheta_1^{\alpha_1} \dots \partial \vartheta_n^{\alpha_n}} \right)^2 \right]^{p/2} d\vartheta \right\}, \end{aligned}$$

where we again used (A.14) and (A.15). This inequality remains valid if in its right hand member we replace the factor δ^{lp} of the second integral by 1. But then the right hand member becomes $\gamma^p \|\xi\|_{W_p^l(I_\delta)}$ which proves that γ is a bound for λ.

We now return to the Sobolev domain Ω considered in theorem 4.2 and suppose that $I_\delta \subset \Omega$. Then the following facts are easily verfied: if we map the element $x \in W_p^l(I_\delta)$ on $\tilde{x}=\tilde{x}(t)$ defined by

$$\begin{aligned} \tilde{x}(t) &= x(t) \quad \text{for} \quad t \in I_\delta \\ &= 0 \qquad \text{for} \quad t \in \Omega - I_\delta, \end{aligned}$$

we obtain an isometric imbedding of $W_p^l(I_\delta)$ in $W_p^l(\Omega)$, and $W_p^l(I_\delta)$ thus considered as subset of $W_p^l(\Omega)$ is a closed linear subspace of $W_p^l(\Omega)$. Consequently a linear bounded functional λ on $W_p^l(I_\delta)$ can be extended to a bounded linear functional on $W_p^l(\Omega)$. Therefore we may and will consider the $\lambda_{\beta_1 \ldots \beta_n}$ appearing in (A.16) as bounded linear functionals on $W_p^l(\Omega)$.

Now let I_δ^j $(j=1, \ldots, N)$ and Ω_N be as described in the statement of theorem 4.2. If then $\xi \in W_p(\Omega)$, then the restriction of ξ to I_δ^j belongs to $W_p^l(I_\delta^j)$. Therefore (A.16) holds for each I_δ^j. Since the $\lambda_{\beta_1 \ldots \beta_n}$ now depend on j we use the notation $\lambda^j_{\beta_1 \ldots \beta_n}$ for them. If we add the inequalities thus obtained over j from 1 to N, we obtain the asserted inequality (4.1).

Appendix B

Another application of theorem 4.3.

Theorem. *Let B be a bounded set in $W_p^l(\Omega)$, and let ${}_\rho x$ $(\rho=1, 2, \ldots)$ be a sequence of elements of B. Then there exists a subsequence ρ_i, such for $m=0, 1, \ldots, l-1$, $v=1, 2, \ldots, \sigma_m$ the sequence of derivatives ${}_{\rho_1}x_v^{(m)}$, ${}_{\rho_2}x_v^{(m)}$, … converges strongly in L_p.*

Proof. Since B is bounded, there exists an R such that $B \subset B_R$. Since B_R is weakly compact and therefore closed in the weak topology of W_p^l, we have $\bar{B} \subset \bar{B}_R = B_R$ where the bar denotes closure in the weak topology. As a closed subset of the compact set B_R, $\bar{B}$ is also compact in the weak topology. It follows by a well known theorem (see *e.g.* [*6*; p. 430]) that B is weakly sequentially compact, which means by definition: every sequence ${}_\rho x$ in B contains a subsequence ${}_{\rho_i}x$ to which there exists an ${}_0x \in W_p^l$ such that

$$\lim_{i \to \infty} \lambda({}_{\rho_i}x) = \lambda({}_0x) \tag{B.1}$$

for every bounded linear functional λ on W_p^l. Now ${}_{\rho_i}x - {}_{\rho_j}x \in B_{2R}$ since ${}_\rho x \in B \in B_R$ for all ρ. We see therefore from theorem 4.3 that $\|{}_{\rho_i}x - {}_{\rho_j}x\|_p^p$ is majorized by the right member of (4.11) with σL_1 replaced by 1, and $x-a$ by ${}_{\rho_i}x - {}_{\rho_j}x$. Arguing as in the lines following (4.11), we see that to given positive ε we can choose first η and then δ in such a way that the first two terms of the modified right member of (4.11) together are not greater than $2\varepsilon/3$. It now follows from (B.1) that there exists an i_0 such that the third term becomes less than $\varepsilon/3$ for i and j greater than i_0.

Remark. If B is closed in the weak topology, then ${}_0x \in B$. Sufficient for the weak closure of B is that B is convex and strongly closed (see *e.g.* [*6*; p. 422]).

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