Spectral Theory of Taylor Vortices

Part I. Structure of Unstable Modes

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1. Introduction

In studying the axisymmetric vortices that form in the flow (Couette flow) between concentric rotating cylinders as a result of instability, TAYLOR assumed these vortices to be nonoscillatory. This assumption has been made by all subsequent investigators of the subject, but has never been proved. Furthermore, TAYLOR and subsequent investigators have studied only the first mode of these vortices. The implicit assumption that the first mode must be the most unstable among all possible modes has also never been proved.

The above mentioned vortices have been called Taylor vortices, and we shall adopt this term here for simplicity. This paper is a study of Taylor vortices in Couette flow with the cylinders rotating in the same direction. Since the flow is known to be stable if the circulation increases outwards, we shall assume that it decreases outwards. The main purpose of this paper is to show that for cylinders rotating in the same direction, with the circulation decreasing outward, all modes, stable or unstable, are nonoscillatory. In the process of obtaining this main result, a good deal of information is obtained, including the structure of the Taylor vortices when they are unstable.* The theory so developed will be called the spectral theory of Taylor vortices.

* Some relevant work (for $\sigma \geqq -\lambda^2$, λ being the wave number) falling short of our main goal has been done by YUDOVICH [7] and IVANILOV & IAKOVIEV [8]; see the end of Section 8. The author is indebted to Professor D. D. JOSEPH for bringing these Russian works to his attention, after this work had been completed. The results for $\sigma \geqq -\lambda^2$ are useful because the structure of unstable Taylor vortices are intimately related to the results for $\sigma \geqq -\lambda^2$.

DiPrima and Habetler [1] have shown that, for any given Reynolds number (the R to be defined later) there are infinitely many eigenvalues for the growth rate σ, and that the corresponding generalized eigenfunctions are complete. But they did not show that σ must be real in the case specified here, nor did they show that the eigenvalues of σ are simple.

This paper consists of two parts. Part I gives the structure of unsteady non-oscillatory Taylor vortices, and Part II gives the proof that Taylor vortices are nonoscillatory provided the square of the circulation decreases outwards.

From the mathematical point of view this paper can be considered to be a study of eigenvalue problems governed by a non-selfadjoint linear differential system. Its approach and methodology are therefore not restricted to the particular physical problem considered here, but may find applications to a wide variety of problems. The theory is a sort of Sturm-Liouville theory for differential equations of orders higher than 2, and is a revival of a branch of classical mathematics. However, it is its bearing on hydrodynamic phenomena that is emphasized here.

2. The Differential System Governing the Formation of Taylor Vortices

Consider the flow of a homogeneous incompressible fluid in the annular space between two infinitely long concentric cylinders, the inner one of which has the radius r_1 and rotates with angular velocity Ω_1 and the outer one of which has the radius r_2 and rotates with angular velocity Ω_2. The z-axis is assumed to coincide with the common axis of the cylinders, and the distance r is measured radially from the z-axis. The following dimensionless variables will be used:

$$r' = \frac{r}{r_1}, \qquad z' = \frac{z}{r_1}, \qquad t' = t\,\Omega_1,$$

but henceforth we shall drop the accents on these variables. The mean velocity V, in the azimuthal direction, is then given by

$$\frac{V}{\Omega_1 r_1^2} = A r + \frac{B}{r},$$

in which

$$A = \frac{\Omega_2 r_2^2 - \Omega_1 r_1^2}{(r_2^2 - r_1^2)\,\Omega_1}, \qquad B = \frac{(\Omega_1 - \Omega_2)\, r_2^2}{\Omega_1 (r_2^2 - r_1^2)}. \tag{1}$$

It should be noted that

$$A + B = 1, \tag{2}$$

and that $A + \frac{B}{r^2}$ is always positive if the cylinders rotate in the same direction.

Taylor [2] assumed the components of the velocity perturbation due to axisymmetric disturbances to have the forms

$$\begin{gathered} u' = u(r) \cos \lambda z\, e^{\sigma t}, \qquad v' = v(r) \cos \lambda z\, e^{\sigma t}, \\ w' = w(r) \sin \lambda z\, e^{\sigma t}, \end{gathered} \tag{3}$$

in which u', v', and w' are the radial, azimuthal, and longitudinal component, respectively, λ is the wave number characterizing the spacing of the rings of disturbance or Taylor vortices, and σ is the growth rate, which may be complex. We shall write

$$\sigma = \sigma_r + i\sigma_i. \tag{4}$$

Using the equation of continuity and the linearized equation of motion, TAYLOR obtained

$$(L-\lambda^2-\sigma)(L-\lambda^2)u = 4\lambda^2 R^2 \left(A+\frac{B}{r^2}\right) v, \tag{5}$$

$$(L-\lambda^2-\sigma)v = Au, \tag{6}$$

in which

$$L = D^2 + \frac{1}{r} D - \frac{1}{r^2} = D \frac{1}{r} D(r\ ), \qquad D = \frac{d}{dr},$$

and

$$R = \Omega_1 r_1^2/\nu,$$

ν being the kinematic viscosity. The boundary conditions can be expressed in terms of u and v, and are

$$u = Du = 0 = v \quad \text{at} \quad r=1 \quad \text{and} \quad r = r_2/r_1 = b\text{(say)}. \tag{7}$$

For the derivation of (5) and (6), see LIN [3]. The σ here appears there as σR. This amounts to an insignificant change of time scale, which is r_1^2/ν here but Ω_1^{-1} in LIN's book. Also the u here is $2Ru$ in LIN's book.

It has been assumed by TAYLOR and subsequent workers that for non-negative σ_r, corresponding to growing or neutrally stable disturbances, σ_i is zero. This assumption, when true, is often called the "principle of exchange of stabilities." The truth of this assumption is very little in doubt. Nevertheless, it has never been convincingly demonstrated, although DAVIS [4] showed by an expansion that TAYLOR's assumption is indeed true for a limited range of positive values of Ω_2/Ω_1 and a limited range of r_2/r_1. TAYLOR's assumption will be proved for $\Omega_2/\Omega_1 \geqq 0$ in Part II, with no restriction on geometry. In Part I we shall concentrate on the structure of unstable Taylor vortices.

In the following we shall assume, once and for all, that the cylinders rotate in the same direction, so that $A+\frac{B}{r^2}$ is always positive (at least zero at the outer cylinder, if it is stationary). We also assume once and for all that A is negative, for it is known (SYNGE [5]) that if $A \geqq 0$ the flow is stable, and we are only interested in undamped Taylor vortices.

3. The Transitory Differential System

The differential equations (5) and (6) can be combined to give

$$(L-\lambda^2-\sigma)^2(L-\lambda^2)v = 4\lambda^2 R^2 A\left(A+\frac{B}{r^2}\right) v, \tag{8}$$

or

$$(L-\lambda^2-\sigma)^2(L-\lambda^2)v = 4\lambda^2 R^2 A\left[1-B\left(1-\frac{1}{r^2}\right)\right] v. \tag{9}$$

We shall make use of the "transitory equation"

$$(L-\lambda^2-\sigma)^2(L-\lambda^2)v=4\lambda^2 R^2 A\left[1-\varepsilon B\left(1-\frac{1}{r^2}\right)\right]v. \tag{10}$$

When $\varepsilon=1$, (9) and (10) are identical. When $\varepsilon=0$, (10) is sufficiently simple to provide us some very important information. Equation (10) is obtained from the simultaneous transitory equations

$$(L-\lambda^2-\sigma)(L-\lambda^2)u=4\lambda^2 R^2\left[1-\varepsilon B\left(1-\frac{1}{r^2}\right)\right]v, \tag{11}$$

$$(L-\lambda^2-\sigma)v=Au. \tag{12}$$

The singular point $r=0$ for (10) is regular. By use of standard methods it is found that (10) possesses the six fundamental solutions:

$$\begin{aligned}
\phi_1&=\sum_{n=-\infty}^{\infty} a_{2n-1}^{(1)} r^{2n-1}, & \phi_2&=\sum_{n=2}^{\infty} a_{2n-1}^{(2)} r^{2n-1},\\
\phi_3&=\sum_{n=3}^{\infty} a_{2n-1}^{(3)} r^{2n-1}, & \phi_4&=\sum_{n=-\infty}^{\infty} a_{2n-1}^{(4)} r^{2n-1}+\phi_1\ln r,\\
\phi_5&=\sum_{n=-\infty}^{\infty} a_{2n-1}^{(5)} r^{2n-1}+\phi_2\ln r, & \phi_6&=\sum_{n=-\infty}^{\infty} a_{2n-1}^{(6)} r^{2n-1}+\phi_3\ln r.
\end{aligned} \tag{13}$$

The recursion formulas (not exhibited here) connecting the a's in (13) show definitely that for any finite values of λ^2, σ, A, B, R, and ε, all the six series converge uniformly and absolutely in the interval $1\leqq r\leqq b$. They therefore vary *continuously* with these parameters. They can also be differentiated with respect to these parameters any number of times.

4. The Approach

Since the subsequent developments and arguments are complex, it is helpful to state briefly the method of approach. For this purpose we first define a few terms to be used for convenience:

(a) The original system: this consists of (5), (6), and (7).

(b) The transitory system: this consists of (11), (12), and (7).

(c) The starting system, which is defined to be the transitory system with $\varepsilon=0$.

(d) The auxiliary system: this consists of (11), (12), and the boundary conditions

$$u=Lu=0=v \quad \text{at} \quad r=1 \quad \text{and} \quad r=b. \tag{14}$$

(e) The intermediate system: this consists of (11), (12), and the boundary conditions

$$u=Lu=0=v \quad \text{at} \quad r=1, \tag{15}$$

$$u=Du=0=v \quad \text{at} \quad r=b. \tag{16}$$

Our approach will be as follows. First we shall show that, for any ε from zero to 1, $\sigma_i=0$ and R^2 is positive for the auxiliary system, as long as $\lambda^2+\sigma_r$ is non-negative. Then we show that the auxiliary system can be transformed to a homogeneous integral equation of the Fredholm type, with a symmetric kernel, and hence has infinitely many real (and non-negative real $\lambda^2+\sigma$) eigenvalues of R^2, which we shall denote by E_n'', with $n=1, 2, 3$, etc. Next we shall demonstrate that each of the eigenfunctions for u and v for the n-th eigenvalue E_n'' has exactly $n-1$ internal zeros. In the process of this demonstration, we shall show that for the same non-negative $\sigma+\lambda^2$, eigenvalues E_n' for the intermediate system exist for any ε from zero to 1, and that

$$E_1''<E_1'<E_2''<E_2'<E_3''<E_3' \qquad \text{etc.}$$

We then show that between E_n' and E_{n+1}' there is exactly one eigenvalue E_n of the transitory system, so that

$$E_1'<E_1<E_2'<E_2<E_3'<E_3<E_4'<E_4<E_5' \qquad \text{etc.} \tag{17}$$

There are no complex eigenvalues for R^2 as long as $\lambda^2+\sigma$ is non-negative. The eigenfunctions for u and v corresponding to E_n each have $n-1$ internal zeros. All these eigenvalues are simple for non-negative $\lambda^2+\sigma$.

The proof of the nonoscillation of Taylor vortices when the square of the circulation decreases outwards will be given in Part II of our work.

5. The Auxiliary System

The boundary conditions (14) can be written in terms of v as follows

$$v=Lv=L^2v=0 \quad \text{at} \quad r=1 \quad \text{and} \quad r=b, \tag{14a}$$

as can be easily verified by using (12). Multiplying (10) by rv^* (the asterisk indicating the complex conjugate), integrating by parts between 1 and b and using (14a) whenever necessary, we have

$$\begin{aligned} -\{J_3+(3\lambda^2+2\sigma)J_2+(\lambda^2+\sigma)(3\lambda^2+\sigma)J_1+\lambda^2(\lambda^2+\sigma)^2J_0\} \\ =4\lambda^2R^2A\int r\left[1-\varepsilon B\left(1-\frac{1}{r^2}\right)\right]|v|^2, \end{aligned} \tag{18}$$

in which

$$J_3=\int\frac{1}{r}|D(rLv)|^2, \quad J_2=\int r|Lv|^2, \quad J_1=\int\frac{1}{r}|D(rv)|^2, \quad J_0=\int r|v|^2. \tag{19}$$

In obtaining (18) it is convenient to recall that $L=D\frac{1}{r}D(r\ )$. The limits of integration of the integrals in (18) and (19), being understood to be 1 and b, are omitted together with dr. The imaginary part of (18) is

$$\sigma_i[2J_2+(4\lambda^2+2\sigma_r)J_1+2\lambda^2(\lambda^2+\sigma_r)J_0]=0, \tag{20}$$

since R^2 and λ^2, as well as A and B, are real. Hence σ_i must vanish for non-negative σ_r for the auxiliary system.

For real non-negative σ, (18) shows that R^2 must be positive, since A is negative. We shall now show that there are infinitely many eigenvalues E_n'' for R^2 for the auxiliary system.

Consider the Green's function $G(r, \xi)$ satisfying the differential equation

$$(L-\lambda^2-\sigma)^2(L-\lambda^2)\,G=0 \tag{21}$$

everywhere except at $r=\xi$, where L^3G becomes infinite in such a way that

$$\int_{\xi-\varepsilon'}^{\xi+\varepsilon'} rL^3G\,dr=1 \tag{22}$$

however small ε' is. In other words

$$(L-\lambda^2-\sigma)^2(L-\lambda^2)\,G(r,\xi)=\frac{\delta(r,\xi)}{r}, \tag{23}$$

in which $\delta(r, \xi)$ is the Dirac function. Furthermore $G(r, \xi)$ satisfies

$$G=LG=L^2G=0 \quad \text{at} \quad r=1 \quad \text{and} \quad r=b, \tag{24}$$

for all ξ. The solution of (10) and (14a) is then

$$v(r)=4\lambda^2R^2A\int_1^b \xi\left[1-\varepsilon B\left(1-\frac{1}{\xi^2}\right)\right]G(r,\xi)\,v(\xi)\,d\xi. \tag{25}$$

The Green's function $G(r, \xi)$ is symmetric in the sense that

$$G(r,\xi)=G(\xi,r). \tag{26}$$

To prove (26), consider (23) together with

$$(L-\lambda^2-\sigma)^2(L-\lambda^2)\,G(r,\eta)=\frac{\delta(r,\eta)}{r}. \tag{27}$$

Multiplying (23) by $rG(r, \eta)$, integrating between 1 and b, using (24) whenever necessary, and writing

$$M=\{J_3'+(3\lambda^2+2\sigma)J_2'+(\lambda^2+\sigma)(3\lambda^2+\sigma)J_1'+\lambda^2(\lambda^2+\sigma)^2J_0'\}$$

where

$$J_3'=\int\frac{1}{r}D[rLG(r,\xi)]\,D[rLG(r,\eta)], \qquad J_2'=\int rLG(r,\xi)\,LG(r,\eta),$$

$$J_1'=\int\frac{1}{r}D[rG(r,\xi)]\,D[rG(r,\eta)], \qquad J_0'=\int rG(r,\xi)\,G(r,\eta),$$

we have

$$M=G(\xi,\eta). \tag{28}$$

Similarly, multiplying (27) by $rG(r, \xi)$ and integrating, we have

$$M=G(\eta,\xi). \tag{29}$$

Equations (28) and (29) show that

$$G(\xi,\eta)=G(\eta,\xi),$$

which is the same thing as (26).

If now we write

$$\phi(r) = \sqrt{f(r)}\, v(r), \tag{30}$$

in which

$$f(r) = r\left[1 - \varepsilon B\left(1 - \frac{1}{r^2}\right)\right] \tag{31}$$

is non-negative for $0 \leqq \varepsilon \leqq 1$, we can write (25) as

$$\phi(r) = 4\lambda^2 A R^2 \int_1^b K(r, \xi)\, \phi(\xi)\, d\xi, \tag{32}$$

where

$$K(r, \xi) = \sqrt{f(r) f(\xi)}\, G(r, \xi) \tag{33}$$

is obviously symmetric since $G(r, \xi)$ is symmetric. According to a well-known theorem in the theory of integral equations (TRICOMI [6] pp. 103–105), (32) either has an infinite number of eigenvalues of R^2 for nonzero λ^2, or has a Pincherle-Goursat kernel and thus has only a finite number of eigenvalues. The latter possibility is ruled out by considering the secular equation $\Delta = 0$ obtained by writing

$$v(r) = \sum_{n=1}^{6} c_n \phi_n(r) \tag{34}$$

and requiring $v(r)$ to satisfy (14a) and the c's to be not all zero. This Δ in the secular equation is transcendental and not a polynomial in R^2. Hence (32) does not have a Pincherle-Goursat kernel, for which Δ would be a polynomial in R^2.

These eigenvalues are positive, as we have shown, as long as σ is non-negative. They must increase toward infinity. For otherwise there would be a finite limit point for these eigenvalues, according to the Weierstrass-Bolzano theorem. Since Δ is a continuous and infinitely differentiable function of R^2, at the limit point Δ and all its derivatives up to the n^{th} order with respect to R^2 are zero, no matter how large n is. Thus Δ would be identically zero, which is absurd. Hence the infinite number of eigenvalues increase toward infinity, and furthermore there can be no finite limit points.

We shall now study in detail how these eigenvalues are reached. In the process of that study important progress will be made toward our original goals. Before doing so, however, we shall summarize the results of this section in two theorems:

Theorem 1. *For the auxiliary system $\sigma_i = 0$ if σ_r is non-negative.*

Theorem 2. *For non-negative real $\sigma + \lambda^2$, the auxiliary system has infinitely many positive eigenvalues for R^2, increasing without a finite limit point toward infinity, provided λ^2 is positive.*

In the following sections we shall assume once and for all that λ^2 is not zero.

6. Evolution of the Eigenfunctions of the Auxiliary and Intermediate Systems

We shall now see how the eigenvalues E_n'' for the auxiliary system are reached and how the corresponding eigenfunctions evolve. The development from now up to the end of Section 8 is for non-negative $\sigma + \lambda^2$. All the lemmas and theorems up to Theorem 13 will be obtained under this condition, though it will not always be explicitly stated. For any real and non-negative $\sigma+\lambda^2$, we start with $R^2=0$ and impose on (11) and (12) the conditions

$$u=Lu=0=v \quad \text{at} \quad r=1, \tag{35}$$

$$u=0=v \quad \text{and} \quad bDv=-1 \quad \text{at} \quad r=b. \tag{36}$$

Multiplying (11) by ru and integrating between 1 and b, using (35) and (36) whenever necessary, we obtain

$$-A(I_1+\lambda^2 I_0)-b\,D\,v(b)\,Lu(b)=0, \tag{37}$$

where

$$I_1=\int\frac{1}{r}[D(ru)]^2, \qquad I_0=\int ru^2, \tag{38}$$

the limits of integration being understood, and dr being omitted for brevity. Equation (37) shows clearly that neither $Dv(b)$ nor $Lu(b)$ can possibly vanish for $R^2=0$, and in particular we can indeed demand $bDv(b)=-1$ as we did in (36). (This remark is necessary to remove the possibility that $R^2=0$ may be an eigenvalue for (11), (12), (35), and (36), with $Dv(b)=0$ in (36) instead of $bDv(b)=-1$.) If we impose $bDv(b)=-1$, we see from (37) that $Lu(b)$ is negative.

We shall now show that the u and v satisfying (11), (12), (35), and (36) for $R^2=0$ are positive throughout except at the end-points, where they vanish. We know v to be positive near $r=b$ (for $r<b$), since $bDv(b)=-1$. Thus there are two points at which $v=0$, between which v is positive. Hence u cannot be negative throughout (see the arguments in connection with (41)). If u were anywhere negative in $1<r<b$ there would be at least one minimum, at which u is negative and Lu non-negative, and one maximum, at which u is positive and Lu non-positive. Then between these two points there must be at least one point where

$$Q=(L-\lambda^2)u \tag{39}$$

vanishes. Since Q vanishes at $r=1$, there would be at least two points, say P and S, in $1\leqq r\leqq b$ where Q vanishes. Now (11) can be written as

$$(L-\lambda^2-\sigma)Q=0 \tag{40}$$

for zero R^2. Multiplying (40) by rQ and integrating between P and S, we have

$$-\int_P^S\left\{\frac{1}{r}[D(rQ)]^2+(\lambda^2+\sigma)rQ^2\right\}=0,$$

which can be true only in the trivial case $Q\equiv 0$ since $\lambda^2+\sigma\geqq 0$. If $Q=0$ throughout and (35) and (37) are satisfied, we easily conclude that $u=0=v$ throughout. Therefore in non-trivial cases u must be non-negative throughout. If u is non-negative throughout, then so must be v. For if not, there would be two points where

$v=0$ and between which v is negative. Multiplying (12) by rv and integrating between these two points, one would have

$$-\int\left\{\frac{1}{r}[D(rv)]^2+(\lambda^2+\sigma)rv^2\right\}=A\int uv, \tag{41}$$

which is a contradiction (except in the trivial case $u=0=v$) since A is negative, u positive, and v negative between the two points. Hence v must be non-negative throughout at $R^2=0$.

We now consider the two auxiliary systems

$$(L-\lambda^2-\sigma)(L-\lambda^2)u_1=4\lambda^2 R_1^2\left[1-\varepsilon B\left(1-\frac{1}{r^2}\right)\right]v_1, \tag{42}$$

$$(L-\lambda^2-\sigma)v_1=Au_1, \tag{43}$$

where

$$u_1=Lu_1=0=v_1 \quad \text{at} \quad r=1 \tag{44}$$

$$u_1=0=v_1 \quad \text{and} \quad bDv_1=-1 \quad \text{at} \quad r=b, \tag{45}$$

and

$$(L-\lambda^2-\sigma)(L-\lambda^2)u_2=4\lambda^2 R_2^2\left[1-\varepsilon B\left(1-\frac{1}{r^2}\right)\right]v_2, \tag{46}$$

$$(L-\lambda^2-\sigma)v_2=Au_2, \tag{47}$$

where

$$u_2=Lu_2=0=v_2 \quad \text{at} \quad r=1, \tag{48}$$

$$u_2=0=v_2 \quad \text{and} \quad bDv_2=-1 \quad \text{at} \quad r=b. \tag{49}$$

Multiplying (42) by rv_2 and integrating between 1 and b, we have, upon using the boundary conditions on v_2 and u_1,

$$\int r(L-\lambda^2)u_1(L-\lambda^2-\sigma)v_2\,dr-bDv_2(b)Lu_1(b)=R_1^2 J_{m4}, \tag{50}$$

in which

$$J_{m4}=4\lambda^2\int r\left[1-\varepsilon B\left(1-\frac{1}{r^2}\right)\right]v_1 v_2\,dr. \tag{51}$$

Using (47), and $bDv_2=-1$, we can integrate the left-hand side of (50) further and obtain

$$-A(I_{m1}+\lambda^2 I_{m0})+Lu_1(b)=R_1^2 J_{m4}, \tag{52}$$

where, with limits of integration and dr understood,

$$I_{m1}=\int\frac{1}{r}D(ru_1)D(ru_2), \qquad I_{m0}=\int ru_1u_2. \tag{53}$$

Similarly, multiplying (46) by rv_1 and integrating, one obtains eventually

$$-A(I_{m1}+\lambda^2 I_{m0})+Lu_2(b)=R_2^2 J_{m4}. \tag{54}$$

Taking the difference between (54) and (52) and letting R_1^2 approach R_2^2, we have

$$\frac{d}{dR^2} Lu(b)=J_4 \tag{55}$$

where

$$J_4=4\lambda^2 \int r\left[1-\varepsilon B\left(1-\frac{1}{r^2}\right)\right] v^2>0. \tag{56}$$

Equation (55) implies

Theorem 3. *If u and v satisfy* (11), (12), *and the boundary conditions* (35) *and* (36), *then $Lu(b)$ increases with R^2.*

The following lemmas will be useful for later developments:

Lemma 1. *If u and v are continuous and satisfy* (12) *and if u is positive (or negative) between two consecutive zeros of v, then v must be positive (or negative) between these zeros.*

Lemma 2. *If u and v satisfy* (11) *and* (12), *there cannot be two zeros of Q [defined by* (39)*] between which Q is positive (or negative), provided v is positive (or negative) between these zeros of Q.*

Lemma 3. *If u and v satisfy* (12) *and $v=0=Dv$ at any point P, then u and v must have opposite signs in the neighborhood of P.*

The proof of Lemma 1 is given by the argument leading to the absurdity of (41), since v is either entirely positive or entirely negative between two consecutive zeros. The proof of Lemma 2 is as follows. Suppose there were two such zeros, say at P and S. Writing (11) as

$$(L-\lambda^2-\sigma)Q=4\lambda^2 R^2\left[1-\varepsilon B\left(1-\frac{1}{r^2}\right)\right] v, \tag{57}$$

multiplying (57) by rQ and integrating between P and S, we obtain

$$-\int\left\{\frac{1}{r}[D(rQ)]^2+(\lambda^2+\sigma)rQ^2\right\}=4\lambda^2 R^2\int r\left[1-\varepsilon B\left(1-\frac{1}{r^2}\right)\right] vQ, \tag{58}$$

which is an absurdity if $vQ>0$ between P and S. Hence Lemma 2 is true.

Lemma 3 is proved in the following way. If u does not vanish at the point P, then (12) states that $D^2 v$ is of the opposite sign from u at P, where $v=0=Dv$. Then, since $Dv=0$ at P, v and u must differ in sign in the neighborhood of P. If $u=0$ at P, then if $D^n u$ is the first derivative of u that does not vanish at P, the first non-vanishing derivative of v at P is $D^{n+2}v$, and it differs in sign from $D^n u$ according to (12). Hence again the lemma is true.

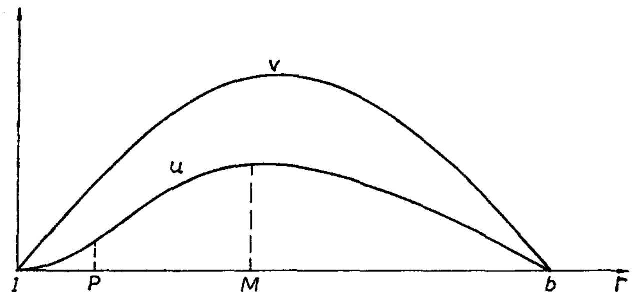

Fig. 1. Impossibility of the vanishing of $Du(1)$ when $Lu(1)$ vanishes

Let u and v satisfy the system consisting of (11), (12), (35), and (36). For $R^2=0$ we have seen that u and v are positive throughout and that $Lu(b)$ is negative. If we hold $bLu(b)$ at the value -1 and increase R^2, working with (50) and the corresponding relation for R_2, we see that

$$\frac{d}{dR^2}Dv(b)<0,$$

so that $Dv(b)$ decreases and will never reach zero. Hence $Dv(b)$ will not vanish as long as $Lu(b)$ is negative, as R^2 increases. It is evident that however small the magnitude of $Lu(b)$, the above inequality always holds, so that the possibility of $Lu(b)$ and $Dv(v)$ vanishing simultaneously is also ruled out. Holding $bDv(b)$ at the value -1 and increasing R^2, $Lu(b)$ increases according to (55), and we know that it will increase to zero as R^2 reaches E_1'', the existence of which is guaranteed by Theorem 2. When we have reached a positive value* of $bLu(b)$, say 1, we can consider the system consisting of (11), (12), (35), and

$$u(b)=0=v(b), \qquad bLu(b)=1,$$

and increase R^2. We see from Lemma 3 that neither $Dv(1)$ nor $Dv(b)$ can vanish as R^2 increases, before u becomes negative in the neighborhood of 1 or b, respectively. Hence $Du(b)$ must vanish before $Dv(b)$ as R increases. It can obviously not vanish twice before $Dv(b)$ does, since $Lu(b)$ is held positive.

We shall now show that in the evolution process just described, as long as v is non-negative $Du(1)$ must remain positive as R^2 is increased. For u and v are positive throughout to start with (except at the end points where they vanish), and v cannot become partly negative before u does. Suppose that $Du(1)=0$ at a value of R^2. Then Q is zero at $r=1$ as specified, but negative at M (Figure 1). In the neighborhood of 1, since $u(1)=Du(1)=0=Lu(1)$, and since u is positive in the neighborhood of 1 (because we are considering the situation just before u starts to become negative near 1), a Taylor series shows that both u and Lu increase

* Any positive value reached by $bLu(b)$ can always be taken to be 1, since we can then abandon the value -1 for $bDv(b)$, and u and v can be multiplied by any constant. See also the footnote following Theorem 6.

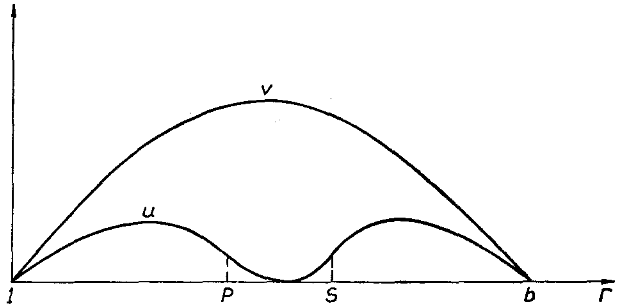

Fig. 2. Impossibility of creation of internal zeros of u

with r, with Lu increasing faster. Thus Q first *increases* with r at $r=1$. Then (Figure 1) there would be a point P between 1 and M, such that Q is zero at P and positive between 1 and P. Lemma 2 rules out this possibility, so that the supposition $Du(1)=0$ is false and $Du(1)$ must remain positive as R^2 increases, at least up to $R^2=E_1''$. (Later we shall show that $Du(1)>0$ for all R^2, as R^2 increases in the process of evolution of the eigenvalues and eigenfunctions, to be described below.)

The proof that u, Du, and Lu cannot all vanish at $r=1$ while u is non-negative throughout can be used intact to show that u, Du, and Lu cannot all vanish at $r=b$ while u is non-negative throughout. We know that $Lu(b)$ vanishes before or when R^2 reaches E_1', at which value we have $Du(b)=0$. Since $Du(b)$ and $Lu(b)$ cannot both vanish, $Lu(b)$ must vanish before $Du(b)$ as R^2 is increased. At $R^2=E_1''$, when $Lu(b)$ vanishes, $Du(b)$ is still negative. We have called the eigenvalues of the intermediate system E_n'. Thus we have just shown that $E_1''<E_1'$. The existence of E_1' is proved in the following way. If it did not exist, then $Du(b)$ would remain negative as R^2 increased, so that, *a fortiori*, $Dv(b)$ would remain negative. Then since R^2 is beyond E_1'' and $Lu(b)$ is positive, it will forever increase with R^2, with $Dv(b)$ kept at -1, say, and will never again vanish, contradicting the fact that infinitely many E_n'' exist. Arguing in this way, we find not only the existence of E_1' but also that of E_n'. This argument will be repeated for emphasis when absolute clarity seems especially important.

It is easy to show that as R^2 increases from zero to E_1', no internal zeros of u and v can develop. For any functions u and v satisfying (11), (12), (35), (36) are well-defined and continuous functions of R^2 up to $R^2=E_1'$, at which $Dv(b)$ remains negative (and fixed arbitrarily at -1). Hence if u were to become negative before $R^2=E_1'$, there would be a $R^2<E_1'$ for which u has a multiple zero of order $2n$ ($n=1, 2$, etc.) at an internal point. Then two points P and S (Figure 2) would exist, at which Q vanishes and between which Q is positive, contradicting Lemma 2. If v were to become negative at $R^2=E_1'$, there would be a $R^2<E_1'$ for which v has a zero of order $2n$ (Figure 3), contradicting Lemma 3.

We have treated the possibilities of u and v becoming negative near the end points or in the interior quite separately. It is easy to see that the possibility of u and v becoming negative simultaneously in the interior or near one or both of

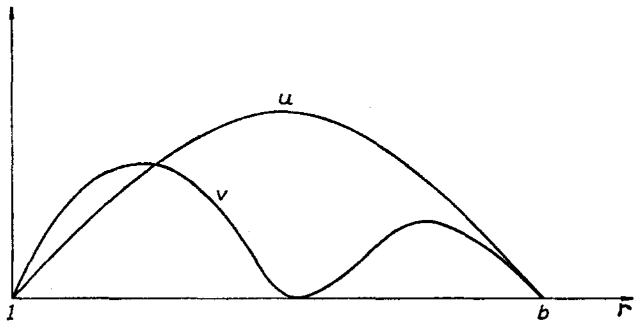

Fig. 3. Impossibility of creation of internal zeros of v

the end points can be ruled out, *a fortiori*, and we shall not present any further details.

The results given in the preceding five paragraphs can be summarized in the following theorems:

Theorem 4. *For the first mode corresponding to $R^2 = E_1''$, u and v are everywhere positive between the end points, at which they vanish. Furthermore Du and Dv are positive at* 1 *and negative at b.*

Theorem 5. $E_1'' < E_1'$.

We shall now let R^2 increase beyond E_1' and study the further evolution of u and v, which are supposed to satisfy (11), (12), and (35). Since we know that $Lu(b)$ is positive at $R^2 = E_1'$, we can now fix $bLu(b)$ at 1 and replace (36) by

$$u = 0 = v \quad \text{and} \quad bLu = 1 \quad \text{at} \quad r = b. \tag{59}$$

Then from (50), instead of (52) and (54) we obtain

$$-A(I_{m1} + \lambda^2 I_{m0}) - Dv_2(b) = R_1^2 J_{m4} \tag{60}$$

and

$$-A(I_{m1} + \lambda^2 I_{m0}) - Dv_1(b) = R_2^2 J_{m4}. \tag{61}$$

Taking the difference of (61) and (60), and letting R_1^2 approach R_2^2, we have

$$\frac{d}{dR^2} Dv(b) = J_4 > 0. \tag{62}$$

Hence we have

Theorem 6. *If u and v satisfy* (11), (12), (35), *and* (59), *then $Dv(b)$ increases with R^2.*

This theorem is a complement of Theorem 3.

As we increase R^2, $Dv(b)$ will eventually increase to zero at some finite R^2. For if not, $Dv(b)$ would remain negative, and we could fix it at a negative number

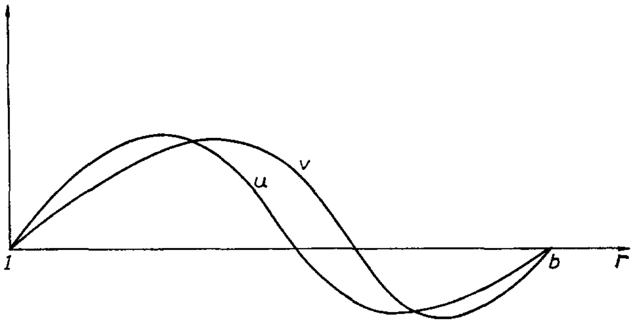

Fig. 4. Eigenfunctions of the second mode of the auxiliary system

and increase R^2 indefinitely, and $Lu(b)$ would increase indefinitely according to Theorem 3, and we should have only one eigenvalue E_1'' of the auxiliary system, in contradiction to Theorem 2. Furthermore, before $Dv(b)$ vanishes $Lu(b)$ cannot again vanish, for otherwise $Dv(b)$ would be negative throughout before $Lu(b)$ supposedly vanishes, and fixing $b\,Dv(b)$ at -1 we see that $Lu(b)$ would *increase*, not decrease from its positive value at $R^2=E_1'$ to zero. Now $Dv(b)$ will increase to 1, for if we fix $bLu(b)$ at 1 and increase R^2 toward the second eigenvalue E_2'', which we know to exist, $Dv(b)$ will increase indefinitely*, assuring its passage through b^{-1} at some R^2. Fixing $Dv(b)$ at b^{-1} afterwards, and increasing R^2 further, $Lu(b)$ will decrease to zero at $R^2=E_2''$ without encountering the difficulty of a vanishing $Dv(b)$, since, as long as $Lu(b)$ remains positive, $Dv(b)$ can only *increase*. We have then reached the second eigenvalue for the auxiliary system. Further evolution of the eigenvalues and eigenfunctions is now clear.

We have seen that after R^2 passes through E_1'' and before it passes through E_2'', $Du(b)$ must vanish once and only once. After E_1' is passed and before E_2'' is reached, $Lu(b)$ remains positive, so that $Du(b)$ cannot vanish *before* $Lu(b)$. It also cannot vanish simultaneously with $Lu(b)$. The proof of this impossibility is by the use of Lemma 2, and is similar to the proof for the first mode ($R^2=E_1''$), the only difference being that one has to pursue the arguments from one "arch" of the eigenfunction u to the next to reach the conclusion. (There are two "arches" of u for $R^2=E_2''$, one of which is "inverted", where u is negative.) Hence $Du(b)$ vanishes only once between E_1'' and E_2'', so that between these there is only one E_1'. The same argument applies to higher modes.

Just as in the case of the first mode, for the second mode $Du(b)$ also cannot be zero. Unlike the first mode, however, the second mode has a *positive* $Du(b)$ because when $Du(b)$ vanishes at $R^2=E_2'$, $Lu(b)$ is already negative, which implies that E_2'' has already been passed. Hence

$$E_1''<E_1'<E_2''<E_2'.$$

* Toward infinity, since $Dv(b)/Lu(b)$ approaches ∞ as R^2 approaches E_2''.

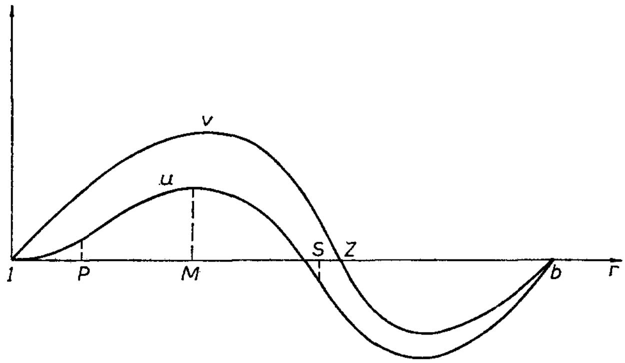

Fig. 5

We pause to note that at E_1', u and v are still non-negative throughout, since the arguments leading to Theorem 4 show that u and v are positive between the end points all the way up to E_1'. During the passage from E_1' to E_2'', $Du(b)$ will become positive; then $Dv(b)$ will become zero and then positive (before $Lu(b)$ again vanishes) at E_2''. The eigenfunctions u and v are as shown in Figure 4. The following lemmas hold for u and v satisfying (11), (12), (35), and (36) or (59) as the evolution progresses.

Lemma 4. *During the transition from E_1'' to E_2'', $Du(1)$ and $Dv(1)$ remain positive.*

Lemma 5. *During the transition from E_1'' to E_2'', no new zeros of u or v can be created in the interior of the interval $1 \leqq r \leqq b$, nor are multiple zeros possible in the interior.*

Lemma 6. *A new zero of u will be created in the form of a double zero at $r=b$ during the transition from E_1'' to E_2'', then a new zero of v. When E_2'' is reached, Du and Dv are both positive at $r=1$ and at $r=b$, and there is exactly one simple interior zero for u, and one for v.*

Lemma 6 follows from the results given in the paragraph preceeding. The proof of Lemma 5 is similar to that given for the transition from $R^2=0$ to $R^2=E_1''$, although it is a little more complicated. The same complication is involved in the proof of Lemma 4, and requires a kind of an "argument of pursuit". For these reasons, and because Lemma 4 is crucial later on, we shall give the proof of Lemma 4 only.

To prove Lemma 4, we note that $Dv(1)$ cannot vanish before $Du(1)$, by virture of Lemma 3. It is then sufficient to show that $Du(1)$ cannot vanish. If it did, Q would increase with r (Figure 5) in the neighborhood of 1, because $Lu(r)$ increases faster than $u(r)$, the point $r=1$ being a triple zero. Thus there would be a point P between 1 and M at which $Q=0$, with Q positive between P and 1. Furthermore, there would exist at least one point S such that (i) $Q=0$ at S and (ii) $Q<0$ between S and the nearest zero of Q to the left of S (this zero of Q is shown as P in Fig. 5, though this need not always be the case). Whether P lies to the left or right of the internal zero (Z) of v, Lemma 2 would be violated if $Du(1)=0$, if not in the arch of v then necessarily in its inverted arch (this is the "argument of pursuit"), since $Q=0$ at $r=1$ also. Hence Lemma 4 is true. It can be shown similarly that Lemma 5 holds.

Further evolution of the eigenvalues E_n'' and the eigenfunctions of the auxiliary system and concurrently those of the immediate system is now clear. The same types of arguments can be used to establish the following theorems:

Theorem 7. *For the auxiliary system constructed by the process described above, Du and Dv are always positive at $r=1$. They are negative at $r=b$ for the eigenfunctions corresponding to E_{2n-1}'' ($n=1, 2, 3$, etc.) and positive for the other eigenfunctions. The eigenfunctions u and v for E_n'' have each exactly $n-1$ internal zeros, all simple.*

Theorem 8. *For the intermediate system, all the statements in Theorem 7 hold, except that $Du(b)=0$ by definition of the system, and E_n'' should be replaced by E_n'. Furthermore, $Lu(b)$ is positive for E_{2n-1}' ($n=1, 2, 3$, etc.), and negative for the other eigenfunctions.*

Theorem 9. $E_1''<E_1'<E_2''<E_2'\cdots<E_n''<E_n'<E_{n+1}''\cdots$.

We shall next show how the eigenvalues of the transitory system evolve as we study the evolution of the eigenfunctions of the intermediate system from a slightly different point of view.

7. Evolution of Eigenvalues and Eigenfunctions for the Transitory System

The development in Section 6 show that there are infinitely many eigenvalues E_n' ($n=1, 2$, etc.) for the intermediate system. Furthermore $Lu(b)$ is positive for n odd and negative for n even, and $Du(1)$ and $Dv(1)$ are both positive. We now multiply the eigenfunctions for even n by -1 so that the new series of eigenfunctions are characterized by a positive $Lu(b)$ for all modes, positive $Du(1)$ and $Dv(1)$ for n odd, and negative $Du(1)$ and $Dv(1)$ for n even. Furthermore we can divide the eigenfunctions by the (new) positive $Lu(b)$. Then the eigenfunctions of the intermediate system are characterized by

$$u(1)=0=v(1), \tag{63}$$

$$u(b)=Du(b)=0=v(b), \tag{64}$$

$$Lu(1)=0, \tag{65}$$

and

$$Lu(b)=1. \tag{66}$$

The eigenfunctions of the first two modes so specified are depicted in Figures 6 and 7.

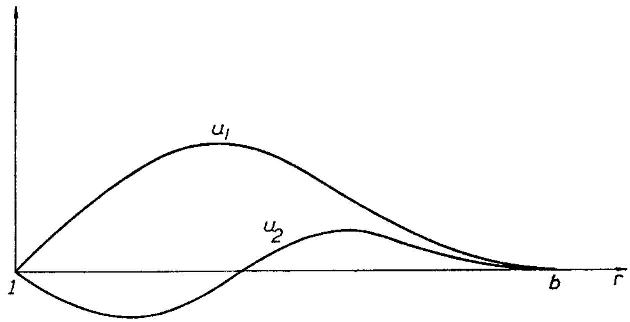

Fig. 6. The u function for the first two modes of the intermediate system, the subscript indicating the mode

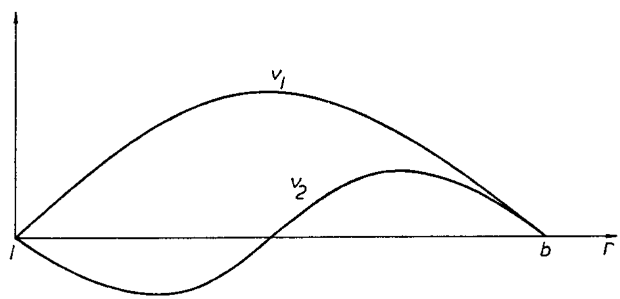

Fig. 7. The v function for the first two modes of the intermediate system, the subscript indicating the mode

It is now crucial to demonstrate that there is a *continuous* evolution of the eigenfunctions so specified. That is, there is a well-defined system whose solutions u and v "pass through" the eigenfunctions described above as R^2 increases. In the process of the evolution of these eigenfunctions by means of this well-defined system, the eigenfunctions of the transitory system (and, in particular, those of the original system) will evolve, as is evident from the fact that $Du(1)$ changes sign repeatedly.

We may use the system consisting of the differential equations (11) and (12) and the boundary conditions (63), (64), and (66). To qualify as a well-defined system, however, we have to be sure that $Lu(b)$ never vanishes* as R^2 increases. That such a system is well-defined will now be shown.

According to Theorems 7 and 8, $Du(1)$ is never zero for the auxiliary or the intermediate system, for both of which $Lu(1)=0$. Indeed, $Du(1)$ is never zero as we increase R^2 through the eigenvalues E_n'' and E_n'. (The proof for this statement is similar to the proof of Lemma 4, although the "argument of pursuit" referred to before has to be employed.) Thus $Du(1)$ is nonzero not only at these eigenvalues

* In other words, the system consisting of (11), (12), (63), (64), and $Lu(b)=0$ never has any positive eigenvalue for R^2.

but between them, throughout the entire range of positive R^2. This means that if

$$v(1)=u(1)=0=v(b)=u(b) \tag{66a}$$

and

$$Lu(1)=0,$$

then $Du(1)$ cannot vanish. That is to say, if (66a) holds and

$$Du(1)=0,$$

then $Lu(1)$ cannot vanish. Since there is nothing to impede the symmetry of arguments as the roles of the end points $r=1$ and $r=b$ are interchanged, we conclude that if (66a) holds and

$$Du(b)=0,$$

then $Lu(b)$ cannot vanish. That is to say, (11), (12), (63), (64), and (66) constitute a well-defined system. We shall call this system W.

We now recall that for the new series of eigenfunctions defined by (11), (12), and (63) to (66), $Du(1)$ and $Dv(1)$ are positive for odd n in E_n' and negative for even n. It is important to keep in mind that as R^2 increases from E_n' to E_{n+1}', the sign of $Lu(1)$ remains the same, the only values of R^2 at which $Lu(1)$ vanishes being E_n' and E_{n+1}', for the simple reason that E_n' and E_{n+1}' are consecutive zeros of $Lu(1)$ for the intermediate system. Furthermore, between E_1' and E_2' the sign of $Lu(1)$ is positive, because if $Du(1)$ vanishes as u and v vary continuously with R^2 from the non-negative u and v at $R^2=E_1'$, $Lu(1)$ must be positive there. Similarly $Lu(1)$ must be negative between E_2' and E_3'. In general $Lu(1)$ is positive between E_{2m-1}' and E_{2m}' and negative between E_{2m}' and E_{2m+1}', where m is any positive integer.

The well-defined System W is available for evolution of the eigenvalues and eigenfunctions of the transitory system. We know that the eigenvalues E_n' ($n=1$, 2, etc.) exist for the intermediate system, and we know the properties of the corresponding eigenfunctions. As we increase R^2 from E_1' to E_2', since $Du(1)$ changes sign in the process, $Du(1)$ must vanish at least once, and furthermore $Lu(1)$ is positive, so that there is at least one eigenvalue E_1 between E_1' and E_2', for the transitory system, and for which u and hence v are non-negative throughout. Similarly, as R^2 increases from E_n' to E_{n+1}', there must be at least one eigenvalue E_n between E_n' and E_{n+1}', for which $Lu(1)$ has the sign of $(-1)^{n-1}$ and u and v both have exactly $n-1$ simple (because of Theorem 8) internal zeros*. Hence we have

Theorem 10. *For any real $\sigma \geqq -\lambda^2$ and ε in $0 \leqq \varepsilon \leqq 1$, real eigenvalues of R^2 exist for the transitory system. There is at least one eigenvalue E_n between E_n' and*

* The statement concerning internal zeros of u and v in Theorem 8 holds not merely for $R^2=E_n'$ (for any n), but also throughout the transition as R^2 varies from E_n' to E_{n+1}'. This situation continues to hold in our present transition from E_n' to E_{n+1}' (by keeping $Du(b)$ equal to zero). Hence for any eigenvalue E_n between E_n' and E_{n+1}', both u and v have exactly $n-1$ simple internal zeros.

E'_{n+1}, and for any such eigenvalue both u and v have exactly $n-1$ simple internal zeros.

This theorem is a cornerstone of the spectral theory. Since in particular ε can be 1, we have the

Corollary. *For any real $\sigma \geqq -\lambda^2$, real eigenvalues of R^2 exist for the original system.*

We shall show later that between E'_n and E'_{n+1} there is exactly one eigenvalue E_n for the transitory system.

8. Number of Eigenvalues of the Transitory System between E'_n and E'_{n+1}

We shall now consider the eigenvalues of the starting system, that is, the transitory system with $\varepsilon=0$. Equations (11) and (12) become

$$(L-\lambda^2-\sigma)(L-\lambda^2)u=4\lambda^2 R^2 v, \tag{67}$$

$$(L-\lambda^2-\sigma)v=Au. \tag{68}$$

The boundary conditions are still (7). If the parameters λ^2 and R^2 are assumed real and positive, it is easy to show that σ must be real if A is negative.

Using an asterisk to denote the complex conjugate, and multiplying (67) by ru^* and integrating between 1 and b, we have by appropriate integration by parts and use of (7)

$$I_2+(2\lambda^2+\sigma)I_1+\lambda^2(\lambda^2+\sigma)I_0=4\lambda^2 R^2 \int rvu^*, \tag{69}$$

where

$$I_2=\int r|Lu|^2, \quad I_1=\int \frac{1}{r}|D(ru)|^2, \quad I_0=\int r|u|^2, \tag{70}$$

the limits of integration being understood and dr being omitted in (69) and (70). Using (68) and the boundary conditions on v given in (7), we may write the integral on the right-hand side of (69) as

$$-J_1-(\lambda^2+\sigma^*)J_0,$$

with J_1 and J_0 defined by (19). Thus (69) can be written as

$$I_2+(2\lambda^2+\sigma)I_1+\lambda^2(\lambda^2+\sigma)I_0=-4\lambda^2 R^2 A^{-1}[J_1+(\lambda^2+\sigma^*)J_0]. \tag{71}$$

The imaginary part of (71) can be written as

$$\sigma_i(I_1+\lambda^2 I_0-A^{-1}\lambda^2 R^2 J_0)=0. \tag{72}$$

Since A is negative, this says that $\sigma_i=0$ for the starting system, or σ is real. Note also that, for real σ, (71) demands that R^2 be real for real λ^2.

The existence of eigenvalues E''_n or E'_n for the auxiliary system or the intermediate system of the starting system is known, of course, since Theorems 7 to 9 are valid for any ε in $0\leqq\varepsilon\leqq 1$. Also the existence of E_n for the starting system is stated by Theorem 10. However, for the starting system we can show that between E_n' and E_{n+1}' there is exactly one E_n, for any non-negative $\sigma+\lambda^2$.

Fixing $\sigma+\lambda^2$ at any non-negative real value, we consider two systems. The first consists of

$$(L-\lambda^2-\sigma)(L-\lambda^2)u_1=4\lambda^2 R_1^2 v_1, \tag{73}$$

$$(L-\lambda^2-\sigma)v_1=Au_1. \tag{74}$$

The second consists of

$$(L-\lambda^2-\sigma)(L-\lambda^2)u_2+4\lambda^2 R_2^2 v_2, \tag{75}$$

$$(L-\lambda^2-\sigma)v_2=Au_2. \tag{76}$$

In (73) to (76), u_1, u_2, v_1, and v_2 satisfy the conditions

$$u_i(b)=Du_i(b)=0=v_i(b), \quad u_i(1)=0=v_i(1) \quad \text{for } i=1 \text{ or } 2, \tag{77}$$

and one more condition to be specified.

Multiplying (73) by ru_2 and integrating between 1 and b, and using (76) to evaluate the u_1 on the right-hand side, we have

$$\begin{aligned} I_{m2}+(2\lambda^2+\sigma)I_{m1}+\lambda^2(\lambda^2+\sigma)I_{m0}+Du_2(1)Lu_1(1) \\ =-4\lambda^2 R_1^2 A^{-1}[J_{m1}+(\lambda^2+\sigma)J_{m0}], \end{aligned} \tag{78}$$

where

$$\begin{aligned} I_{m2}=\int rLu_1Lu_2, \quad I_{m1}=\int\frac{1}{r}D(ru_1)D(ru_2), \quad I_{m0}=\int ru_1u_2, \\ J_{m1}=\int\frac{1}{r}D(rv_1)D(rv_2), \quad J_{m0}=\int rv_1v_2, \end{aligned} \tag{79}$$

the m standing for "mixed". Similarly, multiplying (75) by ru_1 and integrating, using (74) to evaluate the u_1 on the right-hand side, we have

$$\begin{aligned} I_{m2}+(2\lambda^2+\sigma)I_{m1}+\lambda^2(\lambda^2+\sigma)I_{m0}+Du_1(1)Lu_2(1) \\ =-4\lambda^2 R_2^2 A^{-1}[J_{m1}+(\lambda^2+\sigma)J_{m0}]. \end{aligned} \tag{80}$$

For the special case of $R_1=R_2=0$, we shall drop the subscripts; then either (78) or (80) shows that $Du(1)\,Lu(1)$ must be negative. We shall, to begin with, take $Du(1)$ to be 1; then $Lu(1)$ is negative at $R=0$. The difference of (80) and (78) is, for $Du_1(1)=1=Du_2(1)$,

$$Lu_2(1)-Lu_1(1)=-4\lambda^2 A^{-1}(R_2^2-R_1^2)[J_{m1}+(\lambda^2+\sigma)J_{m0}]. \tag{81}$$

If we let R_1 approach R_2, we have

$$\frac{d}{d(R^2)}Lu(1)=-4\lambda^2 A^{-1}[J_1+(\lambda^2+\sigma)J_0], \tag{82}$$

with J_1 and J_0 defined by (19). This shows that as long as $Du(1)$ is positive and fixed at 1, $Lu(1)$ increases with R^2. We know from Section 6 that E_1' exists. Therefore $Lu(1)$ will increase to zero. We know also that when it becomes zero $Du(1)$ is still positive. After E_1' has been passed, $Lu(1)$ will increase to 1 if $Du(1)$ is kept at 1 as R^2 increases indefinitely. For otherwise E_1 would not exist, and from Section 7 we know that at least one E_1 exists between E_1' and E_2'. (Existence of E_1 demands that as it is approached, $Lu(1)$ approaches infinity if $Du(1)$ is kept at 1.) When $Lu(1)$ becomes 1, we keep it at 1. Then from (80) and (78) we have, if both $Lu_1(1)$ and $Lu_2(1)$ are kept at 1,

$$Du_2(1)-Du_1(1)=4\lambda^2 A^{-1}(R_2^2-R_1^2)\,[J_{m1}+(\lambda^2+\sigma)\,J_{m0}], \tag{83}$$

which in the limit becomes

$$\frac{d}{d(R^2)}\,Du(1)=4\lambda^2 A^{-1}\,[J_1+(\lambda^2+\sigma)\,J_0]. \tag{84}$$

Thus $Du(1)$ will decrease monotonically as R^2 increases, and since we know E_1 exists, it will decrease to zero. The corresponding value of R^2 is E_1. When E_1 is passed, $Du(1)$ becomes negative; it will decrease to negative infinity if we fix $Lu(1)$ at 1 and increase R^2 toward E_2'. It will therefore reach -1. Fixing $Du(1)$ at -1, we see that $Lu(1)$ will decrease monotonically, first to zero, when R^2 reaches E_2', and then to -1. The argument continues, and we see that between E_n' and E_{n+1}' there is exactly one eigenvalue E_n for the starting system. We summarize the result obtained so far in this section in the following theorem:

Theorem 11. *For the starting system, with negative A, σ is real if R^2 is real, and vice versa, and exactly one eigenvalue E_n for R^2 exists between E_n' and E_{n+1}'.*

We shall now show that for the starting system both the eigenvalues E_n and E_n' are simple, for any positive integral value of n. Their simplicity is indeed already evident from (82) and (84). If the secular equation relevant to E_n' is

$$F_1(R^2,\sigma,\varepsilon)=0, \tag{85}$$

then for E_n' to be a multiple eigenvalue it is necessary that both (85) and

$$\frac{\partial}{\partial R^2}\,F(R^2,\sigma,\varepsilon)=0 \tag{86}$$

be satisfied. This means that if R^2 varies from E_n' to $E_n'+\Delta R^2$ while

$$u(b)=Du(b)=0=v(b)=u(1)=v(1), \qquad Du(1)=1,$$

then $Lu(1)$ is of the order of $(\Delta R^2)^2$, that is,

$$\frac{d}{dR^2}\,Lu(1)=0,$$

which is clearly contradicted by (82) for $\sigma \geqq -\lambda^2$. Hence E_n' is simple for any positive integral value of n. Similarly, (84) shows that E_n is simple. Hence we have

Theorem 12. *For the starting system and for $\sigma \geqq -\lambda^2$, the eigenvalues E_n' and E_n are simple.*

Actually a similar demonstration will show that for the starting system the eigenvalues E_n'' are also simple.

Furthermore, the same reasoning (using (55)) shows that the eigenvalues E_n'' are simple for the auxiliary system, for any ε between zero and 1.

With the help of Theorem 12, we shall now prove

Theorem 13. *For $\sigma \geqq -\lambda^2$ and $0 \leqq \varepsilon \leqq 1$, the transitory system has exactly one eigenvalue E_n of R^2 between E_n' and E_{n+1}', and it is simple.*

Indeed, if there were more than one E_n between E_n' and E_{n+1}', then as $\varepsilon \to 0$ there would be more than one E_n between E_n' and E_{n+1}' for the starting system, in contradiction to Theorem 11, if these do not merge and become complex in the approach to $\varepsilon = 0$. If they do merge and remain complex throughout the approach to $\varepsilon = 0$, the reality of all E_n for the starting system would be contradicted. If they merge and then reappear as real eigenvalues at $\varepsilon = 0$, this would make one of the eigenvalues E_n at $\varepsilon = 0$ (for $n = 1, 2, \ldots$) other than simple, in contradiction to Theorem 12. The same arguments show that E_n must be simple for $0 < \varepsilon \leqq 1$. Hence we have proved Theorem 13.

We note in passing that the secular equation

$$F(R^2, \sigma, \varepsilon) = 0$$

for the transitory system represents, for any finite real values of the arguments R^2, σ, and ε, sheets of continuous surfaces in the space of these parameters or variables, by virtue of the fact that the eigenfunctions, and hence $F(R^2, \sigma, \varepsilon)$, are entire functions of R^2, σ, and ε., and that positive eigenvalues of R^2 exist for $\varepsilon = 0$ and $\sigma > 0$.

All the preceding development is for non-negative $\lambda^2 + \sigma$. Before going on to study damped modes with $\sigma < -\lambda^2$, we shall pause to review what has been done and to discuss other related work. The main results we have obtained are (*a*) that for non-negative $\lambda^2 + \sigma$ there are infinitely many real positive eigenvalues E_n of R^2, (*b*) that these are simple and are all the eigenvalues, there being no complex ones, (*c*) that between E_n' and E_{n+1}' there is exactly one eigenvalue E_n, and (*d*) that the eigenfunctions u and v for E_n each have exactly $n-1$ internal zeros. After these results were found, my attention was drawn to the work of Yudovich [7] and of Ivanilov & Iakoviev [8], where some of my results have already been obtained. The approach used here is entirely new for eigenvalue problems of order higher than 2, and is predominately elementary in comparison with [7] and [8]. The easily found bounds E_n'' may also be of some interest.

Since we have only dealt with the case of non-negative $\lambda^2 + \sigma$, we are far from having proved the nonoscillatory nature of Taylor vortices. The proof given in Part II will show that the preceding development is not necessary in its entirety, so far as that proof is concerned. We have given the preceding development because it is of much interest to know the structure of Taylor vortices when they are unstable, and Theorem 10 gives information regarding the nodal points (internal zeros) of such vortices.

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