# *Existence Theorems Concerning Simple Integrals of the Calculus of Variations for Discontinuous Solutions*

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*Dedicated to James Serrin on his 60^th^ birthday*

## 1. Introduction

In this paper we apply the direct method of the calculus of variations, based on lower semicontinuity and lower closure (see [5]), to prove the existence of optimal solutions $x(t) = (x^1, \ldots, x^n)$, $t_1 \leqq t \leqq t_2$, for which $\alpha$ components $y(t) = (x^1, \ldots, x^\alpha)$ are AC and $n - \alpha$ components $z(t) = (x^{\alpha+1}, \ldots, x^n)$ are BV and not necessarily AC. If $\alpha = 0$ all components of $x$ are BV, and in this situation no growth assumption is made on the integrand function. The cost functional $\mathscr{I}$ is of SERRIN type ([14]), *i.e.* it is obtained from the usual integral expression $I$ by means of a limit process, based on a topology $\tau$, of this integral $I$ over curves $x_k$ whose components are all AC. The topology $\tau$ that we use here is the topology of uniform convergence on the $y_k$ components and pointwise convergence almost everywhere on the $z_k$ components. This pointwise convergence almost everywhere has been used by CESARI in the study of area of discontinuous surfaces ([4], 1936) and in existence theorems concerning simple integrals for AC solutions (see [5], Chapt. 15 and the papers cited there).

In Section 2 we first prove a closure theorem (Theorem 1) for problems in which mere pointwise convergence almost everywhere is adopted. The closure theorem is used, in Section 3, for proving a lower semicontinuity theorem (Theorem 1') based on the topology $\tau$. The same lower semicontinuity theorem allows us to prove that $\mathscr{I}$ is a true extension of $I$, in the sense that $\mathscr{I} = I$ whenever all components are AC.

In Section 3 we prove also an existence theorem for the absolute minimum of extended problems of the calculus of variations with constraints on the direction of the tangent. In Section 4 we derive, as a corollary, an existence theorem for the absolute minimum of problems of optimal control. For a different viewpoint connecting Serrin-type integrals, usual integrals, and Burkill-Cesari integrals we mention the paper [3] by CANDELORO & PUCCI, where also lower semicontinuity theorems are given for solutions which are only continuous and of bounded variation.

Elsewhere ([6b]) the present work on discontinuous solutions will be extended to multiple integrals of the calculus of variations and functions of $\nu > 1$ independent variables. Therefore, the BV concept in [6b] will be the one introduced by CESARI in 1936 ([4]) and shown by KRICKEBERG ([12]) to be equivalent to the one in terms of distributions. Later the functions of bounded variation in the sense of CESARI were briefly denoted as BVC by CONWAY & SMOLLER ([8]), DAFERMOS ([9]) and DIPERNA ([10]). The functions of bounded variation defined in the equivalent terms of distributions were briefly denoted as BV by VOLPERT ([16]) and others. In order that the present work, which concerns functions of one variable, be in harmony with [6b], we use the notations from [6b].

## 2. A closure theorem with components converging only pointwise

Let $A$ be a subset of the $(t, x)$-space $R^{n+1}$ whose projection on the $t$-axis contains the fixed interval $[t_1, t_2]$. Let $Q(t, x)$, $(t, x) \in A$, $Q(t, x) \subset R^n$, or $Q: A \to R^n$, be a given set valued function.

Following CESARI [5] we shall say that the set function $Q$ has *property* $(Q)$ at the point $(\bar{t}, \bar{x})$, with respect to $(t, x)$, if

$$Q(\bar{t}, \bar{x}) = \bigcap_{\delta > 0} \text{cl co} \bigcup_{(t,x) \in B(\bar{t}, \bar{x}; \delta)} Q(t, x)$$

where $B(\bar{t}, \bar{x}; \delta) = \{(t, x) \in A : |(t, x) - (\bar{t}, \bar{x})| \leqq \delta\}$. Let $Q(\bar{t}, \bar{x}; \delta) = \bigcup Q(t, x)$ for $(t, x) \in B(\bar{t}, \bar{x}; \delta)$.

Analogously, $Q$ is said to have *property* $(Q)$ at the point $(\bar{t}, \bar{x})$, with respect to $x$ only, if

$$Q(\bar{t}, \bar{x}) = \bigcap_{\delta > 0} \text{cl co} \bigcup_{x \in B'(\bar{t}, \bar{x}; \delta)} Q(\bar{t}, x)$$

where $B'(\bar{t}, \bar{x}; \delta) = \{(t, x) \in A : |x - \bar{x}| \leqq \delta\}$. The corresponding Kuratowski properties (K) are obtained by writing only cl, instead of cl co, in the relations above.

We mention here that a summable function $x(t)$ from $[t_1, t_2]$ into $R^n$, or $x: [t_1, t_2] \to R^n$, is said to be of *bounded variation in the sense of Cesari,* briefly BVC, if it is equivalent to a BV function $\tilde{x}: [t_1, t_2] \to R^n$. It may well occur that $x$ is equivalent to infinitely many BV functions $\tilde{x}$. In this case, at every point $t_0 \in (t_1, t_2)$ of (first kind) discontinuity for $\tilde{x}$, we may take $\tilde{x}(t_0)$ so that $\tilde{x}(t_0 - 0) \leqq \tilde{x}(t_0) \leqq \tilde{x}(t_0 + 0)$ or the same relations with the sign $\geqq$. Also, we may take $\tilde{x}(t_1) = \tilde{x}(t_1 + 0)$, $\tilde{x}(t_2) = \tilde{x}(t_2 - 0)$. With this choice for $\tilde{x}$ the variation $V(\tilde{x})$ is uniquely determined and it has the minimum value for all $\tilde{x}$ equivalent to $x$. We take, by definition of *generalized variation* $V^*(x)$ the number $V^*(x) = V(\tilde{x})$, for $\tilde{x}$ chosen as stated. Moreover we take, by definition $x' = \tilde{x}'$ (a.e. in $[t_1, t_2]$).

Analogously, $x$ is said to be *absolutely continuous in the generalized sense,* briefly ACg if $x$ is equivalent to an AC function $\tilde{x}$. In this case $\tilde{x}$ is uniquely defined, and for the generalized variation we take $V^*(x) = V(\tilde{x})$.

For further properties of such functions see [4], [6a], [1], [2], [13].

We shall consider the orientor field equation

$$(t, x(t)) \in A, \quad x'(t) \in Q(t, x(t)), \quad \text{a.e. in } [t_1 \ t_2]. \tag{1}$$

that is, the problem of determining a BVC function $x$ satisfying these relations.

We state and prove now a closure theorem which replaces, in the present situation, the closure theorem 15.2.i of [5].

**Theorem 1** (A closure theorem). *Let us assume that* (i) *$A$ is closed*; (ii) *the set valued function $Q$ has closed and convex values*; (iii) *the set valued function $Q$ has property* (Q), *with respect to $(t, x)$, at every point $(\bar{t}, \bar{x}) \in A$, with the exception perhaps of a set of points whose t-coordinate lies in a set $H$ of measure zero in $[t_1, t_2]$.*

*Let $x_k : [t_1, t_2] \to R^n$, $k \in N$, be a sequence of* ACg *solutions of the orientor field* (3), *and assume that $V^*(x_k) \leqq V_0$, $k \in N$, and that $x_k \to x$ pointwise a.e. in $[t_1, t_2]$, with $x \in$* BVC. *Then the function $x$ is a solution of the orientor field relation* (3).

**Proof.** (a) By the hypotheses it follows that

$$(t, x(t)) \in A, \quad \text{a.e. in } [t_1 \ t_2], \tag{2}$$

so we have only to prove that

$$x'(t) \in Q(t, x(t)) \cdot \quad \text{a.e. in } [t_1, t_2]. \tag{3}$$

In order to see that, without loss of generality we can suppose that $x$ is BV and $x_k$ is AC, $k \in N$.

Let $T_0 \subset [t_2, t_2]$ be a set of measure zero such that in $[t_1, t_2] - T_0$ we have

$$\lim_{k\to\infty} x_k(t) = x(t) \quad \text{and} \quad x'(t) = x'_a(t)$$

where $x = x_a + x_s$ denotes the Jordan decomposition of $x$.

(b) Now for every $m \in N$, we divide $[t_1, t_2]$ into $m$ equal parts $I_r^{(m)}$, $r = 1, \ldots, m$, each of length $(t_2 - t_1) m^{-1} = Tm^{-1}$; and denote by $T_1 \subset [t_1, t_2]$ the set of all points of subdivision, so that $T_1$ is denumerable and, therefore, has measure zero.

Let $m \in N$ and $\varepsilon > 0$ be fixed. For every $k \in N$, we consider those intervals $I_r^{(m)}$ if any, such that $\omega(x_k, I_r^{(m)}) \geqq \varepsilon$, where $\omega(x_k, I)$ denotes the oscillation of $x_k$ over $I$.

Let $S_k^{(m)}$ be the system of such intervals, or

$$S_k^{(m)} = \{I_r^{(m)}, r = 1, \ldots, m : \omega(x_k, I_r^{(m)}) \geqq \varepsilon\}.$$

We now proceed to the determination of a suitable set $\Sigma^{(m)}$ and to the extraction of a suitable subsequence of $(x_k)_{k\in N}$. First, if $I_1^{(m)} \in S_k^{(m)}$ for all $k$ sufficiently large, we put $I_1^{(m)}$ in $\Sigma^{(m)}$; if not then there are infinitely many $k \in N$ such that $I_1^{(m)} \notin S_k^{(m)}$ and we denote by $(k_{1s})_{s\in N}$ such sequence, i.e. $I_1^{(m)} \notin S_{k_{1s}}^{(m)}$, $s \in N$. If $I_2^{(m)} \in S_{k_{1s}}^{(m)}$ for all $s$ sufficiently large, we put $I_2^{(m)}$ in $\Sigma^{(m)}$; if not then there are infinitely many $s \in N$ such that $I_2^{(m)} \notin S_{k_{1s}}^{(m)}$. We denote such a sequence by $(k_{2s})_{s\in N}$; then $(k_{2s}) \subset (k_{1s})$ and $I_2^{(m)} \notin S_{k_{2s}}^{(m)}$, $s \in N$.

We proceed as indicated for $I_3^{(m)}, \ldots, I_m^{(m)}$.

At the end we have a set $\Sigma^{(m)}$ made up of all points of certain intervals $I_r^{(m)}$, $r = 1, \ldots, m$, say, for simplicity $\Sigma^{(m)} = \{I_i^{(m)}, i = 1, \ldots, \nu\}$, and a final sequence $(k_{ms})_{s\in N}$ with $(k_{ms})_{s\in N} \subset (k_{m-1,s})_{s\in N} \subset \cdots \subset (k_{1s})_{s\in N}$.

Note that for all $k_{ms}$ sufficiently large we have

$$\omega(x_{k_{ms}}, I_i^{(m)}) \geqq \varepsilon, \quad i = 1, \ldots, \nu,$$

and hence

$$\nu\varepsilon \leqq \sum_{i=1}^{\nu} \omega(x_{k_{ms}}, I_i^{(m)}) \leqq \sum_{r=1}^{m} \omega(x_{k_{ms}}, I_r^{(m)}) \leqq V(x_{k_{ms}}) \leqq V_0,$$

i.e. $\nu \leqq V_0 \varepsilon^{-1}$.

This implies that

$$\text{meas}\,(\Sigma^{(m)}) = \sum_{i=1}^{\nu} \text{meas}\,(I_i^{(m)}) = \nu T m^{-1} \leqq V_0 T/\varepsilon m.$$

Hence, for every $\varepsilon > 0$ we can choose an integer $m_\varepsilon$ sufficiently large that

$$\text{meas}\,(\Sigma^{(m_\varepsilon)}) \leqq V_0 T/\varepsilon m_\varepsilon < \varepsilon.$$

Now we take $\varepsilon$ ranging in succession over the values $(1/2^\lambda)_{\lambda\in N}$. Thus, for $\lambda = 1$ then $\varepsilon = 1/2$ and, starting from the original sequence $(k)_{k\in N}$, we obtain from the above an integer $m_\varepsilon$, which we denote by $m_1$, a set $\Sigma^{(m_\varepsilon)}$, which we denote by $\Sigma^{(1)}$, and a subsequence $(k_{ms})_{s\in N}$ that we denote by $(k_s^1)_{s\in N}$.

For $\lambda = 2$ then $\varepsilon = 1/2^2$ and, starting from the sequence $(k_s^1)_{s\in N}$ we obtain, as before, an integer $m_2$, a set $\Sigma^{(2)}$ and a sequence $(k_s^2)_{s\in N}$.

Proceeding as indicated for the generic $\lambda \in N$, we see that $\varepsilon = 1/2^\lambda$ and, starting from the sequence $(k_s^{\lambda-1})_{s\in N}$, we obtain an integer $m_\lambda$, a set $\Sigma^{(\lambda)}$ and a subsequence $(k_s^\lambda)_{s\in N}$ as before.

It is not restrictive to assume that $(m_\lambda)_{\lambda\in N}$ is an increasing sequence. We consider now the sets

$$\Sigma_n = \bigcup_{\lambda=n}^{\infty} \Sigma^{(\lambda)}, \quad n \in N \quad \text{and} \quad \Sigma_0 = \bigcap_{n\in N} \Sigma_n.$$

We have,

$$\text{meas}\,\Sigma_n \leqq \sum_{\lambda=n}^{\infty} \text{meas}\,(\Sigma^{(\lambda)}) \leqq \sum_{\lambda=n}^{\infty} 1/2^\lambda = 1/2^{n-1}1, \qquad \text{meas}\,\Sigma_0 = 0.$$

(c) Let us now take any point $t_0 \in [t_1\ \ t_2] - (\Sigma_0 \cup T_0 \cup T_1 \cup H)$. Then there is a real $\sigma > 0$ and an integer $n_0$ such that $t_1 < t_0 - \sigma < t_0 < t_0 + \sigma < t_2$ and $t_0 \notin \Sigma^{(\lambda)}$ for every $\lambda \geqq n_0$.

For every given $\varepsilon > 0$ we take $\lambda \in N$ sufficiently large that $1/2^\lambda < \varepsilon/2$ and $t_0 \notin \Sigma^{(\lambda)}$. Consequently $t_0 \in (I^{(\lambda)})^0$ with $I^{(\lambda)} \notin S_{k_s^\lambda}^{(\lambda)}$, $s \in N$; hence

$$\omega(x_{k_s^\lambda}, I^{(\lambda)}) < 1/2^\lambda < \varepsilon/2 \quad \text{for every } s \in N. \tag{4}$$

Since $\lambda$ is fixed now, for simplicity we shall write $(k_s^\lambda)_{s\in N} = (k_s)_{s\in N}$. For every $0 < h < \sigma$ we consider the averages

$$m_h = h^{-1} \int_0^h x'(t_0 + \tau)\, d\tau = h^{-1}[x_a(t_0 + h) - x_a(t_0)]$$

$$m_{k_s h} = h^{-1} \int_0^h x'_{k_s}(t_0 + \tau)\, d\tau = h^{-1}[x_{k_s}(t_0 + h) - x_{k_s}(t_0)].$$

Now, for an arbitrary fixed $\eta > 0$ and for all $0 < h < \sigma$ sufficiently small we have

$$|m_h - x'_a(t_0)| < \eta/2 \quad \text{and} \quad |x_s(t_0 + h) - x_s(t_0)| < h\eta/4. \tag{5}$$

Thus we fix $0 < h < \min(\varepsilon, \sigma)$ in such a way that relation (5) holds and moreover $t_0 + h \notin T_0 \cup T_1$ and $[t_0, t_0 + h] \subset I^{(\lambda)}$. From (4), for every $t_0 \leqq t \leqq t_0 + h$, we have

$$|x_{k_s}(t) - x_{k_s}(t_0)| \leqq \omega(x_{k_s}, I^{(\lambda)}) < \varepsilon/2, \quad s \in N. \tag{6}$$

Since $t_0, t_0 + h \notin T_0$, we can find an integer $s$ such that we have

$$|x_{k_s}(t_0) - x(t_0)| \leqq \min\{\varepsilon/2, \eta h/8\} \tag{7}$$

and

$$|x_{k_s}(t_0 + h) - x(t_0 + h)| < \eta h/8.$$

Therefore from (6) and (7), for every $t_0 \leqq t \leqq t_0 + h$, we have

$$|x_{k_s}(t) - x(t_0)| \leqq |x_{k_s}(t) - x_{k_s}(t_0)| + |x_{k_s}(t_0) - x(t_0)| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \tag{8}$$

By hypothesis we have

$$x'_{k_s}(t) \in Q(t, x_{k_s}(t)) \quad \text{for a.a. } t_0 \leqq t \leqq t_0 + h;$$

hence, from (8) and because $h < \varepsilon$, we have

$$x'_{k_s}(t) \in \operatorname{cl} \operatorname{co} \bigcup_{\substack{t\in[t_0,t_0+\varepsilon]\\ x\in B(x(t_0),\varepsilon)}} Q(t, x) = \operatorname{cl} \operatorname{co} Q(t_0, x(t_0), \varepsilon) \tag{9}$$

for a.a. $t_0 \leqq t \leqq t_0 + h$.

Finally we observe that the average $m_{k_s h}$ is also a point of the same closed and convex set ([5], p. 288), i.e.

$$m_{k_s h} \in \operatorname{cl} \operatorname{co} Q(t_0, x(t_0), \varepsilon). \tag{10}$$

Now by virtue of (7) and (5) we derive that

$$\begin{aligned} |m_{k_s h} - m_h| &= h^{-1} |(x_{k_s}(t_0 + h) - x_{k_s}(t_0)) - (x_a(t_0 + h) - x_a(t_0))| \\ &\leqq h^{-1} |x_{k_s}(t_0 + h) - x(t_0 + h)| + h^{-1} |x_{k_s}(t_0) - x(t_0)| \\ &\quad + h^{-1} |x_s(t_0 + h) - x_s(t_0)| \\ &\leqq h^{-1} \eta h/8 + h^{-1} \eta h/8 + h^{-1} \eta h/4 = \eta/2. \end{aligned} \tag{11}$$

Thus, by (5) and (11), it follows that,

$$|x'_a(t_0) - m_{k_s h}| \leqq |x'_a(t_0) - m_h| + |m_h - m_{k_s h}| \leqq \eta/2 + \eta/2 = \eta; \qquad (12)$$

and, from (10) and (12), that

$$x'_a(t_0) = x'(t_0) \in [\text{cl co } Q(t_0, x(t_0), \varepsilon)]_\eta .$$

Because $\eta$ is arbitrary, it follows that for every $\varepsilon > 0$

$$x'(t_0) \in \text{cl co } Q(t_0, x(t_0), \varepsilon) . \qquad (13)$$

Now the function $Q$ satisfies property (Q) at $(t_0, x(t_0))$; hence from (13) we derive

$$x'(t_0) \in \bigcap_{\varepsilon>0} \text{cl co } Q(t_0, x(t_0), \varepsilon) = Q(t_0, x(t_0)) .$$

This completes the proof of (2).

*Remark 1.* In Theorem 1 condition (Q) cannot be replaced by the weaker condition (K) as the following example, from Section 8.8 of [5], shows. We report the example here with some simplifications for the convenience of the reader. Let $n = 1$ and $A = [0, 1] \times R$, let $C$ be a closed Cantor subset of [0, 1] whose measure $|C|$ is positive, and let $C' = [0, 1] - C$. Then $C'$ is the countable union of disjoint subintervals of [0, 1], $C' = \bigcup_{n \in N} I_n$. Let $\sigma(t): C' \to R^+$ be a continuous and integrable function, which tends to $+\infty$ whenever $t$ tends to an end point of any interval $I_n$. Moreover let us suppose that $\lim_{n \to +\infty} \min \sigma/I_n = +\infty$.

Let $Q(t) = \{-1\}$ if $t \in C$, and $Q(t) = \{z \in R : z \geqq \sigma(t)\}$ if $t \in C'$. Let us extend the function $\sigma$ by taking $\sigma(t) = 0$, for $t \in C$, and consider the decomposition of [0, 1] into $k$ intervals of equal length: $J_k^s = [t_{k,s-1}, t_{k,s}]$, $s = 1, \ldots, k$, $t_{k,s} = s/k$. Define $\xi_k$ by taking $\xi_k(t) = \sigma(t) + v_k(t)$, where $v_k(t) = -1$ if $t \in C$, and $v_k(t) = |C \cap J_k^s|/|C' \cap J_k^s|$ if $t \in C' \cap J_k^s$. Then $\xi_k$ is integrable in [0, 1], and $\xi_k(t) \in Q(t)$ for every $t \in [0, 1]$ and $k \in N$.

Let $x_k(t) = \int_0^t \xi_k(\tau)\, d\tau$, $0 \leqq t \leqq 1$, or $x_k(t) = x(t) + y_k(t) = \int_0^t \sigma(\tau)\, d\tau + \int_0^t v_k(\tau)\, d\tau$. Here $\int_{J_k^s} v_k(t)\, dt = 0$; hence $y_k(t_{k,s}) = 0$ for all $s$ and $k$, and $|y_k(t)| \leqq 2/k$. Hence $x_k \to x$ uniformly on [0, 1], as $k \to +\infty$; moreover all $x_k$ and $x$ are AC with $x'_k(t) \in Q(t)$, $t \in [0, 1]$. Now $x'(t) = 0$ a.e. in $C$, while $Q(t) = \{-1\}$ for $t \in C$. Thus $x'(t) \notin Q(t)$ on a subset $C$ of positive measure in [0, 1]. Note that $\int_{J_k^s} |v_k(t)|\, dt = 2\,|J_k^s \cap C|$; hence $V(y_k) \leqq 2$ and $V(x_k) \leqq \int_0^1 \sigma(t)\, dt + 2 = V_0$, a constant, for all $k$. Here the sets $Q(t, x)$ have property (K) on [0, 1]; moreover they have property (Q) both on $C$ and on $C'$ but not on [0, 1].

## 3. An existence theorem of the calculus of variations

### 3a. The integral $\mathscr{I}$

Let $\alpha, n$ be integers such that $0 \leqq \alpha \leqq n$, $n \geqq 1$, and for every $x \in R^n$ we write $x = (y, z)$ with $y \in R^\alpha$, $z \in R^{n-\alpha}$. Let $A \subset R^{n+1}$ and $Q: A \to R^n$ be defined as before. Let $M \subset R^{2n+1}$ denote the set $M = \{(t, x, \zeta): (t, x) \in A, \zeta \in Q(t, x)\}$, and let $F_0(t, x, \zeta)$, or $F_0: M \to R$ be a given function.

Let $\Omega$ be a class of admissible functions, *i.e.* functions $x(t) = (y(t), z(t))$, or $x: [t_1, t_2] \to R^n$, such that (i) $y$ is ACg and $z$ is BVC; (ii) $(t, x(t)) \in A$, $x'(t) \in Q(t, x(t))$ a.e. in $[t_1, t_2]$; (iii) $F_0(\cdot, x(\cdot), x'(\cdot)) \in L_1$.

We consider the functional $\mathscr{I}: \Omega \to \tilde{R}$ defined by

$$\mathscr{I}(x) = \mathscr{I}(y, z) = \inf_{\Gamma(x)} \varliminf_{k\to\infty} \int_{t_1}^{t_2} F_0(t, y_k(t), z_k(t), y_k'(t), z_k'(t))\, dt$$

$$= \inf_{\Gamma(x)} \varliminf_{k\to\infty} I(y_k, z_k),$$

where $\Gamma(x)$ denotes the class of all sequences $(x_k)_{k\in N}$ such that (a) $x_k = (y_k, z_k) \in \mathrm{ACg} \cap \Omega$, $k \in N$; (b) $y_k \to y$ uniformly and $z_k \to z$ pointwise a.e. in $[t_1, t_2]$, and where, as stated in (i), $y$ is ACg and $z$ is BVC. If $\Gamma(x) = \emptyset$ we put $\mathscr{I}(x) = +\infty$. We may think of $F_0$ as extended to all of $R^{2n+1}$ by taking $F_0 = +\infty$ on $R^{2n+1} - M$. We denote by $\tilde{R}$ the reals with the addition of $+\infty$. Note that if $x \in \mathrm{ACg} \cap \Omega$, then $\Gamma(x) \neq \emptyset$. Moreover, if $A$ is convex, $Q(t, x) = R^n$, $(t, x) \in A$, and the integral means $(x_h)_{h>0}$ belong to $\Omega$, then $\Gamma(x) \neq \emptyset$.

The class $\Omega$ is said to be *closed* if $\Omega$ has the following property (c): if $x_k(t) = (y_k, z_k)$ is any sequence of (admissible) pairs in $\Omega$ satisfying (a) and (b) above, and if $x = (y, z)$ is admissible, then $x$ belongs to $\Omega$.

The functional $\mathscr{I}$ is modeled on Lebesgue area theory for nonparametric discontinuous surfaces (see Cesari [4]) and it is also close to the concept of integral in the sense of Serrin [14] when the present mixed convergence is used, uniform on $y$ and pointwise a.e. on $z$.

Note that the generalized weighted variation and length for a BVC curve are particular cases of the functional $\mathscr{I}$. In fact, let $F_0(t, v): [t_1, t_2] \times R^n \to R_0^+$ be defined by $F_0(t, v) = |\varphi(t) \cdot v|$, where $\varphi(t): [t_1, t_2] \to R^n$ is continuous. Then (see [1], [13])

$$\mathscr{I}(z) = \inf_{\Gamma(x)} \varliminf_{k\to\infty} \int_{t_1}^{t_2} |z_k'(t) \cdot \varphi(t)|\, dt = \inf_{\Gamma(x)} \varliminf_{k\to\infty} V_\varphi^*(z_k) \geqq V_\varphi^*(z),$$

and moreover

$$\mathscr{I}(z) \leqq \lim_{h\to 0} \int_{t_1}^{t_2} |z_h'(t) \cdot \varphi(t)|\, dt = V_\varphi^*(z),$$

where $z_h(t) = \dfrac{1}{h} \int_0^h z(t+\tau)\, d\tau$ is the integral mean of $z$. Analogous considerations hold for the length.

*Remark 2.* In order to deal with the minimization of the functionals under consideration when initial and terminal values for $x$ are involved, we carry over the definition of the functional $\mathscr{I}$ to the following setting.

Let $\mathscr{N}$ denote a family of subsets $N \subset [t_1, t_2]$, with $|N| = 0$, which is closed under countable unions. We shall write briefly $\mathscr{N}$-a.e. when we refer to a neglected null set $N \in \mathscr{N}$. Moreover we shall denote by $\mathscr{N}$-AC and $\mathscr{N}$-BVC the family of all the functions which are $\mathscr{N}$-a.e. equal to an AC function or a BV function, respectively.

We consider now the class $\Omega_{\mathscr{N}}$ of all the functions $x(t) = (y(t), z(t))$, such that

i) $y \in \mathscr{N}$-AC and $z \in \mathscr{N}$-BVC,

ii) $(t, x(t)) \in A$, $\mathscr{N}$-a.e. and $x'(t) \in Q(t, x(t))$, a.e. in $[t_1, t_2]$,

iii) $F_0(\cdot, x(\cdot), x'(\cdot)) \in L_1$.

For every $x \in \Omega_{\mathscr{N}}$, we shall denote by $\Gamma_{\mathscr{N}}(x)$ the class of all the sequences $(x_k)_{k \in N}$ such that

(a) $x_k = (y_k, z_k) \in \mathscr{N}$-AC, $k \in N$,

(b) $y_k \to y$ uniformly and $z_k \to z$ pointwise $\mathscr{N}$-a.e.

Thus we shall take $\mathscr{I}_{\mathscr{N}} : \Omega_{\mathscr{N}} \to \tilde{R}$ with

$$\mathscr{I}_{\mathscr{N}} = \inf_{\Gamma_{\mathscr{N}}(x)} \varliminf_{k \to \infty} I(x_k) \quad \text{if } \Gamma_{\mathscr{N}}(x) \neq \emptyset; \quad \mathscr{I}_{\mathscr{N}} = +\infty \text{ otherwise}.$$

Note that, if $\mathscr{N}$ is the family of *all* the null sets in $[t_1, t_2]$, then $\mathscr{I}_{\mathscr{N}}$ is exactly the functional $\mathscr{I}$. Furthermore, if $\mathscr{N}_1 \subset \mathscr{N}_2$, then $\Omega_{\mathscr{N}_1} \subset \Omega_{\mathscr{N}_2}$, $\Gamma_{\mathscr{N}_1}(x) \subset \Gamma_{\mathscr{N}_2}(x)$ and $\mathscr{I}_{\mathscr{N}_1}(x) \geqq \mathscr{I}_{\mathscr{N}_2}(x)$, for every $x \in \Omega_{\mathscr{N}_1}$. Note that $>$ sign may hold, as the Example 2 in Section 3 shows. In this way we get a "spectrum" of integral functionals whose lower and upper lines are $\mathscr{I}$ and $\mathscr{I}_{\{\emptyset\}}$, respectively. Observe that, if $F_0(v) = |v|$, then $\{\mathscr{I}_{\mathscr{N}}\}_{\mathscr{N}}$ is the "spectrum of variations" whose lower and upper lines are the generalized variation and the classic one, respectively.

In the following, for simplicity, we shall deal with the functional $\mathscr{I}$, but all our results hold for any other functional $\mathscr{I}_{\mathscr{N}}$, as well. In fact, we shall make systematic use of Helly's theorem which guarantees convergence at *all* points $t \in [t_1, t_2]$.

Note that, in this way, we treat also minimization which involves given initial and terminal data for $x$, say $x(t_1) \in B_1$, $x(t_2) \in B_2$, with $B_1$, $B_2$ closed sets in $R^n$. This is the case when the family $\mathscr{N}$ is such that $\bigcup_{N \in \mathscr{N}} N = (t_1, t_2)$. Thus, in the computation of the variation $V^*(x)$ of $x$ in $[t_1, t_2]$ we always take note of the values of $x$ at $t_1$ and $t_2$. Concerning the convergence of the trace operator, see the known results mentioned in [6a].

*Remark 3.* We shall see now that, for every $x = (y, z) \in \Omega$, with $\Gamma(x) \neq \emptyset$, there is a sequence $(x_k)_{k \in N} \in \Gamma(x)$ such that

$$\lim_{k \to \infty} I(x_k) = \mathscr{I}(x).$$

If $\mathscr{I}(x) \in R$, the proof is analogous to that for the case in which $\mathscr{I}(x) = +\infty$. Observe that, by definition, for every $m \in N$ there is a sequence $(y_n^m, z_n^m)_{n \in N} \in$

$\Gamma(x)$ such that

$$\left|\lim_{n\to\infty} I(y_n^m, z_n^m) - \mathscr{I}(x)\right| < 1/m;$$

moreover, by virtue of Severini-Egoroff theorem, we know that $z_n^m \xrightarrow[n\to\infty]{} z$ almost uniformly, $m \in N$. Thus we can find a set $T_m \subset [t_1, t_2]$ and an integer $n_m$ such that $\text{meas}\,(T_m) > (t_2 - t_1) - 1/2^m$ and, for every $n \geqq n_m$, we have $|I(y_n^m, z_n^m) - \mathscr{I}(x)| < 2/m$, $|y_n^m(t) - y(t)| < 1/m$, $t \in [t_1, t_2]$ and $|z_n^m(t) - z(t)| < 1/m$, $t \in T_m$.

Let us denote by $(x_k)_{k\in N}$ the sequence $x_k = (y_k, z_k) = (y_{n_k}^k, z_{n_k}^k)$, $k \in N$. We shall see that $(x_k)_{k\in N}$ is the sequence we were looking for. To do so, we put $T^k = \bigcap_{m\geqq k} T_m$ and $T = \bigcup_{k\in N} T^k$, then $\text{meas}\,(T^k) > (t_2 - t_1) - 1/2^{k-1}$ and $\text{meas}\,(T) = t_2 - t_1$. Thus, for every fixed $\varepsilon > 0$ and $t \in T$, there is an integer $\bar{k} > 1/\varepsilon$ such that $t \in T_k$ for every $k \geqq \bar{k}$ and therefore $|z_k(t) - z(t)| < 1/k < \varepsilon$. In other words $z_k \to z$ pointwise on $T$. Obviously $y_k \to y$ uniformly in $[t_1, t_2]$ and hence $(x_k)_{k\in N} \in \Gamma(x)$. Finally, having fixed $\varepsilon > 0$ and having taken $k_\varepsilon \in N$ such that $k_\varepsilon > 2/\varepsilon$, then for every $k \geqq k_\varepsilon$, we have $|I(x_k) - \mathscr{I}(x)| < 2/k_\varepsilon < \varepsilon$. The proof is complete.

### *3b. A lower semicontinuity property of $I$ and $\mathscr{I}$*

As is well known ([5]), closure theorems can be reworded into lower closure theorems and into lower semicontinuity theorems. From the closure Theorem 1 of Section 2 we derive here a lower semicontinuity theorem for the integral $I$ and the relevant inequality $I(x) \leqq \mathscr{I}(x)$, under the assumption $V^*(x_k) \leqq V_0$, $k \in N$, and the topology under consideration, namely uniform convergence on the components $y^i$ and pointwise convergence almost everywhere on the components $z^j$.

For the lower semicontinuity theorem we shall need the auxiliary sets

$$\tilde{Q}(t, x) = \{(z^0, \xi) \in R^{n+1} : z^0 \geqq F_0(t, x, \xi), \xi \in Q(t, x)\}, \quad (t, x) \in A, \tag{1}$$

or "augmented" set-valued function $\tilde{Q}: A \to R^{n+1}$.

**Theorem 1'** (*A lower semicontinuity theorem*). *Let* $1 \leqq \alpha \leqq n - 1$, *and assume that*: (i) *$A$ is closed*; (ii) *the sets $\tilde{Q}(t, x)$ are closed, convex and have property* (Q) *with respect to $(t, x)$ at every point $(\bar{t}, \bar{x}) \in A$, with the exception perhaps of a set of points whose $t$-coordinate lies in a set $H$ of measure zero*; (iii) *$F_0(t, y, z, u, v)$ is lower semicontinuous in $M$ and $\lambda \in L_1$ exists such that $F_0(t, y, z, u, v) \geqq \lambda(t)$ for all $(t, y, z, u, v) \in M$*; (iv) *a sequence of vector functions is given*

$$x(t) = (y, z),\ x_k(t) = (y_k, z_k),\ t \in [t_1, t_2],\ y, y_k, z_k \in \text{ACg},\ z \in \text{BVC},$$

*$y_k \to y$ uniformly, $z_k \to z$ pointwise a.e. in $[t_1, t_2]$, such that $(t, x_k(t)) \in A$, $x_k'(t) \in Q(t, x_k(t))$ a.e. in $[t_1, t_2]$*; (v) *$V^*(x_k) \leqq V_0$ for all $k$ and some constant $V_0$. Then $(t, x(t)) \in A$, $x'(t) \in Q(t, x(t))$, a.e. in $[t_1, t_2]$ and $\varliminf_{k\to+\infty} I(x_k) \geqq I(x)$. Thus,*

*if* $(x_k)_{k\in N}$ *is a sequence as in the definition of* $\mathscr{I}(x)$, i.e. $x_k \in \mathrm{ACg} \cap \Omega$, *and* $V^*(x_k) \leq V_0$, $k \in N$, *then*

$$\varliminf_{k\to\infty} I(x_k) \geqq \mathscr{I}(x) \geqq I(x).$$

*For* $\alpha = 0$ *this statement concerns sequences* $z_k(t) = (z_k^1, \ldots, z_k^n)$, $z(t) = (z^1, \ldots, z^n)$, $t \in [t_1, t_2]$, $z_k \in \mathrm{ACg}$, $z \in \mathrm{BVC}$, *and the conclusions are still valid.* *For* $\alpha = n$, *this statement concerns sequences* $y_k(t) = (y_k^1, \ldots, y_k^n)$, $y(t) = (y^1, \ldots, y^n)$, $t \in [t_1, t_2]$, $y, y_k \in \mathrm{ACg}$, *and the problem reduces to those discussed in Theorem* 10.8.i *of* [5].

*Remark 4.* As for Theorem 1, condition (Q) cannot be replaced by the weaker condition (K), as the following example shows. This example is only a modification of the one in Remark 1. Indeed, we take $n = 1$, $F_0 = 0$ and $\tilde{Q}(t) = \{(z^0, z) : z^0 \geqq 0, z \in Q(t)\}$, where the sets $Q(t)$ are defined in Remark 1. Then, for $x$, $x_k$ as in Remark 1, we have $x_k'(t) \in Q(t)$, but $x'(t) \notin Q(t)$, for $t \in C$, a set of positive measure.

**Proof** of Theorem 1′. Without loss of generality we can suppose that $i = \varliminf_{k\to+\infty} I(x_k) = \lim_{k\to+\infty} I(x_k) < +\infty$, where $i$ is finite because of (iii). Take $F_k(t) = F_0(t, y_k(t), z_k(t), y_k'(t), z_k'(t))$, $t \in [t_1, t_2]$, and note that for $F_k(t) = F_k^+(t) - F_k^-(t)$, $|F_k(t)| = F_k^+ + F_k^-$, $F_k^+, F_k^- \geqq 0$, we have $0 \leqq F_k^-(t) \leqq |\lambda(t)|$; hence $0 \leqq \int_{t_1}^{t_2} F_k^-(t)\, dt \leqq \int_{t_1}^{t_2} |\lambda(t)|\, dt$. Since $i$ is finite, we have that $I(x_k) \leqq W_0$ for all $k$ and some constant $W_0$. Finally,

$$\int_{t_1}^{t_2} |F_k(t)|\, dt = \int_{t_1}^{t_2} (F_k^+ + F_k^-)(t)\, dt = \int_{t_1}^{t_2} F_k(t)\, dt + 2 \int_{t_1}^{t_2} F_k^-(t)\, dt$$

$$\leqq W_0 + 2 \int_{t_1}^{t_2} |\lambda(t)|\, dt < +\infty.$$

Let us consider the AC functions $z_k^0 : [t_1, t_2] \to R$ defined by $z_k^0(t) = \int_{t_1}^{t} F_0(\tau, x_k(\tau), x_k'(\tau))\, d\tau$, $k \in N$. Then $z_k^0(t_1) = 0$ and

$$V(z_k^0) = \int_{t_1}^{t_2} |F_k(t)|\, dt \leqq W_0 + 2 \int_{t_1}^{t_2} |\lambda(t)|\, dt = V,$$

$V$ a constant. By Helly's theorem there is a subsequence, say still $(k)$, such that $z_k^0(t) \to z^0(t)$ pointwise everywhere in $[t_1, t_2]$, with $z^0 \in \mathrm{BV}$, $z^0$ not necessarily continuous, and $z^0(t_1) = 0$. Note that the functions $(z_k^0, x_k)_{k\in N}$ are ACg solutions of the orientor field

$$(t, x_k(t)) \in A,\ (z_k^{0\prime}(t), x_k'(t)) \in \tilde{Q}(t, x_k(t)),\ \text{a.e. in } [t_1, t_2], \tag{2}$$

where $\tilde{Q} : A \to R^{n+1}$ is the set-valued function defined by (1). As an application of Theorem 1 we now prove that the limit function $(z^0, y, z) = (z^0, x)$:

$[t_1, t_2] \to R^{n+1}$ is again a solution of the orientor field (2), *i.e.*

$$(t, x(t)) \in A, \quad (z^{0\prime}(t), x'(t)) \in \tilde{Q}(t, x(t)) \text{ a.e. in } [t_1, t_2],$$

or

$$(t, x(t)) \in A, \quad x'(t) \in Q(t, x(t)), \quad z^{0\prime}(t) \geqq F_0(t, x(t), x'(t)) \text{ a.e. in } [t_1, t_2].$$

Note that $z^{0\prime}(t) \geqq F_0(t) \geqq \lambda(t)$; hence $F_0(t, x(t), x'(t))$ is summable in $[t_1, t_2]$ because it lies between summable functions, and $x: [t_1, t_2] \to R^n$ is admissible. Finally, if we take

$$Z^0(t) = z^0(t) - \int_{t_1}^{t} \lambda(\tau)\, d\tau, \quad t \in [t_1, t_2],$$

we see that $Z^0(t)$ is monotone non-decreasing in $[t_1, t_2]$; hence

$$\begin{aligned} Z^0(t_2) &= Z^0(t_2) - Z^0(t_1) = z^0(t_2) - \int_{t_1}^{t_2} \lambda(t)\, dt \\ &= V(Z^0) \geqq \int_{t_1}^{t_2} |z^{0\prime}(t) - \lambda(t)|\, dt = \int_{t_1}^{t_2} (z^{0\prime}(t) - \lambda(t))\, dt \\ &\geqq \int_{t_1}^{t_2} (F_0(t) - \lambda(t))\, dt = \int_{t_1}^{t_2} F_0(t)\, dt - \int_{t_1}^{t_2} \lambda(t)\, dt. \end{aligned}$$

Hence $z^0(t_2) \geqq \int_{t_1}^{t_2} F_0(t)\, dt$, and finally

$$I(x) = \int_{t_1}^{t_2} F_0(t)\, dt \leqq z^0(t_2) = \lim_{k \to +\infty} z_k^0(t_2) = \lim_{k \to +\infty} I(x_k).$$

Theorem 1′ is thereby proved.

Theorem 1′ has an important consequence concerning the concept of integral $\mathscr{I}(x)$ defined at the beginning of Section 3. Indeed, as long as we define $\mathscr{I}(x)$ by means of sequences $x_k = (y_k, z_k) \in \text{ACg}$, $k \in N$, with equibounded variation, then we can well say that $\mathscr{I}$ is an extension to BVC of the functional $I$; in other words, if $x = (y, z), y, z$ both ACg, then $\mathscr{I}(x) = I(x)$. In fact, from Theorem 1′ we have[1] $I(x) \leqq \mathscr{I}(x)$, but $\Gamma(x)$ contains now the sequence of repetitions $x_k = (y, z)$, $k \in N$; hence $\mathscr{I}(x) \leqq I(x)$, and finally $I(x) = \mathscr{I}(x)$.

### *3c. The existence theorem*

We now state and prove an existence theorem of the calculus of variations for the integral $\mathscr{I}$. In other words we have to prove that $\mathscr{I}$ has an absolute minimum in $\Omega$. That is we have to prove, under the assumptions that

(a) the infimum $i$ of $I(x)$ in $\text{ACg} \cap \Omega$ is finite; hence there are minimizing sequences $x_k = (y_k, z_k)$, $k \in N$, in $\Omega$, both $y_k, z_k \in \text{ACg}$, such that $I(x_k) \to i$;

(b) for some subsequence, say $(k)$ again, and elements $x = (y, z) \in \Omega$, $y \in \text{ACg}$, $z \in \text{BVC}$, we have $y_k \to y$ uniformly, $z_k \to z$ pointwise a.e. in $[t_1, t_2]$.

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[1] This is not true, in general, if assumption (v) of Theorem 1′ is dropped, as Example 3 below shows.

Since we shall assume that there are such sequences $x_k = (y_k, z_k)$ with equibounded variations $V(x_k)$, then by Theorem 1′ we know that $I(x) \leqq \mathcal{I}(x) = i$.[2]

To state and prove our existence theorem, we denote by $(\gamma_1)$, $(\gamma_2)$, $(\gamma_3)$ the following alternative assumptions on the function $F_0$.

$(\gamma_1)$ There is a scalar function $\phi(\zeta)$, $0 \leqq \zeta < +\infty$, or $\phi: R_0^+ \to R$, bounded below, with $\phi(\zeta)/\zeta \to +\infty$ as $\zeta \to +\infty$, such that $F_0(t, y, z, u, v) \geqq \phi(|u|)$ for all $(t, y, z, u, v) \in M$.

$(\gamma_2)$ For any $\varepsilon > 0$ there is an integrable scalar function $\psi_\varepsilon(t) \geqq 0$, or $\psi_\varepsilon: [t_1, t_2] \to R_0^+$, such that $|u| \leqq \psi_\varepsilon(t) + \varepsilon F_0(t, y, z, u, v)$ for all $(t, y, z, u, v) \in M$.

$(\gamma_3)$ For every $\alpha$-vector $p \in R^\alpha$ there is an integrable scalar function $\phi_p(t) \geqq 0$, or $\phi_p: [t_1, t_2] \to R_0^+$, such that $F_0(t, y, z, u, v) \geqq \langle p, u \rangle - \phi_p(t)$ for all $(t, y, z, u, v) \in M$.

Note that under condition $(\gamma_1)$ certainly $\phi(\zeta) \geqq \lambda$ for some real constant $\lambda$, and then $F_0(t, y, z, u, v) \geqq \phi(|u|) \geqq \lambda$ for all $(t, y, z, u, v)$. Under condition $(\gamma_2)$ and $\varepsilon = 1$ we have $|u| \leqq \psi_1(t) + F_0(t, y, z, u, v)$; hence $F_0(t, y, z, u, v) \geqq -\psi_1(t)$, a summable function in $[t_1, t_2]$. Under condition $(\gamma_3)$ and $p = 0$, we have $F_0(t, y, z, u, v) \geqq -\phi_0(t)$, a summable function in $[t_1, t_2]$.

**Theorem 2** (An existence theorem). *Let* $1 \leqq \alpha \leqq n - 1$, *and assume that* (i) *A is compact and M is closed*; (ii) *the sets* $\tilde{Q}(t, x)$ *are closed and convex and have property* (Q) *with respect to* $(t, x)$ *at every point* $(t, x)$ *of A* (*with the exception perhaps of a set of points whose t-coordinate lies on a set of measure zero on the t-axis*); (iii) $F_0(t, y, z, u, v)$ *is lower semicontinuous in M*; (iv) $F_0$ *satisfies one of the growth conditions* $(\gamma_1)$, *or* $(\gamma_2)$, *or* $(\gamma_3)$. *Also we assume that the class* $\Omega$ *is nonempty and closed, and* (v) *there exists a constant* $W_0$ *such that for every element* $x = (y, z) \in$ ACg $\cap\, \Omega$, *then* $V^*(z) \leqq W_0$.

*Then the functional* $\mathcal{I}$ *has an absolute minimum* $x = (y, z)$ *in* $\Omega$, $y \in$ ACg, $z \in$ BVC, and $I(x) \leq \mathcal{I}(x) = i$.

*For* $\alpha = 0$, *then* $x = z$, *requirements* $(\gamma_1)$, *or alternatively* $(\gamma_2)$, $(\gamma_3)$ *do not apply, yet the conclusion is still valid if we know that* (iv)′ *there is a summable scalar function* $\lambda: [t_1, t_2] \to R$ *such that* $F_0(t, z, \zeta) \geqq \lambda(t)$ *for all* $(t, z, \zeta) \in M$.

*For* $\alpha = n$, *then* $x = y$, $\Omega$ *is a nonempty and closed class of* ACg *functions* $y(t) = (y^1, \ldots, y^n)$, $t \in [t_1, t_2]$, *condition* (v) *does not apply, and the problem reduces essentially to those discussed in* Theorems 11.1.i *and* ii *of* [5].

Note that the condition in (ii) concerning property $(Q)$ for the sets $\tilde{Q}(t, x)$, not only implies that the same sets $\tilde{Q}(t, x)$ are closed and convex, but also that their projections, the sets $Q(t, x)$, also are convex, and that $F_0(t, y, z, u, v)$ is convex in $(u, v)$.

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[2] See Note 1.

Note that for $0 \leqq \alpha \leqq n-1$, if (v′) there are scalar functions $\psi_i \in L_1([t_1, t_2])$, $i = \alpha+1, \ldots, n$, such that $(t, y, z, u, v) \in M$ implies $v^i \geqq \psi_i(t)$ a.e. in $[t_1, t_2]$, then (v) certainly holds. (*Cf.* part (g) of proof of Theorem 2 below).

Note that for $1 \leqq \alpha \leqq n-1$, the sets $\tilde{Q}(t, x)$ closed and convex, and $(\gamma_1)$ holds, if (v″) there are constants $L_i$ such that, a.e. in $[t_1, t_2]$, $(t, y, z, u, v) \in M$ implies $v^i \geqq L_i$, $i = \alpha+1, \ldots, n$, then both (ii) and (v) hold. (*Cf.* part (h) of proof of Theorem 2 below).

*Remark 5.* If $\alpha = n$, $x = y$, note that Theorems 11.1,i, ii of [5] are proved under weaker assumptions on the function $F_0$ and definitively less information on the sets $\tilde{Q}(t, x)$. In particular in Theorem 11.1.ii, under none of the assumptions $(\gamma_1)$, $(\gamma_2)$, $(\gamma_3)$ is it needed to verify that the sets $\tilde{Q}(t, x)$ have property (Q). Indeed, a different topology is used on the functions $y_k$, namely $y_k \to y$ uniformly and $y'_k \to y'$ weakly in $L_1$. Then, in terms of the equivalence theorem ([5], Theorem 10.3.i; see also Cesari & Pucci [7]), these sets $\tilde{Q}(t, y)$ are shown to have augmented sets $\tilde{\tilde{Q}}(t, y)$ which have property (Q) with respect to $y$, a.e. in $t$ (see [5], proof of Theorem 10.7.i.).

**Proof** of Theorem 2. Without loss of generality we can suppose that there is an element $x \in \Omega$ such that $\Gamma(x) \neq \emptyset$. Let $i = \inf_{\Omega} \mathscr{I}(x)$; then $-\infty \leqq i < +\infty$. Let $(\bar{x}_k)_{k \in N}$ be a minimizing sequence, *i.e.* $\mathscr{I}(\bar{x}_k) \to i$ as $k \to \infty$. We divide the proof into parts.

(a) First note that we can find a sequence $(x_k)_{k \in N}$ in ACg $\cap\, \Omega$ such that $I(x_k) \to i$ as $k \to \infty$. In order to see this, note that for every $k \in N$, there is an integer $n_k$ such that $|\mathscr{I}(\bar{x}_{n_k}) - i| < 1/k$, and moreover (see Remark 2) there is a sequence $(x_n^k)_{n \in N}$ in ACg $\cap\, \Omega$ such that $I(x_n^k) \to \mathscr{I}(\bar{x}_{n_k})$ as $n \to \infty$. Thus, for every $k \in N$, there is an integer $\bar{n} = \bar{n}(k)$ such that $|I(x_{\bar{n}}^k) - i| < 2/k$ (or $I(x_{\bar{n}}^k) < k$ if $i = -\infty$). Let $x_k = x_{\bar{n}}^k$, $k \in N$. Then $x_k \in$ ACg $\cap\, \Omega$, and $|I(x_k) - i| < 2/k$, $k \in N$. Without loss of generality we can suppose that $x_k$ is AC, $k \in N$, and, since $A$ is compact, the sequence $(x_k)_{k \in N}$, $x_k = (y_k, z_k)$, is equibounded.

(b) By virtue of hypothesis $(\gamma_1)$, we prove, as in Theorem 10.4.i of [5] that the sequence $(y_k)_{k \in N}$ is equi-absolutely continuous. Thus, $V(y_k) \leqq V_1$ for all $k \in N$ and some constant $V_1$. Moreover, since $(y_k)_{k \in N}$ is equibounded, there is a subsequence, say still $(y_k)_{k \in N}$, such that $y_k \to y$ uniformly in $[t_1, t_2]$, with $y \in$ AC.

Moreover, the assumption (v) shows that $V(z_k) \leqq W_0$ for all $k \in N$. Hence, by Helly's theorem (see [5], Theorem 15.1.i) there is a subsequence, say still $(z_k)_{k \in N}$, such that $z_k(t) \to z(t)$ pointwise everywhere in $[t_1, t_2]$ with $z \in$ BV, not necessarily continuous.

(c) Now the function $\phi$ in $(\gamma_1)$ is bounded below, say $\phi(\zeta) \geqq \lambda$, $\lambda \in R$; hence $F_k(t) = F(t, y_k(t), z_k(t), y'_k(t), z'_k(t)) \geqq \phi(|y'_k(t)|) \geqq \lambda$ for $k \in N$ and $t \in [t_1, t_2]$. Consequently, for $F_k(t) = F_k^+(t) - F_k^-(t)$, $F_k^- \geqq 0$, $F_k^+ \geqq 0$, $|F_k(t)| = F_k^+(t) +$

$F_k^-(t)$, we have $F_k(t) \geqq \lambda$, $F_k^-(t) \leqq |\lambda|$. Since $\int_{t_1}^{t_2} F_k(t)\,dt \to i$ as $k \to \infty$, we derive $i \geqq \lambda(t_2 - t_1)$, and thus $i$ is finite. Without loss of generality we can assume that $\int_{t_1}^{t_2} F_k(t)\,dt \leqq i + 1$ for all $k \in N$, and then

$$\int_{t_1}^{t_2} F_k^+(t)\,dt = \int_{t_1}^{t_2} F_k(t)\,dt - \int_{t_1}^{t_2} F_k^-(t)\,dt \leqq i + 1 + |\lambda|\,(t_2 - t_1),$$

$$\int_{t_1}^{t_2} |F_k(t)|\,dt \leqq i + 1 + 2\,|\lambda|\,(t_2 - t_1).$$

Let us consider now the AC functions $z_k^0 : [t_1, t_2] \to R$ defined by $z_k^0(t) = \int_{t_1}^{t} F_0(\tau, x_k(\tau), x_k'(\tau))\,d\tau$, $k \in N$. Then $V(z_k^0) = \int_{t_1}^{t_2} |F_k(t)|\,dt \leqq i + 1 + 2\,|\lambda|\,(t_2 - t_1) = V$, $k \in N$. Again, by Helly's theorem there is a subsequence, say still $(z_k^0)_{k \in N}$, such that $z_k^0(t) \to z^0(t)$ pointwise everywhere in $[t_1, t_2]$, with $z^0 \in \mathrm{BV}$ (not necessarily continuous).

(d) Note that the functions $(z_k^0, x_k)_{k \in N}$ are AC solutions of the orientor field $(t, x_k(t)) \in A$, $(z_k^{0\prime}(t), x_k'(t)) \in \tilde{Q}(t, x_k(t))$, a.e. in $[t_1, t_2]$, where $\tilde{Q} : A \to R^{n+1}$ is the set-valued function defined by $\tilde{Q}(t, x) = \{(r, \zeta) : \zeta \in Q(t, x),\ r \geqq F_0(t, x, \zeta)\} = \operatorname{epi} F_0(t, x, \cdot)_{/Q(t,x)}$. By (ii) the sets $\tilde{Q}(t, x)$ have property $(Q)$ with respect to $(t, x)$ in $A$, for a.e. $t$.

(e) As an application of Theorem 1, we see that the limit function $(z^0, y, z) = (z^0, x) : [t_1, t_2] \to R^{n+1}$ is again a solution of the orientor field $(t, x(t)) \in A$, $(z^{0\prime}(t), x'(t)) \in \tilde{Q}(t, x(t))$, a.e. in $[t_1, t_2]$, i.e. $(t, x(t)) \in A$, $x'(t) \in Q(t, x(t))$, $z^{0\prime}(t) \geqq F(t, x(t), x'(t))$, a.e. in $[t_1, t_2]$. Note that $z^{0\prime}(t) \geqq F_0 \geqq \lambda$; hence $F_0(t, x(t), x'(t))$ is summable in $[t_1, t_2]$ and the function $x : [t_1, t_2] \to R^n$ is admissible. Since the class $\Omega$ is closed, we conclude that $x \in \Omega$.

(f) Finally, by definition, $(x_k)_{k \in N} \in \Gamma(x)$ and $i \leqq \mathscr{I}(x) \leqq \lim_{k \to \infty} I(x_k) = i$, or $\mathscr{I}(x) = i$ and the proof is complete.

(g) Let us prove now that, for $0 \leqq \alpha \leqq n - 1$, $A$ compact and property (v′), then $V(z_k) \leqq V_2$, $k \in N$, for some constant $V_2$. Indeed, since $z_k$ is AC, we have (writing $z_k$ for any of its components $z_k^i$, $i = \alpha + 1, \dots, n$),

$$z_k(t_2) - z_k(t_1) = \int_{t_1}^{t_2} z_k'^{+}(t)\,dt - \int_{t_1}^{t_2} z_k'^{-}(t)\,dt$$

and

$$V(z_k) = \int_{t_1}^{t_2} z_k'^{+}(t)\,dt + \int_{t_1}^{t_2} z_k'^{-}(t)\,dt.$$

Thus, by condition (v′), we have

$$V(z_k) = 2 \int_{t_1}^{t_2} z_k'^{-}(t)\,dt + z_k(t_2) - z_k(t_1) \leqq 2 \int_{t_1}^{t_2} \psi^{-}(t)\,dt + \operatorname{diam} A.$$

(h) Let us prove that, if $1 \leqq \alpha \leqq n-1$, $A$ compact, the sets $\tilde{Q}(t, x)$ closed and convex, and $(\gamma_1)$ and (v″), then not only (v) but also (ii) holds, that is, the sets $\tilde{Q}(t, x)$ have property (Q) with respect to $(t, x)$. Indeed, we have $F_0(t, x, u, v) \geqq \phi(|u|)$, $v^i \geqq L_i$, $i = \alpha+1, \dots, n$, for all $(t, x, u, v) \in M$, and by virtue of Theorem 10.5.ii (second version) of [5] (with the variable $x$ replaced by $(t, x)$), the thesis follows.

*Remark 6*. The hypothesis (v) can be replaced by the weaker assumption (v‴) the level sets $L_K = \{x = (y, z) \in \mathrm{ACg} \cap \Omega : I(x) \leqq K\}$ are equibounded in variation. In fact, as can be seen by the proof, we use condition (v) only to guarantee that every minimizing sequence of ACg curves $x = (y, z)$ is equibounded in variation.

We shall illustrate now three situations in which even condition (v‴) can be dropped.

a) Let $F_0 : R \to R_0^+$ be a convex function then $F_0$ is the least upper bound of its support straight lines, i.e.

$$F_0(v) = \sup \{\varphi(v) : \varphi(w) = aw + b \leqq F_0(w),\ w \in R\}.$$

Thus, either $F_0(v) = \text{const}$, and then Theorem 2 is trivial, or $F_0(v) \geqq av + b$, $v \in R$, $a \neq 0$. But in this last case, it is easy to see that every minimizing sequence of AC curves is equibounded in variation.

b) Let consider now an integrand $F_0$ which does not depend on the variable $z$ and does satisfy the condition

$$|v_1| \leqq |v_2| \text{ implies } F_0(t, v_1) \leqq F_0(t, v_2).$$

Then condition (v‴) in Theorem 2 can be omitted, provided we suppose that the sets $Q(t, x)$ are such that if $v \in Q(t, x)$ and $|w| \leqq |v|$, then $w \in Q(t, x)$. In fact, in the present case, we can find a minimizing sequence of AC curves with equibounded variation. In order to see this, given any sequence of AC curves such that $I(z_k) \to i$ as $k \to +\infty$, it is sufficient to alter the sequence $(z_k)_{k \in N}$ in the following way. For simplicity we write $z_k$ for any of its components $z_k^i$, $i = 1, \dots, n$. Let us suppose first that $z_k(t_1) < z_k(t_2)$, take

$$\bar{t} = \max \{t \in [t_1, t_2] : z_k(t) = z_k(t_1)\},\ \bar{\bar{t}} = \min \{t \in [\bar{t}, t_2] : z_k(t) = z_k(t_2)\}$$

and

$$\bar{z}_k(t) = \begin{cases} z_k(t_1), & t \in [t_1, \bar{t}], \\ \max \{z_k(\tau),\ \tau \in [\bar{t}, t]\}, & t \in [\bar{t}, \bar{\bar{t}}], \\ z_k(t_2), & t \in [\bar{\bar{t}}, t_2]. \end{cases} \tag{3}$$

If $z_k(t_1) > z_k(t_2)$, we define $\bar{z}_k$ analogously by substituting min for max in (3). Finally, if $z_k(t_1) = z_k(t_2)$, we take $\bar{z}_k(t) = z_k(t_1)$, $t \in [t_1, t_2]$. Observe that, in any case, $\bar{z}_k$ is again AC and moreover it is monotone and $|\bar{z}_k'(t)| \leqq |z_k'(t)|$, a.e. in $[t_1, t_2]$. Therefore $V(\bar{z}_k) \leqq \operatorname{diam} A$, $k \in N$, and, by virtue of the assumption on the integrand $F$, we have $I(\bar{z}_k) \leqq I(z_k)$, $k \in N$. This proves that $(\bar{z}_k)_{k \in N}$ is still a minimizing sequence.

c) Let $F_0(t, z, v): M \to R^+$ subjected to the growth condition $F_0(t, z, v) \geqq a|v| + b(t)$, with $a > 0$ and $b \in L_1$. In this case condition (v‴) is trivially satisfied.

*Remark 6′*. Note that we may drop the requirement that $A$ be bounded if we know that there is a minimizing sequence $x_k = (y_k, z_k)$, $y_k, z_k \in \mathrm{AC}$, with $I(x_k) \to i$, which is equibounded. Thus the assumptions "$A$ compact and (v)" can be replaced by the weaker conditions: $A$ closed and

(v′) the level sets $L_K = \{x = (y, z) \in \mathrm{ACg} \cap \Omega : I(x) \leqq K\}$ are bounded in the norm $\|x\| = |x_e(t_1)| + V^*(x)$, where $x_e(t_1) = \lim\limits_{t \to t_1^+} \operatorname{ess} x(t)$.

*Remark 7*. Note that we consider the infimum $i$ of $I(x)$ in the class $\mathrm{ACg} \cap \Omega$ and we prove in Theorem 2 under the hypotheses that there is some element $x = (y, z)$ in $\Omega$, $y \in \mathrm{ACg}$, $z \in \mathrm{BVC}$, and some sequence $x_k = (y_k, z_k)$, $y_k$, $z_k \in \mathrm{ACg}$, $k \in N$, in $\Omega$ with $I(x_k) \to i$, and $I(x) \leqq \mathcal{I}(x) = i$.

In other words, under the assumptions of Theorem 2, the infimum $i$ is attained by $\mathcal{I}$, or $\mathcal{I}(x) = i$, while $I(x)$ may have a value equal to or less than $i$. In Examples 4 and 5 below $I(x) = \mathcal{I}(x) = i$. However, it may well happen that $I(x) < \mathcal{I}(x) = i$ as Example 1 below shows. Note that if we denote by $i_0$ the infimum of $I(x)$ in the class $\Omega$, then $\mathrm{ACg} \cap \Omega \subset \Omega$; hence $i_0 \leqq i$. We shall see in Example 2 below that possibly $i_0 < i$, and that both can be attained, say $I(x) = i_0$ and $I(\bar{x}) \leqq \mathcal{I}(\bar{x}) = i$, possibly by different $x$, $\bar{x} \in \Omega$. Also note that for $\bar{x}$ optimal for $\mathcal{I}$ under the assumptions of the present paper, we certainly have $i_0 \leqq I(\bar{x}) \leqq \mathcal{I}(\bar{x}) = i$.

*Example 1*. Let us show that, if $i$ is the infimum of $I(x)$ in $\mathrm{ACg} \cap \Omega$ (and therefore the infimum of $\mathcal{I}(x)$ in $\Omega$), and $x \in \Omega$ is a minimizing element, then it may happen that $I(x) < \mathcal{I}(x) = i$.

Let us consider the problem of minimizing the length of the plane curves $z^1 = z^1(t)$, $z^2 = z^2(t)$, $0 \leqq t \leqq 1$, joining two given points, say (0, 0) and (1, 1), or

$$I(x) = \int_0^1 [(z^{1\prime}(t))^2 + (z^{2\prime}(t))^2]^{1/2}\, dt,$$

$$z^1(0) = 0, \quad z^2(0) = 0, \quad z^1(1) = 1, \quad z^2(1) = 1.$$

Here, for $z^1, z^2 \in \mathrm{AC}$, the infimum of $I$ is $i = \sqrt{2}$, and this infimum is attained not only by the obvious solution $z^1(t) = z^2(t) = t$, $0 \leqq t \leqq 1$, but also by the infinitely many solutions $z^1(t) = z^2(t) = \xi(t)$, $0 \leqq t \leqq 1$, $\xi \in \mathrm{AC}$, monotone nondecreasing with $\xi(0) = 0$, $\xi(1) = 1$; hence $z^1(0) = 0$, $z^2(0) = 0$, $z^1(1) = 1$,

$z^2(1) = 1$, and

$$I(z) = \sqrt{2} \int_0^1 |\xi'| \, dt = \sqrt{2} \int_0^1 \xi' \, dt = \sqrt{2} = i.$$

On the other hand, let us consider the usual ternary Cantor function $\varphi(t)$, $0 \leqq t \leqq 1$, $\varphi(0) = 0$, $\varphi(1) = 1$, continuous, monotone non decreasing, with derivative zero a.e. in $[0, 1]$, $\varphi$ BV and not AC. Let $\xi_k(t)$, $0 \leqq t \leqq 1$, $k \in N$, be a sequence of monotone nondecreasing AC approximations of $\varphi$ with $\xi_k(0) = 0$, $\xi_k(1) = 1$, and $\xi_k \to \varphi$ uniformly in $[0, 1]$. Now we take the sequence of AC functions $z_k^1(t) = z_k^2(t) = \xi_k(t)$, $0 \leqq t \leqq 1$, $k \in N$. For $z_k = (z_k^1, z_k^2) = (\xi_k, \xi_k)$, $z = (z^1, z^2) = (\varphi, \varphi)$, we have

$$I(z_k) = \sqrt{2} \int_0^1 |\xi_k'(t)| \, dt = \sqrt{2} \int_0^1 \xi_k'(t) \, dt = \sqrt{2};$$

$z_k \to z$, i.e. $z_k^1 \to z^1$, $z_k^2 \to z^2$ uniformly; hence pointwise in $[0, 1]$

$$I(z_k) \to i = \sqrt{2}.$$

Thus, in the terms of the beginning of Section 3, $\mathscr{I}(\varphi) = i = \sqrt{2}$, $I(\varphi) = 0$; hence $I(\varphi) < \mathscr{I}(\varphi) = i$.

*Example 2.* Suppose

$$A = [0, 1]^2 \cap \{(t, x) \in R^2 : t - \tfrac{1}{3} \leqq z \leqq t + \tfrac{1}{3}\}, Q(t, x) = R, n = 1,$$

and let $F_0(v): R \to R$ be defined by $F_0(v) = |v|$. In this case (see Remark 2) the functional $\mathscr{I}$ is the generalized variation. Let $\varphi(t) : [0, 1] \to [0, 1]$ be the usual ternary Cantor function. Then $\varphi$ is continuous, BV and not ACg, with $\varphi'(t) = 0$, a.e. in $[0, 1]$, and graph $\varphi \subset A$. Thus $I(\varphi) = 0$, *i.e.*; if $i_0$ denotes the infimum of $I(x)$ in $\Omega$, then $I(\varphi) = i_0 = 0$. On the other hand, if $i$ denotes the infimum of $I(x)$ in ACg $\cap\, \Omega$ then by Theorem 2, $\mathscr{I}$ also attains its infimum at some minimizing element $\bar{z} \in \Omega$, and $\mathscr{I}(\bar{z}) = i$. Certainly $i > 0$ (hence $i > i_0$), since $i = 0$ would imply $\bar{z}'(t) = 0$ a.e., $\bar{z}(t) =$ const. a.e., and this is not possible given the shape of the set $A$. It is easy to see that $i = 2/3$ and that a minimizing element is $\bar{z}(t) = 0$ for $0 \leqq t \leqq 1/3$, $\bar{z}(t) = t - 1/3$ for $1/3 < t \leqq 1$, with $I(\bar{z}) = \mathscr{I}(\bar{z}) = 2/3$. Note that if we consider the same problem with boundary data $z(0) = 0, = z(1) = 1$, then $\varphi$ satisfies the same data;h ence $I(\varphi) = i_0 = 0$ as before. On the other hand, it is easy to see that the new infimum is now $i = 1$, and that a minimizing element is $\bar{z}(t) = 0$ for $0 \leqq t \leqq 1/3$, $\bar{z}(t) = t - \frac{1}{3}$ for $1/3\ t < 1$, $\bar{z}(1 - 0) = 2/3$, $\bar{z}(1) = 1$ with a jump of $1/3$ at $t = 1$, and $\mathscr{I}(\bar{z}) = 1 = i$, $I(\bar{z}) = 2/3$ and again $I(\bar{z}) < \mathscr{I}(\bar{z}) = 1$ as in Example 1. Thus $0 = i_0 = I(\varphi) < I(\bar{z}) < \mathscr{I}(\bar{z}) = i = 1$.

*Example 3.* We show here an example in which occurs the "non natural" situation $i = \mathscr{I}(x) < I(x)$. In this example $A$ is compact, but the sets $\tilde{Q}$ do not have the property ($Q$), and for a minimizing sequence of AC functions $x_k = (y_k, z_k)$, $k \in N$, the total variations $V^*(z_k)$ are not equibounded, and the sequence $z_k'$, $k \in N$, is not equibounded below.

Let $A = [0, 2\pi] \times [-1, 1]^2$, $Q(t, x) = R^2$, $(t, x) \in A$, and let $F_0(z, v)$: $[-1, 1]^2 \times R^2 \to R^+$ be defined by

$$F_0(z_1, z_2, v_1, v_2) = \exp(z_1 v_2 - z_2 v_1).$$

We consider now the sequence $(z_k)_{k \in N}$ given by

$$z_k^1(t) = r_k \sin kt, \quad z_k^2(t) = r_k \cos kt, \quad 0 \leqq t \leqq 2\pi, k \in N,$$

where $r_k = k^{-1/3}$. Note that $z_k^{1'}(t) = r_k k \cos kt$, $z_k^{2'}(t) = -r_k k \sin kt$, and $z_k^1 z_k^{2'} - z_k^2 z_k^{1'} = -r_k^2 k = -k^{1/3} \to -\infty$. Therefore we have

$$0 \leqq \mathscr{I}(z_k) \leqq I(z_k) = \int_0^{2\pi} F_0(z_k(t), z_k'(t))\, dt = 2\pi \exp(-k^{1/3}) \to 0,$$

as $K \to \infty$. Thus $i = 0$. Note that $V(z_k^1) = V(z_k^2) = 4r_k k = 4k^{2/3} \to +\infty$ as $k \to \infty$. Since $r_k \to 0$ as $k \to \infty$, if we take $z = (z^1, z^2)$ with $z^1(t) = z^2(t) = 0$, $t \in [t_1, t_2]$, then $z$ is AC in $[0, 2\pi]$ and $z_k \to z$ uniformly, hence $\mathscr{I}(z) = 0$. But $I(z) = \int_0^{2\pi} \exp(0)\, dt = 2\pi > 0$. Moreover, for every $z \in$ BVC we certainly have

$$I(z) = \int_0^{2\pi} \exp(z_1(t) z_2'(t) - z_2(t) z_1'(t))\, dt > 0.$$

We shall give now two examples which illustrate Theorem 2 and show that, in general, the minimum of $\mathscr{I}$ is attained by a BVC function, not necessarily ACg.

*Example 4.* Let $A = [0, 2] \times [0, 1]$, $n = 1$, $\alpha = 0$, $\psi(t) = 0$. $Q(t, x) = [0, +\infty)$ for $(t, x) \in A$, $M = A \times [0, +\infty)$, with boundary conditions $x(0) = 0$, $x(2) = 1$. Let $F_0(t, v)$ be defined by $F_0(t, v) = |1 - t|\,|v|$ for $(t, v) \in M$. Thus the functional $\mathscr{I}(x)$ is nonnegative. Note that, for the sequence $z_k: [0, 2] \to R$, $k \in N$, defined by $z_k(t) = 0$ for $t \in [0, 1 - 1/k]$; $z_k(t) = 1$ for $t \in [1, 2]$, $z_k(t) = 1 - k + kt$ for $t \in (1 - 1/k, 1)$, we have

$$0 \leqq \mathscr{I}(z_k) = I(z_k) = \int_{1-1/k}^{1} (1 - t)\, k\, dt = 1/2k \to 0 \text{ as } k \to \infty.$$

Thus the infimum $i$ of $\mathscr{I}$ is zero, and $z_k$ is a minimizing sequence. The minimum is attained by the discontinuous function $z: [0, 2] \to R$ defined by $z(t) = 0$ for $t \in [0, 1)$, $z(t) = 1$ for $t \in [1, 2]$. In other words $I(z) = \mathscr{I}(z) = 0$.

*Example 5.* Let $A = [-1, 1] \times [0, 1]$, $n = 1$, $\alpha = 0$, $\psi(t) = 0$, $Q(t, x) = [-1, +\infty)$, $M = A \times [-1, +\infty)$, $F_0(t, v) = |t|\, v^2$, $F_0 \geqq 0$. with boundary conditions $x(-1) = 0$, $x(1) = 1$. The functional $\mathscr{I}(z)$ is nonnegative. Note that for the sequence $z_k: [-1, 1] \to R$, $k \in N$, defined by $z_k(t) = 0$ for $t \in [-1, 1/k]$; $z_k(t) = (\log k)^{-1} \log t + 1$, for $t \in (1/k, 1]$, we have

$$0 \leqq \mathscr{I}(z_k) = I(z_k) = \int_{1/k}^{1} (\log k)^{-2}/t\, dt = (\log k)^{-1} \to 0 \quad \text{as } k \to \infty.$$

Thus the infimum $i$ of $\mathscr{I}$ is zero and $(z_k)_{k\in N}$ is a minimizing sequence. The minimum is attained by the discontinuous function $z:[-1,1]\to R$, defined by $z(t)=0$ for $t\in[-1,0]$, $z(t)=1$ for $t\in(0,1]$. In other words, $I(z)=\mathscr{I}(z)=i$. For this example *cf.* [5], Section 1.1, no. 4.

## 4. An existence theorem for problems of optimal control

As above, let $\alpha, n$, $0\leqq\alpha\leqq n$, be given integers and, for every $x\in R^n$, let $x=(y,z)$ with $y\in R^\alpha$ and $z\in R^{n-\alpha}$. Let $A$ be a compact subset of the $(t,x)$-space such that its projection onto the $t$-axis contains the fixed interval $[t_1,t_2]$. Let $U(t,x)$, $(t,x)\in A$, $U(t,x)\subset R^m$, or $U: A\to R^m$ be a given set-valued function and let $M_0$ denote the set

$$M_0=\{(t,x,w):(t,x)\in A,\ w\in U(t,x)\}\subset R^{1+n+m}.$$

Let $f_0(t,x,w), f(t,x,w)=(f_1,\dots,f_n)$ be given functions defined on $M_0\subset R^{1+n+m}$. Let $\Omega_0$ be a class of admissible systems $(y(t), z(t), w(t))$, $t\in[t_1,t_2]$, i.e. functions $x(t)=(y(t),z(t))$, or $x:[t_1,t_2]\to R^n$, $w:[t_1,t_2]\to R^m$, such that (i) $y\in$ ACg, $z\in$ BVC, $w$ is measurable; (ii) $(t,y(t),z(t))\in A$, $w(t)\in U(t,y(t),z(t))$, a.e. in $[t_1,t_2]$; (iii) $x'(t)=f(t,x(t),w(t))$, a.e. in $[t_1,t_2]$, $f_0(\cdot,x(\cdot),w(\cdot))\in L_1([t_1,t_2])$.

We consider the functional $\mathscr{I}_0:\Omega_0\to R$ defined by

$$\mathscr{I}_0(x)=\mathscr{I}_0(y,z)=\inf_{\Gamma_0(x)}\varliminf_{k\to\infty}\int_{t_1}^{t_2} f_0(t,y_k(t),z_k(t),w_k(t))\,dt$$

$$=\inf_{\Gamma_0(x)}\varliminf_{k\to\infty} I_0(y_k,z_k,w_k),$$

where $\Gamma_0(x)$ denotes the class of all sequences $(x_k,w_k)_{k\in N}$ in $\Omega_0$ such that (a) $x_k=(y_k,z_k)\in$ ACg, $k\in N$; (b) $y_k\to y$ uniformly and $z_k\to z$ pointwise a.e. in $[t_1,t_2]$. If $\Gamma_0(x)=\emptyset$ we take $\mathscr{I}_0(x)=+\infty$. The class $\Omega_0$ is said to be *closed* if it has the following property (c): If $(y_k,z_k,w_k)_{k\in N}$ is a sequence of admissible systems, all in $\Omega_0$, satisfying (a) and (b), and if there exists a measurable $w$ such that $(y,z,w)$ is an admissible system, then $(y,z,w)$ belongs to $\Omega_0$.

Note that, if the problem of minimizing the functional above involves given initial or terminal values for $x$, say $x(t_1)\in B_1$, $x(t_2)\in B_2$, then we will proceed as illustrated in Remark 2 of Section 3.

It is well known (see [5], Section 1.13) that the problem of optimal control described above can be deparametrized, and essentially reduced to a problem of calculus of variations as discussed in Section 3. For every $(t,x)\in A$ let $Q(t,x)$ denote the set

$$Q(t,x)=\{\zeta\in R^n:\zeta=f(t,x,w),\ w\in U(t,x)\},$$

and take

$$M=\{(t,x,\zeta)\in R^{2n+1}:(t,x)\in A,\ \zeta\in Q(t,x)\}.$$

Let $F_0(t, x, \zeta)$ denote the scalar function defined on $M$ by taking

$$F_0(t, x, \zeta) = \inf\{z^0 \in R;\ z^0 \geqq f_0(t, x, w),\ \zeta = f(t, x, w),\ w \in U(t, x)\}. \tag{1}$$

If for some $\zeta$ the set in brackets is empty, we take $F_0 = +\infty$. If in (1) inf is actually a minimum for all $(t, x, \zeta) \in M$, then we may replace the problem of optimal control with the problem of the calculus of variations studied in Section 3, concerning the integral functional $\mathcal{I}$ relative to the integrand $F_0$, with constraints $(t, x(t)) \in A$, $x'(t) \in Q(t, x(t))$, a.e. in $[t_1, t_2]$, and where $x = (y, z)$, $y \in$ ACg, $z \in$ BVC. We will apply Theorem 2 of Section 3 to the present problem of the calculus of variations. Of course, we shall assume that the sets $Q(t, x)$ are non-empty and convex and that the scalar function $F_0(t, x, \zeta)$ is lower semicontinuous in $(t, x, \zeta)$ and convex in $\zeta$. Moreover, once we have a solution $x = (y, z)$ of the deparametrized problem, or problem of the calculus of variations, we shall need to know that there exists some measurable function $w(t)$, or $w: [t_1, t_2] \to R^m$ such that

$$w(t) \in U(t, x(t)),\ f_0(t, x(t), w(t)) = F_0(t, x(t), x'(t)), f(t, x(t), w(t)) = x'(t), \quad \text{a.e. in } [t_1, t_2]. \tag{2}$$

This is a consequence of the implicit function theorems. For instance, if $f_0$ and $f$ are continuous on the closed set $M_0$, then the existence of a measurable $w(t)$ satisfying (2) follows from the McShane-Warfield implicit function theorem ([5], Theorem 8.2.iii). In [5], Sections 8.2., 8.3, a great many situations are depicted for which some implicit function theorem applies. Concerning the $n$-vector function $f(t, x, w) = (f_1, \ldots, f_n)$, we write $\tilde{f}_1 = (f_1, \ldots, f_\alpha)$ and $\tilde{f}_2 = (f_{\alpha+1}, \ldots, f_n)$. We shall need the following alternative assumptions:

($g_1$) There is a scalar function $\phi(\zeta)$, $0 \leqq \zeta < +\infty$, or $\phi: R_0^+ \to R$ bounded below, such that $\phi(\zeta)/\zeta \to +\infty$, as $\zeta \to +\infty$, and $f_0(t, x, w) \geqq \phi(|\tilde{f}_1(t, x, w)|)$ for all $(t, x, w) \in M_0$.

($g_2$) For every $\varepsilon > 0$ there is a summable scalar function $\psi_\varepsilon(t) \geqq 0$ such that $|\tilde{f}_1(t, x, w)| \leqq \psi_\varepsilon(t) + \varepsilon f_0(t, x, w)$ for all $(t, x, w) \in M_0$.

($g_3$) For any $\alpha$-vector $p \in R^\alpha$ there is a summable scalar function $\phi_p(t) \geqq 0$, such that $f_0(t, x, w) \geqq \langle p, \tilde{f}_1(t, x, w)\rangle - \phi_p(t)$, for all $(t, x, w) \in M_0$.

Note that, under condition ($g_1$), certainly $\phi(\zeta) \geqq \lambda$ for some real constant $\lambda$, and then $f_0(t, x, w) \geqq \phi(|\tilde{f}_1(t, x, w)|) \geqq \lambda$ for all $(t, x, w) \in M_0$. Under condition ($g_2$) and $\varepsilon = 1$, we have $|f_1| \leqq \psi_1(t) + f_0(t, x, w)$; hence $f_0(t, x, w) \geqq -\psi_1(t)$, a summable function. Under condition ($g_3$) and $p = 0$. we have $f_0(t, x, w) \geqq -\phi_0(t)$, a summable function.

**Theorem 3** (An existence theorem for problems of Optimal Control). *Let* $1 \leqq \alpha \leqq n-1$, *and assume that* (i) *$A$ is compact and $M_0$ is closed*; (ii) *the sets $\tilde{Q}(t, x)$ are closed, convex and satisfy property* (Q) *with respect to $(t, x)$ at every point $(t,x)$ of $A$ (with the exception perhaps of a set of points whose t-coordinate lies on a set of measure zero on the t-axis)*; (iii) *the functions $f$ and $f_0$ are continuous and satisfy one of the growth conditions* ($g_1$), ($g_2$), ($g_3$). *Also we assume that* *the class* $\Omega_0$ *is nonempty and closed, and* (iv) *there is a constant* $W_0$ *such that for every element* $x = (y, z) \in \Omega_0 \cap$ ACg, *then* $V^*(z) \leqq W_0$. *Then the functional* $\mathcal{J}_0$ *has an absolute minimum* $x = (y, z)$ *in* $\Omega_0$.

*For* $\alpha = 0$, *then* $x = z$, *the requirements* $(g_1)$ *or* $(g_2)$ *or* $(g_3)$ *do not apply, yet the conclusion is still valid if we know that* (iii′) *there is a summable scalar function* $\lambda: [t_1, t_2] \to R$ *such that* $f_0(t, x, w) \geqq \lambda(t)$, *for all* $(t, x, w) \in M_0$.

*For* $\alpha = n$, *then* $x = y$, $\Omega_0$ *is a nonempty and closed class of* ACg *functions* $y(t) = (y^1, \ldots, y^n)$, $t \in [t_1, t_2]$. *condition* (iv) *does not apply, and the problem reduces essentially to those discussed in Theorems* 11.4.i. *and* ii *of* [5].

Statement 3 is a corollary of Theorem 2.

Note that for $0 \leqq \alpha \leqq n - 1$, if (iv′) there are scalar functions $\psi_i \in L_1([t_1, t_2]$, $i = \alpha + 1, \ldots, n$, such that $(t, y, z, u, v) \in M_0$ implies $v^i \geqq \psi_i(t)$ a.e. in $[t_1, t_2]$, then (iv) certainly holds.

Note that, for $1 \leqq \alpha \leqq n - 1$, the sets $\tilde{Q}(t, x)$ are closed and convex, $(g_1)$ holds, and if (iv″) there exist constants $L_i$ such that a.e. in $[t_1, t_2]$, $(t, y, z, u, v) \in M_0$ implies $v^i \geqq L_i$, $i = \alpha + 1, \ldots, n$, then both (ii) and (iv) hold.

See also Remarks 6 and 6′.

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