Power Series Spaces $\Lambda(\alpha)$ and Associated $\Lambda(\alpha)$ -Nuclearity^{*}

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Introduction

The class of strongly nuclear spaces was introduced by Martineau [8] and also rediscovered by Brudovskii [1, 2]. The definition of strongly nuclear maps is related to the space s of rapidly decreasing sequences (see [9]). In this paper we introduce the class of $\Lambda(\alpha)$ -nuclear maps, where $\Lambda(\alpha)$ is a general power series space, known to include as special cases the familiar spaces s and Γ , the space of entire functions (see [3]). We define then the notion of $\Lambda(\alpha)$ -nuclear spaces and obtain a criterion for the $\Lambda(\alpha)$ -nuclearity of sequence spaces $\lambda(P)$. Köthe [7] has proved such a criterion for the s -nuclearity of $\lambda(P)$. Some applications of this criterion are given and finally we obtain a product of several copies of $[\Lambda(\alpha)]_b$ as a universal $\Lambda(\alpha)$ -nuclear space.

Preliminaries

For terminology not explained here, see Köthe [5].

Suppose $a = (a_n)$ and $b = (b_n)$ are two sequences of scalars. Write $a < b$ if there exists a $M > 0$ such that $|a_n| \leq M |b_n|$ for all n . Let P be a set of sequences $a = (a_n)$ such that (i) $a_n \geq 0$ for each n , (ii) for each n there exists an $a \in P$ such that $a_n \neq 0$ and (iii) P is directed by $<$. Define now the sequence space $\lambda(P) = \{x = (x_n) : p_a(x) = \sum |x_n| a_n < \infty \text{ for each } a \in P\}$. The (normal) topology on $\lambda(P)$ is generated by the seminorms $\{p_a\}$.

We recall here the Grothendieck-Pietsch criterion for the nuclearity of $\lambda(P)$ with the above topology (see Pietsch [9], p. 88).

Lemma 1. $\lambda(P)$ is nuclear if and only if for each $a \in P$ there exists a $b \in P$ and a $c \in l^1$ such that $a_n \leq b_n c_n$ for all n .

Consider next a sequence $\alpha = (\alpha_n)$ such that $0 \leq \alpha_0 \leq \alpha_1 \leq \dots \leq \alpha_n \dots, \uparrow \infty$. Let P be the set of sequences (ϱ^{a_n}) , $0 < \varrho < \varrho_0$ (fixed). The sequence space $\lambda(P)$ obtained from this set P is called the power series space $\Lambda(\alpha)$. Throughout this work $\varrho_0 = \infty$ and $\Lambda(\alpha)$ is provided with the topology indicated earlier and generated by the seminorms $p_R(x) = \sum |x_n| R^{a_n}$, $R = 1, 2, \dots$.

The case $\alpha_n = \log(n+1)$ gives the space s of rapidly decreasing sequences and $\alpha_n = n$ yields the space Γ of entire functions, discussed extensively by

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Ganapathy Iyer [3]. The space $\Lambda(\alpha)$ has received brief mention in the unpublished thesis of Somasundaram [10]. Observe that $\Lambda(\alpha) \subset l^1$.

Condition G. The sequence (α_n) is defined to satisfy condition G if the following growth condition holds: there exists a $R > 1$ such that $\sum R^{-\alpha_n} < \infty$.

The spaces s and Γ mentioned above meet this requirement. If $\alpha_n = \log \log(n+k)$, k fixed, then the condition G is not met.

We shall list now some simple consequences of condition G.

In what follows we shall adopt the convention that if $\alpha_n = 0$ for $n < N$ and $\alpha_n \neq 0$ for $n \geq N$, then any statement involving the sequence $(1/\alpha_n)$ is to be interpreted as the corresponding statement on the sequence starting from the N -th term onward. Our assumption that $\alpha_n \uparrow \infty$ guarantees that $\alpha_n \neq 0$ for all $n \geq N$, for some N .

Lemma 2. If $\alpha = (\alpha_n)$ satisfies condition G then $\Lambda(\alpha) = \{a = (a_n) : |a_n|^{1/\alpha_n} \rightarrow 0\}$.

Proof. Let $a = (a_n) \in \Lambda(\alpha)$ and if possible, let $|a_n|^{1/\alpha_n} \not\rightarrow 0$. Then there exists some $\varepsilon > 0$ and a sequence of integers (n_i) such that $|a_{n_i}|^{1/\alpha_{n_i}} > \varepsilon$ for $n = n_1, n_2, \dots$. Choose now R such that $\varepsilon R \geq 1$. Then $\sum_i |a_{n_i}| R^{\alpha_{n_i}} > \sum_i \varepsilon^{\alpha_{n_i}} R^{\alpha_{n_i}} = \infty$ and this violates that $a \in \Lambda(\alpha)$.

Conversely, suppose that $|a_n|^{1/\alpha_n} \rightarrow 0$ and that condition G is fulfilled. Thus we have that $\sum_n u^{\alpha_n} < \infty$ for some $0 < u < 1$. Now given any R , choose $\varepsilon > 0$ such that $\varepsilon R = u$. Since $|a_n|^{1/\alpha_n} \rightarrow 0$, we can find N such that $|a_n|^{1/\alpha_n} < \varepsilon$ for $n \geq N$. Then $\sum_N^\infty |a_n| R^{\alpha_n} < \sum_N^\infty (\varepsilon R)^{\alpha_n} < \infty$ and this completes the proof of the lemma.

Lemma 3. If $\alpha = (\alpha_n)$ satisfies condition G and $\beta = (\beta_n)$ is such that $\beta_n \geq \alpha_n$ for all n then β also satisfies condition G and $\Lambda(\beta) \subset \Lambda(\alpha)$.

Lemma 4. If α satisfies condition G then $\Lambda(\alpha)$ is nuclear.

Lemma 5. If α satisfies condition G then $\left\{ \frac{\log(n+1)}{\alpha_n} \right\}$ is bounded.

The proof of Lemma 3 is trivial while Lemma 4 follows from Lemma 1. Lemma 5 follows easily from the inequalities $\sum_{i=0}^n R^{-\alpha_i} \leq M$ for some $R > 1$ and for some $M > 1$ and therefore $(n+1)R^{-\alpha_n} \leq M$ for all n .

Lemma 6. If α satisfies condition G and the sequence $b = (b_n)$ is such that $\{|b_n|^{1/\alpha_n}\}$ is bounded then $b \in [\Lambda(\alpha)]'$, the topological dual of $\Lambda(\alpha)$.

Lemma 6 is an immediate consequence of the following result of Köthe [6] and of Lemma 4.

Lemma 7. If $\lambda(P)$ is nuclear then the strong topological dual $\lambda'_b = \lambda'(Q)$ where $Q = \{a \in \lambda(P), \alpha_n \geq 0\}$ and $\lambda'(Q)$ has its normal topology.

$\Lambda(\alpha)$ -nuclear Maps

Throughout the rest of the paper we assume that (α_n) satisfies condition G.

Let E and F be two normed linear spaces and T be a linear map on E into F . Suppose T admits the representation

$$Tx = \sum_{n=0}^{\infty} \lambda_n \langle x, a_n \rangle y_n, \quad \text{for each } x \in E \quad (1)$$

where $(\lambda_n) \in \Lambda(\alpha)$, $a_n \in E'$, $\|a_n\| \leq 1$ and $y_n \in F$ with $\|y_n\| \leq 1$. Then T is said to be a $\Lambda(\alpha)$ -nuclear map on E into F . It is easy to see that each $\Lambda(\alpha)$ -nuclear map is continuous, nuclear and therefore precompact.

We define next a linear map T on E into F to be quasi- $\Lambda(\alpha)$ -nuclear if there exist $a_n \in E'$ such that $(\|a_n\|) \in \Lambda(\alpha)$ and $\|Tx\| \leq \sum |\langle x, a_n \rangle|$ for each $x \in E$.

It is trivial that each $\Lambda(\alpha)$ -nuclear map is quasi- $\Lambda(\alpha)$ -nuclear. In the opposite direction we prove the following result.

Proposition 1. *Suppose the Banach space F has the property that if S is an arbitrary closed subspace of $\Lambda(\alpha)$ and L is an arbitrary quasi- $\Lambda(\alpha)$ -nuclear map of S into F then L has a continuous linear extension to the whole of $\Lambda(\alpha)$. Then for arbitrary normed linear spaces E , each quasi- $\Lambda(\alpha)$ -nuclear map of E into F is indeed $\Lambda(\alpha)$ -nuclear.*

Proof. Let T be a quasi- $\Lambda(\alpha)$ -nuclear map on E into F . Determine (a_n) , $a_n \in E'$, $(\|a_n\|) \in \Lambda(\alpha)$ such that $\|Tx\| \leq \sum |\langle x, a_n \rangle|$. Define the maps $T_1 : E \rightarrow l_{\infty}$, by $T_1 x = (\langle x, a_n / \|a_n\| \rangle)$ and $T_2 : l_{\infty} \rightarrow \Lambda(\alpha)$ by $T_2[(y_n)] = (\sqrt{\|a_n\|} y_n)$. Then T_1 is linear and continuous and T_2 is a $\Lambda(\alpha)$ -nuclear map since $(\sqrt{\|a_n\|}) \in \Lambda(\alpha)$, by

Lemma 2 and $T_2[(y_n)] = \sum_{i=1}^{\infty} \langle (y_n), \sqrt{\|a_i\|} e_i \rangle e_i$, where e_i is the usual unit vector

with 1 in the i -th coordinate. Consider the subspace $S = T_2 T_1(E)$ of $\Lambda(\alpha)$ and on this space define $\Phi : S \rightarrow F$ by $\Phi(T_2 T_1 x) = Tx$. Then Φ is quasi- $\Lambda(\alpha)$ -nuclear since $\|\Phi(T_2 T_1 x)\| = \|Tx\| \leq \sum_n |\langle x, a_n \rangle| = \sum_n |\langle T_2 T_1 x, \sqrt{\|a_n\|} e_n \rangle|$. Now using

first the Hahn-Banach theorem extend Φ to $\bar{S}$ and obtain the extension $\hat{\Phi}$ as a quasi- $\Lambda(\alpha)$ -nuclear map on S into F . Use next the hypothesis to obtain a continuous linear extension of $\hat{\Phi}$ to the whole of $\Lambda(\alpha)$ and call this extension T_3 . Now $T = T_3 T_2 T_1$, where T_2 is $\Lambda(\alpha)$ -nuclear and the other two are linear and continuous and therefore T is $\Lambda(\alpha)$ -nuclear.

Corollary. *If F has the bounded extension property then each quasi- $\Lambda(\alpha)$ -nuclear map of a normed linear space E into the Banach space F is $\Lambda(\alpha)$ -nuclear.*

For any linear, continuous map T on E into F define the n -th approximation number $\alpha_n(T)$ by $\alpha_n(T) = \inf \|T - A_n\|$, where the infimum is taken over all linear maps A_n of E into F which have a range of dimension at most n . Obviously $\alpha_n(T) \geq \alpha_{n+1}(T)$ for each n . If $(\alpha_n(T))$ belongs to a specified sequence space λ we say that T is of type λ .

Proposition 2. *Each $\Lambda(\alpha)$ -nuclear map of a normed linear space E into a normed linear space F is of type $\Lambda(\alpha)$.*

Proof. Let T be $\Lambda(\alpha)$ -nuclear and have the representation (1). Consider maps A_i defined by $A_i x = \sum_{n=0}^{i-1} \lambda_n \langle x, a_n \rangle y_n$. Now $\|T - A_i\| \leq \sum_{n=i}^{\infty} |\lambda_n|$ and thus $\alpha_i(T) \leq \sum_{n=i}^{\infty} |\lambda_n|$. For $R \geq 1$, $\sum_{i=0}^{\infty} \alpha_i(T) R^{\alpha_i} \leq \sum_{i=0}^{\infty} \left(\sum_{n=i}^{\infty} |\lambda_n| \right) R^{\alpha_i} = \sum_{n=0}^{\infty} |\lambda_n| \sum_{i=0}^n R^{\alpha_i} \leq \sum_{n=0}^{\infty} |\lambda_n| (n+1) R^{\alpha_n}$. Consider now the sequence $\{(n+1)^{1/\alpha_n}\}$. From Lemma 5 it follows that this sequence is bounded and thus $\sum_{n=0}^{\infty} |\lambda_n| (n+1) R^{\alpha_n} < \infty$ since $(\lambda_n) \in \Lambda(\alpha)$.

Consider next a bounded sequence (d_n) , $d_n > 0$ and the diagonal transformation D on l^1 into itself defined by $D[(x_n)] = (d_n x_n)$. Let Π denote a permutation of the positive integers I^+ .

Köthe [7] proves the following result.

Lemma 8. *The map D is s -nuclear on l^1 into l^1 if and only if there exists a permutation Π of I^+ such that $(d_{\Pi(n)}) \in s$.*

The proof of the sufficiency in the above lemma is immediate. In proving the necessity, Köthe observes that a s -nuclear map on l^1 into l^1 is compact and consequently $d_n \rightarrow 0$. Since a permutation Π can be found so that $d_{\Pi(n)}$ is monotone decreasing, it suffices to prove that for a positive monotone decreasing sequence (d_n) converging to zero, the corresponding diagonal transformation D is s -nuclear implies $(d_n) \in s$. This last statement is achieved by showing $\alpha_n(D) = d_{n+1}$.

In view of the above analysis of the proof of Köthe's theorem and proposition 2 above we state below a criterion for the $\Lambda(\alpha)$ -nuclearity of D and omit its proof.

Proposition 3. *A diagonal transformation $D = (d_n)$, $d_n \geq 0$ is a $\Lambda(\alpha)$ -nuclear map on l^1 into l^1 if and only if the d_n which are different from zero can be rearranged into a sequence in $\Lambda(\alpha)$.*

The above proposition provides the proof of the existence of maps which are nuclear but not $\Lambda(\alpha)$ -nuclear. The sequence (2^{-n}) provides a diagonal map of l^1 into l^1 which is nuclear but not Γ -nuclear (see also proposition 7).

$\Lambda(\alpha)$ -nuclear Spaces

Let E be a locally convex space and U be an absolutely convex neighbourhood, with p_U being its gauge. Set $E(U) = E/N(U)$, where $N(U)$ is the null space of p_U . $E(U)$ is normed by p_U . For another such neighbourhood V which is absorbed by U (written $V < U$), the canonical map $\tilde{K}(V, U)$ is the map on $\tilde{E}(V)$ into $\tilde{E}(U)$ which associates to the equivalence class $x(V)$ the element $x(U)$. The map $\tilde{K}$ is continuous.

Now define the locally convex space E to be a $\Lambda(\alpha)$ -nuclear space if a fundamental system $\mathcal{U}$ of absolutely convex neighbourhoods in E (or equivalently, each such system) has the property that for each $U \in \mathcal{U}$ there exists a $V \in \mathcal{U}$ such that $V \prec U$ and the canonical map $\tilde{K}(V, U)$ is $\Lambda(\alpha)$ -nuclear. The equivalence stated above results from the fact that the composition of linear continuous maps with $\Lambda(\alpha)$ -nuclear maps will be $\Lambda(\alpha)$ -nuclear.

We postpone giving examples of $\Lambda(\alpha)$ -nuclear spaces. First we obtain a criterion for a sequence space $\lambda(P)$ to be $\Lambda(\alpha)$ -nuclear. The result is the analogue of the Grothendieck-Pietsch criterion for the nuclearity of $\lambda(P)$ and is motivated by the Köthe's corrected version of a result of Brudovskii [1, 2] on the s -nuclearity of $\lambda(P)$.

Proposition 4. *The sequence space $\lambda(P)$ is $\Lambda(\alpha)$ -nuclear if and only if to each $a \in P$, there exists a $b \in P$ such that $a \prec b$ and such that the sequence (a_k/b_k) with $a_k \neq 0$ can be rearranged into a member of $\Lambda(\alpha)$.*

Proof (Sketch). The sets $U_a = \{x : p_a(x) \leq 1\}$, $a \in P$, form a fundamental system of neighbourhoods in $\lambda(P)$. Let $M(a)$ denote the set of indices k for which $a_k \neq 0$. If now $E = \lambda(P)$, then as is well-known, $\tilde{E}(U_a)$ is norm isomorphic to $l^1[M(a)]$, $a \prec b$ is equivalent to $U_b \prec U_a$ and the canonical map $\tilde{K}(U_b, U_a)$ can be identified with the diagonal transformation $(y_n) \rightarrow (a_n b_n^{-1} y_n)$ of $l^1[M(b)]$ into $l^1[M(a)]$. Now the desired result is a consequence of Proposition 3.

Corollary. *The power series space $\Lambda(\gamma)$ is $\Lambda(\alpha)$ -nuclear if and only if there exists a $R > 1$ and a permutation Π of I^+ such that $\Pi(R^{-\gamma_n}) \in \Lambda(\alpha)$.*

Proof. Suppose $\Lambda(\gamma)$ is $\Lambda(\alpha)$ -nuclear. Since $\Lambda(\gamma)$ is generated by the stufen system (R^{γ_n}) , $R = 1, 2, \dots$ it follows from Proposition 4 that given R , there exists S and a permutation Π such that $\Pi(R^{\gamma_n}/S^{\gamma_n}) \in \Lambda(\alpha)$. But $S^{-\gamma_n} \leq R^{\gamma_n} S^{-\gamma_n}$ and also $\Lambda(\alpha)$ is a normal sequence space. Thus $\Pi(S^{-\gamma_n}) \in \Lambda(\alpha)$.

Conversely, let there exist R and Π such that $\Pi(R^{-\gamma_n}) \in \Lambda(\alpha)$. Now, given any (T^{γ_n}) belonging to the stufen system, let S denote the $\max(R, T)$. Then $S^{-\gamma_n} \leq R^{-\gamma_n}$ and therefore $\Pi(S^{-\gamma_n}) \in \Lambda(\alpha)$. Consider now $(T^{\gamma_n} S^{-2\gamma_n}) = (T^{\gamma_n} S^{-\gamma_n})(S^{-\gamma_n}) \leq (S^{-\gamma_n})$ and therefore $\Pi(T^{\gamma_n} S^{-2\gamma_n}) \in \Lambda(\alpha)$. Since $(S^{2\gamma_n})$ is also a member of the stufen system, Proposition 4 now gives the desired result.

For a result of this type for Schwartz spaces, see Terzioglu [11].

Examples of $\Lambda(\alpha)$ -nuclear spaces are provided by means of the following two simple propositions both of which are consequences of Proposition 4.

Proposition 5. *If $\left(\frac{\alpha_n}{\beta_n}\right) \rightarrow 0$ then $\Lambda(\beta)$ is $\Lambda(\alpha)$ -nuclear.*

The proof follows immediately from Lemma 2 and the corollary to Proposition 4 since $(2^{-\beta_n/\alpha_n}) \rightarrow 0$.

Proposition 6. *The strong dual $[\Lambda(\alpha)]'_b$ of $\Lambda(\alpha)$ is $\Lambda(\alpha)$ -nuclear.*

Proof. If $a = (a_n) \in \Lambda(\alpha)$ and $a_n \geq 0$, it follows easily that $\sqrt{a} = (\sqrt{a_n})$ is also in $\Lambda(\alpha)$. Now the result is a consequence of Lemma 7 and Proposition 4.

It is well known that l^1 is not nuclear; Brudovskii [1] states that s is not s -nuclear. We conjecture that in general it is true that $\Lambda(\alpha)$ is not $\Lambda(\alpha)$ -nuclear. The following proposition supplements the two known results quoted above and the proof is also of some interest.

Proposition 7. *If $\alpha_n = n^m$, $m > 0$ and fixed, then $\Lambda(\alpha)$ is not $\Lambda(\alpha)$ -nuclear.*

Proof. After the corollary to Proposition 4 we need only show that whatever be $R > 1$, there exists no permutation Π such that $\Pi(R^{-\alpha_n}) \in \Lambda(\alpha)$; i.e., for each permutation Π , $(R^{-\alpha_n/\alpha_{\Pi(n)}})$ does not converge to zero. This is equivalent to showing that there exists no permutation Π of I^+ such that $(\alpha_n/\alpha_{\Pi(n)}) \rightarrow \infty$. If possible let there exist a Π such that $(\alpha_n/\alpha_{\Pi(n)}) = (n/\Pi(n))^m \rightarrow \infty$ or equivalently $(\Pi(n)/n) \rightarrow 0$. Then $\Pi(n)/n < 1/k$, for $n \geq N$, where we pick k to be an integer larger than 2 and choose N such that $N \equiv 0 \pmod{k}$. Then

$$\begin{aligned}\Pi(N+i) &< (N/k) + 1, & i = 0, 1, 2, \dots, k; \\ \Pi(N+j) &< (N/k) + 2, & j = k+1, k+2, \dots, 2k,\end{aligned}$$

and proceeding thus we get,

$$\Pi(N+l) < (N/k) + p, \quad l = (p-1)k + 1, \dots, pk.$$

Thus, since Π is a 1-1, onto map of I^+ to I^+ , we have to find pk positive integers less than $(N/k) + p$. Choosing $p = N/k$, we get $N < 2(N/k)$ or $k < 2$, setting up a contradiction.

Remark. It follows from Proposition 5 and 7 that the space $\Lambda(\alpha)$, $\alpha_n = n^m$, while being not $\Lambda(\alpha)$ -nuclear, is $\Lambda(\gamma)$ -nuclear for $\gamma_n = n^{m-\varepsilon}$, $m \geq \varepsilon > 0$; in particular, the space Γ is $\Lambda(\gamma)$ -nuclear for $\gamma_n = n^\delta$, $0 \leq \delta < 1$.

Following Köthe [7], we say that $\lambda(P)$ is uniformly $\Lambda(\alpha)$ -nuclear if there exists a "universal" permutation Π such that for each $a \in P$ there exists a $b \in P$ such that $b \succ a$ and $a_{\Pi(n)} \leq c_n b_{\Pi(n)}$ for a suitable $c = (c_n)$ such that $(c_n) \in \Lambda(\alpha)$. Terzioglu [12] proves that the strong dual of a uniformly s -nuclear space is nuclear and therefore, by a theorem of Köthe [7], strong dual of a metrizable s -nuclear space is nuclear. Martineau [8] and Brudovskii [1] announce that the strong dual of a metrizable s -nuclear space is s -nuclear. In this context we prove now the following result which is analogous to that of Terzioglu and Köthe.

Proposition 8. *The strong dual of a uniformly $\Lambda(\alpha)$ -nuclear sequence space $\lambda(P)$ is $\Lambda(\beta)$ -nuclear for each $\beta = (\beta_n)$ such that $(\alpha_n/\beta_n) \rightarrow \infty$.*

Proof. Since $\lambda(P)$ is uniformly $\Lambda(\alpha)$ -nuclear, to each $a \in P$ there exists a $b \in P$ with $b \succ a$ and $a_{\Pi(n)} \leq c_n b_{\Pi(n)}$, for a suitable $c = (c_n) \in \Lambda(\alpha)$. Consider now the sequence $(2^{-\alpha_n})$. By Lemma 2 it is in $\Lambda(\beta)$; also, by Lemma 6, (2^{α_n}) is in $[\Lambda(\alpha)]'$. Therefore $(c_n 2^{\alpha_n}) \in l^1$.

Consider next an arbitrary $x = (x_n) \in \lambda(P)$, with $x_n \geq 0$. Let $a \in P$ be arbitrary and determine b and c as above. Since $x \in \lambda(P)$ and $b \in P$, we have $\sum x_n b_n = \sum x_{\Pi(n)} b_{\Pi(n)} < \infty$. But $\sum x_{\Pi(n)} a_{\Pi(n)} 2^{\alpha_n} \leq \sum x_{\Pi(n)} b_{\Pi(n)} c_n 2^{\alpha_n} < \infty$. Thus for each $a \in P$, we have $\sum x_n 2^{\alpha_{\Pi^{-1}(n)}} a_n < \infty$ and consequently $(x_n 2^{\alpha_{\Pi^{-1}(n)}}) \in \lambda(P)$.

Also $\Pi\left(\frac{x_n}{x_n 2^{2^{H^{-1}(n)}}}\right) = (2^{-\alpha_n})$ is in $\Lambda(\beta)$. The proof is now complete by appealing to Lemma 7 and Proposition 4.

Remark. The above result yields in particular that the strong dual of a uniformly Γ -nuclear space is $\Lambda(n^{1-\varepsilon})$ -nuclear for each ε such that $1 \geq \varepsilon > 0$, and also s -nuclear.

Universal $\Lambda(\alpha)$ -nuclear Spaces

Komura and Komura [4] exhibit $[s]^4$ as a universal nuclear space and Martineau [8] states that $[s']^4$ is a universal s -nuclear space. We prove the following general imbedding theorem.

Proposition 9. *The locally convex space E is $\Lambda(\alpha)$ -nuclear if and only if it is (topologically) isomorphic to a subspace of an I -fold product $([\Lambda(\alpha)]'_b)^I$.*

Proof. In an (as yet) unpublished work P. Spuhler proves that subspaces and arbitrary products of $\Lambda(\alpha)$ -nuclear spaces are $\Lambda(\alpha)$ -nuclear and since $[\Lambda(\alpha)]'_b$ is $\Lambda(\alpha)$ -nuclear the proof of the sufficiency is complete.

Suppose now that E is $\Lambda(\alpha)$ -nuclear. Let $\mathcal{U} = [U_i, i \in I]$ be a fundamental system of absolutely convex neighbourhoods in E , where I is a suitably determined index set. Since E is $\Lambda(\alpha)$ -nuclear, to each U_i we can find a $V_i \in \mathcal{U}$ such that $V_i \prec U_i$ and the canonical map $\tilde{K}_i(V_i, U_i)$ of $\tilde{E}(V_i)$ onto $\tilde{E}(U_i)$ is $\Lambda(\alpha)$ -nuclear and therefore can be written as

$$x(U_i) = \tilde{K}_i[x(V_i)] = \sum_n \lambda_n^{(i)} \langle x(V_i), a_n^{(i)} \rangle y_n^{(i)},$$

with $\lambda_n^{(i)} = (\lambda_n^{(i)}) \in \Lambda(\alpha)$, $a_n^{(i)} \in [E(V_i)]'$, $\|a_n^{(i)}\| \leq 1$ for each n and $y_n^{(i)} \in \tilde{E}(U_i)$ with $\|y_n^{(i)}\| \leq 1$; thus we can get $b_n^{(i)} \in V_i^0$ for each n , so that

$$P_{U_i}(x) \leq \sum_n |\lambda_n^{(i)}| |\langle x, b_n^{(i)} \rangle| \leq P_{V_i}(x) \sum |\lambda_n^{(i)}|. \quad (2)$$

Define now the map $\Phi_i: E \rightarrow [\Lambda(\alpha)]'_b$ by $\Phi_i(x) = (\langle x, b_n^{(i)} \rangle)$. Since for each $z = (z_n) \in \Lambda(\alpha)$ with $z_n \geq 0$, we have $\sum |\langle x, b_n^{(i)} \rangle| z_n \leq P_{V_i}(x) \sum |z_n|$ it follows from Lemma 7 that $\Phi_i(x) \in [\Lambda(\alpha)]'$.

Next define $\Phi: E \rightarrow \Pi[\Lambda(\alpha)]'_b = [A']^I$ by the relation $\Phi x = [\Phi_i x, i \in I]$. Since Φ and Φ_i are linear, it follows from (2) above that Φ is 1-1. Now given an arbitrary $\gamma = (\gamma_n) \in \Lambda(\alpha)$ or equivalently, an arbitrary seminorm generating the topology of $[\Lambda(\alpha)]'$, we have, for each $i \in I$, $\sum |\gamma_n| |\langle x, b_n^{(i)} \rangle| \leq \varrho P_{V_i}(x)$ where $\varrho = \sum |\gamma_n|$. This proves the continuity of the linear maps Φ_i and since $\Phi_i = p_i \circ \Phi$, where p_i is the projection of $[A']^I$ onto A' we get that Φ is continuous. To prove the continuity of Φ^{-1} consider the neighbourhood

$$N_i = \{z = (z_n) \in A' : \sum |z_n| |\lambda_n^{(i)}| \leq 1\}$$

in A' . Then $\hat{N}_i = p_i^{-1}(N_i)$ is a neighbourhood in the product space. It is easy to verify, using (2), that $\Phi^{-1}(\hat{N}_i) \cap E \subset U_i$ and the proof is complete.

Added in proof. (Oct. 1, 1970). Mr. P. Spuhler has now informed the author that the product of even two $\Lambda(\alpha)$ -nuclear spaces need not be $\Lambda(\alpha)$ -nuclear. In view of this remark, Proposition 9 needs to be revised as follows: If E is $\Lambda(\alpha)$ -nuclear then it is topologically isomorphic to a subspace of $([\Lambda(\alpha)]_b)'$; if arbitrary products of $\Lambda(\alpha)$ -nuclear spaces is $\Lambda(\alpha)$ -nuclear, then the converse of the above statement holds.

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