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# Inclusion Theorems for Absolutely $\lambda$-Summing Maps*

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## 1. Introduction

The concept of absolutely $p$-summing (absolutely $l^p$-summing) maps between Banach spaces was introduced in [4] and it was shown there that each such map is continuous and is absolutely $q$-summing if $q > p$. Using suitably restricted Köthe sequence spaces $\lambda$, the notion of absolutely $\lambda$-summing maps was introduced and studied in [6]. In this paper we study absolutely $\lambda$-summing maps for arbitrary normal sequence spaces $\lambda$ and consider, for two spaces $\lambda$ and $\mu$, the relationship between the set of absolutely $\lambda$-summing maps and the set of absolutely $\mu$-summing maps. We obtain in Section 3 the main result which gives a sufficient condition for an inclusion relation to hold. In Section 4 we apply the above result to pairs of sequence spaces which are echelon spaces and in Section 5 we consider pairs of power series spaces of finite or infinite type. Since the absolutely $l^\infty$-summing maps are precisely the continuous linear maps we are also able to study the relation between absolutely $\lambda$-summing maps and continuous linear maps. In Section 6 we show that given $\lambda$, either every absolutely $\lambda$-summing map is continuous or every linear map is absolutely $\lambda$-summing.

## 2. Definitions and Preliminary Results

All the notations and terminologies not explained below are as in [3]. Throughout the paper the sequence spaces $\lambda$ considered are assumed to be normal and unless otherwise stated, equipped with the topology $\mathfrak{T}_b(\lambda^\times)$.

A sequence space $\lambda$ is called a *step* provided that it is perfect, that $\lambda[\mathfrak{T}_b(\lambda^\times)]$ is a Banach space and that $l^1 \subset \lambda \subset l^\infty$. In [2] it is observed that a perfect sequence space $\lambda$ is a step if and only if $\lambda^\times$ is a step.

Suppose $\lambda$, $\mu$ are sequence spaces. We define

$$\lambda \cdot \mu = \{(x_n y_n): \quad (x_n) \in \lambda, \quad (y_n) \in \mu\}.$$

Let $(\lambda_k)$ be a sequence of steps and $(a^k)$ a sequence of sequences such that

i) $0 < a_i^k < a_i^{k+1}$ for all $i$, $k$,

ii) $\frac{1}{a^{k+1}} \lambda_{k+1} \subset \frac{1}{a^k} \lambda_k$ for all $k$.

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Under these conditions $(a^k, \lambda_k)$ is called an *echelon system* and $\lambda = \bigcap_k \frac{1}{a^k} \lambda_k$ is called the corresponding *echelon space*. It is shown in [2] that $\lambda$ is a perfect sequence space, that $\lambda^\times = \bigcup_k a^k \lambda_k^\times$ and that $\lambda\,[\mathfrak{T}_b(\lambda^\times)]$ is a Fréchet space.

If $\lambda$ and $\mu$ are sequence spaces we shall denote by $D(\mu, \lambda)$ the set of diagonal matrices carrying $\mu$ into $\lambda$. We shall frequently use the following result of Crofts [1].

**Proposition A.** *$D(\mu, \lambda) \subset (\lambda^\times \cdot \mu)^\times$ and if $\lambda$ is perfect we obtain equality.*

Consider now a normed linear space $E$. Let $\lambda$ be a fixed sequence space. We define $\lambda(E) = \{(x_n) : x_n \in E$ for each $n$ and $(\langle x_n, a\rangle) \in \lambda$ for each $a \in E'\}$ and $\lambda\,[E] = \{(x_n) : x_n \in E$ for each $n$ and $(\|x_n\|) \in \lambda\}$.

Let $E$ and $F$ be two normed linear spaces and $T$ be a linear map on $E$ into $F$. $T$ is said to be *absolutely $\lambda$-summing* if for each $x = (x_n) \in \lambda(E)$, the sequence $Tx = (Tx_n) \in \lambda\,[F]$. We emphasise here that in the definition above we have not required $T$ to be continuous; in fact, in Section 6 of this paper we discuss discontinuous absolutely $\lambda$-summing maps.

The absolutely $l^\infty$-summing maps are precisely the continuous linear maps whereas the absolutely $\omega$-summing maps are exactly the linear maps (between the specified normed spaces). When $\lambda = l^p$ we shall use the terminology "absolutely $p$-summing map" to conform with that notion introduced by Pietsch [4] and when $\lambda = l^1$ we shall simply say "absolutely summing", again to conform to standard practice.

It is now easy to see that if $d = (d_n)$ and $d_n > 0$ for each $n$ then $T$ is absolutely $\lambda$-summing if and only if $T$ is absolutely $(d \cdot \lambda)$-summing.

Suppose $J$ is an infinite subsequence of $\mathbb{N}$ and $x$ is a sequence. We define $x_J = (x_n)_{n \in J}$ and $\lambda_J = \{x_J : x \in \lambda\}$.

Throughout the paper the sequence space $v$ is used exclusively to denote $(\lambda^\times \cdot \mu)^\times$. It now follows from Proposition $A$ that $v = D(\mu, \lambda) = D(\lambda^\times, \mu^\times)$ whenever $\lambda$ is perfect.

## 3. Main Result

Let $E$ and $F$ be two arbitrary normed spaces and $\lambda$ and $\mu$ be two normal sequence spaces. A natural question is: under what conditions is every absolutely $\lambda$-summing map an absolutely $\mu$-summing map? In this section we prove a sufficient condition ensuring the above and provide examples to which the result is applicable. Further applications of this result are contained in the next two sections.

**Theorem.** *If $(v \cdot \lambda^\times)^\times \subset \mu$ and $v \cdot \mu \subset \lambda$ then for arbitrary normed spaces $E$ and $F$ each absolutely $\lambda$-summing map on $E$ into $F$ is absolutely $\mu$-summing.*

*Proof.* Let $T$ be absolutely $\lambda$-summing on $E$ into $F$. Let $(x_n) \in \mu(E)$. Let $\xi$ be the sequence defined by $\xi_n = \|Tx_n\|$. Then for each $\alpha \in v$ and $a \in E'$ we have

$$(\langle \alpha_n x_n, a\rangle) = \alpha \cdot (\langle x_n, a\rangle)_n \in v \cdot \mu \subset \lambda\,.$$

Since $T$ is absolutely $\lambda$-summing it follows that $|\alpha| \cdot \xi = (\| T(\alpha_n x_n) \|) \in \lambda$ and since $\lambda$ is normal, $\alpha \cdot \xi \in \lambda$. Thus $\xi \in D(\nu, \lambda)$ and therefore, by Proposition $A$, $\xi \in (\nu \cdot \lambda^\times)^\times \subset \mu$, i.e., $(\| Tx_n \|) \in \mu$. Thus $T$ is absolutely $\mu$-summing. ∎

We list below some simple specific examples to which the theorem is applicable.

1. Let $1 \leqq p \leqq q \leqq \infty$ and $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. If $\lambda = l^p$ and $\mu = l^q$ then by the Hölder inequality

$$\nu = l^r, \quad (\nu \cdot \lambda^\times)^\times = l^q \quad \text{and} \quad \nu \cdot \mu = l^p$$

so that $l^p \subset l^q$ implies that each absolutely $p$-summing map is absolutely $q$-summing. It may be considered that we have obtained a somewhat simple proof of this known result ([4], p. 335).

2. If $\lambda = l^1$ and $\mu$ is any perfect space then

$$\nu = \mu^\times, \quad (\nu \cdot \lambda^\times)^\times = \nu^\times = \mu \quad \text{and} \quad \nu \cdot \mu = \mu^\times \cdot \mu \subset l^1 = \lambda$$

so that, by the theorem, every absolutely summing map is absolutely $\mu$-summing.

3. If $\lambda = l^\infty$ and $\mu = \phi$ then

$$\nu = \omega, \quad (\nu \cdot \lambda^\times)^\times = \phi = \mu \quad \text{and} \quad \nu \cdot \mu = \phi \subset \lambda$$

and consequently every continuous linear map on $E$ into $F$ is absolutely $\phi$-summing. Actually we shall see in § 6, Example 8, that more is true.

4. If $\lambda = \omega$ then the second condition of the theorem is satisfied but $\nu = \omega$ so $(\nu \cdot \lambda^\times)^\times = (\omega \cdot \phi)^\times = \omega$ so the first condition of the theorem is satisfied if and only if $\mu = \omega$.

5. If $\lambda = \mu$ is perfect then $\nu = (\lambda^\times \cdot \lambda)^\times$ so that $(\nu \cdot \lambda^\times)^\times = ((\lambda^\times \cdot \lambda)^\times \cdot \lambda^\times)^\times \subset \lambda^{\times\times} = \lambda$; also, by Proposition $A$, $\nu = D(\lambda, \lambda)$ so $\nu \cdot \lambda \subset \lambda$ and the conditions of the theorem are satisfied. However if $\lambda = \mu = c_0$ then $\nu = l^\infty$ and $\nu \cdot \mu = \lambda$ but $(\nu \cdot \lambda^\times)^\times = l^\infty \not\subset c_0 = \mu$.

Since $\lambda = \mu$ it now follows that the converse of the theorem is false. We shall see later (§ 6) that the converse is false even if one assumes that $\lambda$ and $\mu$ are perfect.

Problem 1. If $\lambda$ and $\mu$ are perfect and if $l^1 \subset \lambda \subset \mu \subset l^\infty$ does it follow that for arbitrary Banach spaces $E$ and $F$ every absolutely $\lambda$-summing map is absolutely $\mu$-summing?

We shall see (§ 5, Remark after Proposition 10) that the answer is negative if we do not assume that $l^1 \subset \lambda$, $\mu \subset l^\infty$.

We now give a few simple applications of the preceding considerations.

**Corollary 1.** *For any perfect space $\lambda$ every absolutely summing map is absolutely $\lambda$-summing.*

**Corollary 2.** *Let $\lambda$ be a perfect sequence space such that not every nuclear space is $\lambda$-nuclear*[1]. *Then there is no positive integer $n$ such that if $T_1, T_2, \ldots, T_n$*

[1] For definitions and examples of this situation see [7].

*are absolutely $\lambda$-summing operators on $l^2$ then the composition $T = T_1\ T_2, \ldots, T_n$ is $\lambda$-nuclear.*

*Proof.* Consider such a $\lambda$ and let $E$ be a locally convex space which is nuclear but not $\lambda$-nuclear. Then there exists a fundamental system $\mathscr{U}$ of neighbourhoods of zero in $E$ such that if $V \in \mathscr{U}$ then there exists a $W \in \mathscr{U}$ such that $W \prec V$, $\hat{E}_V$ and $\hat{E}_W$ are isomorphic to $l^2$ and the canonical map $\hat{E}_W \to \hat{E}_V$ is absolutely summing; by Corollary 1, it is then absolutely $\lambda$-summing. Now if the product of each $n$ absolutely $\lambda$-summing maps in $l^2$ were $\lambda$-nuclear then given $V$ we could apply the above considerations $n$ times to obtain $U \in \mathscr{U}$ such that $U \prec V$ and the canonical map $\hat{E}_U \to \hat{E}_V$ was $\lambda$-nuclear. But this would imply that $E$ is $\lambda$-nuclear which is false. ∎

## 4. Echelon Spaces of Fixed Order

In this section we apply the main result to echelon spaces $\lambda$ and $\mu$ of (the same) fixed order; we obtain a sufficient condition for the hypotheses of the main theorem to be satisfied. We present cases of $\lambda$ in which each continuous linear map is absolutely $\lambda$-summing and also cases where continuous linear maps are not necessarily absolutely $\lambda$-summing.

Suppose $l$ is a step and $a = (a^k)_k$. We consider the echelon space $\Lambda(a, l) = \bigcap_k \frac{1}{a^k} l$.

Recall ([2], p. 189) that $[\Lambda\ (a, l)]^\times = \bigcup_k a^k l^\times$.

**Lemma 1.** *If $l$ is a step then $D\ (l, l) = l^\infty$.*

*Proof.* Since $l$, being a step, is normal, $l^\infty \subset D\ (l, l)$. Conversely, since $l^1 \subset l \subset l^\infty$ we have

$$D\,(l, l) \subset D\,(l^1, l^\infty) = l^\infty\,. \qquad ∎$$

Throughout the rest of this section we let $\lambda = \Lambda\ (a, l)$ and $\mu = \Lambda\ (b, l)$ be two echelon spaces of order $l$ and since $\lambda$ and $\mu$ are perfect it follows from Proposition $A$ that

$$\nu = (\lambda^\times, \mu)^\times = D\,(\mu, \lambda) = D\,(\lambda^\times, \mu^\times)\,.$$

**Lemma 2.** $\nu = \bigcap_{k=1}^\infty \bigcup_{j=1}^\infty \frac{b^j}{a^k} l^\infty\,.$

*Proof.* Let $z \in \bigcap_{k=1}^\infty \bigcup_{j=1}^\infty \frac{b^j}{a^k} l^\infty$ and take $u \in \lambda^\times$. Then there exists a $k$ such that $u \in a^k l^\times$ and so we have some $j$ with $a^k z \in b^j l^\infty$. Hence

$$z \cdot u = a^k z \frac{u}{a^k} \in b^j l^\infty \cdot l^\times = b^j l^\times \subset \mu^\times$$

and therefore $z \in \nu$. Conversely, suppose that $z \in \nu$. Then we have ([1], p. 67) that the map $z: \lambda^\times[\mathfrak{T}_b(\lambda)] \to \mu^\times[\mathfrak{T}_b(\mu)]$ is continuous, so it maps bounded sets into bounded sets. Applying now the characterization of bounded sets in a co-echelon space ([2], Theorem 2, ii)) it follows that for each $k$, there exists a $j$ and $M>0$ such that if $x$ is in the unit ball of $l^\times$ then $\left\| z \frac{a^k}{b^j} x \right\|_{l^\times} \leqq M$. This implies that $z \frac{a^k}{b^j} l^\times \subset l^\times$. Thus $z \cdot a^k \in b^j D(l^\times, l^\times) = b^j D(l, l) = b^j l^\infty$. Thus we have shown that for each $k$ there exists $j$ such that $z \cdot a^k \in b^j l^\infty$ and this implies that $z \in \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{\infty} \frac{b^j}{a^k} l^\infty$. ∎

**Proposition 1.** $\nu \cdot \mu \subset \lambda$.

Proof is immediate from Proposition $A$ since $\mu$ is perfect. ∎

*Definition 1.* If $\lambda = \Lambda(a, l)$ and $\mu = \Lambda(b, l)$ we say that $b$ *dominates* $a$ if for each $j_0$ there exists a $k_0$ such that for each $k \geqq k_0$ there exists a $j$ such that $a^k b^{j_0} \in a^{k_0} b^j l^\infty$.

As we shall see in the next section, this condition is not related by implication to the condition $\lambda \subset \mu$.

**Proposition 2.** *If $b$ dominates $a$ then $(\nu \cdot \lambda^\times) \supset \mu^\times$ and hence every absolutely $\lambda$-summing map is absolutely $\mu$-summing. Moreover if $l = l^1$ then $b$ dominates $a$ if and only if $(\nu \cdot \lambda^\times) \supset \mu^\times$.*

*Proof.* Suppose $b$ dominates $a$ and let $v \in \mu^\times$. Then we have $j_0$ such that $v \in b^{j_0} l^\times$. For this $j_0$ we have a $k_0$ as in Definition 1. Now for $k \geqq k_0$ there exists $j$ such that

$$\frac{b^{j_0}}{a^{k_0}} \in \frac{b^j}{a^k} l^\infty .$$

But this shows, along with the fact that for $k < k_0$ we have $\frac{1}{a^k} l \supset \frac{1}{a^{k_0}} l$, that $\frac{b^{j_0}}{a^{k_0}} \in \bigcap_{k \geqq k_0} \bigcup_{j=1}^{\infty} \frac{b^j}{a^k} l^\infty = \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{\infty} \frac{b^j}{a^k} l^\infty = \nu$ and so $v \in \frac{b^{j_0}}{a^{k_0}} a^{k_0} l^\times \subset \nu \cdot \lambda^\times$.

Thus we have shown that $\mu^\times \subset (\nu \cdot \lambda^\times)$ and since $\mu$ is perfect we have $(\nu \cdot \lambda^\times)^\times \subset \mu$. This, along with Proposition 1, permits the application of the theorem and the assertion that each absolutely $\lambda$-summing map is absolutely $\mu$-summing.

Finally, suppose that $l = l^1$ and $(\nu \cdot \lambda^\times) \supset \mu^\times$. Choose any $j_0$. Then $b^{j_0} \in b^{j_0} \cdot l^\infty = b^{j_0} \cdot l^\times \subset \mu^\times \subset \nu \cdot \lambda^\times$ so that $b^{j_0} \in \nu \cdot \lambda^\times = \nu \cdot \bigcup_{k=1}^{\infty} a^k l^\infty = \bigcup_{k=1}^{\infty} a^k \nu$.

Hence there exists $k_0$ such that $b^{j_0} \in a^{k_0} \nu$ and by Lemma 2 we have

$$\frac{b^{j_0}}{a^{k_0}} \in \nu = \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{\infty} \frac{b^j}{a^k} l^\infty$$

so that for each $k$ there exists $j$ with $a^k b^{j_0} \in a^{k_0} b^j l^\infty$ and this means that $b$ dominates $a$. ∎

Problem 2. In the last part of Proposition 2 can the requirement $l = l^1$ be dropped?

**Corollary 3.** *If $\lambda$ is an echelon space of order $l$ then every absolutely $l$-summing map is absolutely $\lambda$-summing.*

*Proof.* Let $\lambda = \Lambda(b, l)$ and define $a$ by $a^k = ke$. Then clearly $l = \Lambda(a, l)$. On the other hand, given $j_0$ set $k_0 = 1$ and given $k$ take $j \geqq j_0$ so that $b^{j_0} \leqq b^j$ and we have $a^k b^{j_0} = k b^{j_0} \in b^j l^\infty = e\, b^j \cdot l^\infty = a^{k_0} b^j l^\infty$, so $b$ dominates $a$ and Proposition 2 applies. ∎

The choice $l = l^p$ in Corollary 3 yields the following.

**Corollary 4.** *If $1 \leqq p \leqq \infty$ and $\lambda$ is an echelon space of order $p$ then every absolutely $p$-summing map is absolutely $\lambda$-summing.*

*Remark.* We may consider Corollary 4 to be a partial extension of Corollary 1. For $p = 1$, Corollary 4 is less general than Corollary 1. Also for $p = \infty$ we obtain, from Corollary 4, that if $\lambda$ is an echelon space of order $\infty$ then each continuous linear map is absolutely $\lambda$-summing.

The last remark raises the question of which sequence spaces $\lambda$ have the property that each continuous linear map is absolutely $\lambda$-summing. We already know this to be the case for $\lambda = \phi, \omega, l^\infty$ and any echelon space of order $\infty$. We now give some results that provide supplementary information on that question.

**Proposition 3.** *Let $\lambda$ be a normal sequence space which is nuclear and complete in its normal topology. (In particular let $\lambda$ be a nuclear echelon space.) Then every continuous linear map is absolutely $\lambda$-summing.*

*Proof.* Under the given hypothesis we have ([5], p. 103) for any locally convex space $E$.

$$\lambda \otimes_\varepsilon E \equiv \lambda \otimes_\pi E$$

and this implies (see [8]) that $\lambda(E) \equiv \lambda[E]$ so that the identity map $I: E \to E$ is absolutely $\lambda$-summing. Hence if $T \in \mathscr{L}(E, F)$ then $T = TI$ and it is easy to see therefore that $T$ is absolutely $\lambda$-summing. ∎

Further examples of sequence spaces $\lambda$ such that each continuous linear map is absolutely $\lambda$-summing can be constructed using the following device.

Let $J_1$ and $J_2$ be two disjoint infinite subsequences of $\mathbb{N}$ whose union is $\mathbb{N}$. If $\lambda_1$ and $\lambda_2$ are two sequence spaces, define

$$\lambda = \lambda_1 \oplus \lambda_2 = \{x \in \omega : x_{J_i} \in \lambda_i, \quad i = 1, 2\}.$$

**Proposition 4.** *If $E$ and $F$ are normed spaces then $\Pi_{\lambda_1 \oplus \lambda_2}(E, F) = \Pi_{\lambda_1}(E, F) \cap \Pi_{\lambda_2}(E, F)$ where $\Pi_{\lambda_i}(E, F)$ is the set of all absolutely $\lambda_i$-summing (linear) maps on $E$ into $F$ with a similar interpretation for $\Pi_{\lambda_1 \oplus \lambda_2}$.*

*Proof.* Let $T \in \Pi_{\lambda_1 \oplus \lambda_2}(E, F)$. Let $(x_n)$ be a sequence in $E$ such that $(\langle x_n, a\rangle)_{n \in \mathbb{N}} \in \lambda_1$ for each $a \in E'$. Let $J_1 = (n_1, n_2, \ldots, n_k, \ldots)$. Define $(y_n)_{n \in \mathbb{N}} \subset E$

by $y_{n_k}=x_k$, $k=1,2,\ldots$ and $y_n=0$, $n \neq n_k$ for any $k$. Then, for each $a \in E'$, $(\langle y_n, a\rangle)_{n\in\mathbb{N}} \in \lambda$. Hence $(\|Ty_n\|)_{n\in\mathbb{N}} \in \lambda$ so that $(\|Tx_k\|)_{k\in\mathbb{N}} \in \lambda_1$ and $T$ is absolutely $\lambda_1$-summing. Thus $T \in \Pi_{\lambda_1}(E,F)$ and similarly $T \in \Pi_{\lambda_2}(E,F)$.

Conversely suppose $T \in \Pi_{\lambda_1}(E,F) \cap \Pi_{\lambda_2}(E,F)$ and $(x_n) \in \lambda(E)$. Then for each $a \in E'$,

$$(\langle x_n, a\rangle)_{n\in J_1} \in \lambda_1 \quad \text{and} \quad (\langle x_n, a\rangle)_{n\in J_2} \in \lambda_2$$

*so that*

$$(\|Tx_n\|)_{n\in J_1} \in \lambda_1 \quad \text{and} \quad (\|Tx_n\|)_{n\in J_2} \in \lambda_2$$

*and*

$$(\|Tx_n\|)_{n\in\mathbb{N}} \in \lambda . \qquad \blacksquare$$

**Corollary 5.** *If $\lambda_1$ and $\lambda_2$ have the property that each continuous map is absolutely $\lambda_1$ (respectively, $\lambda_2$)-summing then $\lambda_1 \oplus \lambda_2$ has the same property. In particular this is the case for $\phi \oplus \omega$ and $l^\infty \oplus \lambda$, $\lambda$ a nuclear echelon space.*

We can construct also many examples of spaces $\lambda$ for which not every continuous linear map is absolutely $\lambda$-summing. As is known ([4], p. 336) this is the case for $\lambda = l^p$, $p<\infty$; actually a more general result is true.

**Proposition 5.** *Let $\lambda$ be a normal sequence space. Let $l$ be any Banach sequence space with $\phi \subset l$, $\|e^i\| \geqq 1$ for each $i$ and $l' \subset l^\times$. Suppose there exists a sequence $\xi \notin \lambda$ such that $\xi \cdot l^\times \subset \lambda$. Then there exists a continuous linear map which is not absolutely $\lambda$-summing.*

*Proof.* The identity map $T$ on $l$ is linear and continuous. Define $(x_n)$ in $l$ by $x_n = \xi_n e^n$. Then if $a \in l' \subset l^\times$ we have $(\langle x_n, a\rangle)_n = \xi \cdot a \in \xi \cdot l^\times \subset \lambda$ but $\|Tx_n\| = |\xi_n| \cdot \|e^n\| \geqq |\xi_n|$. Hence $(\|Tx_n\|)_n \notin \lambda$. $\blacksquare$

**Corollary 6.** *Suppose $\lambda$ is a normal sequence space with $l^1 \subset \lambda \subset l^\infty$, $\lambda \neq l^\infty$. Then there exists a continuous linear map which is not absolutely $\lambda$-summing.*

*Proof.* Obviously $l=c_0$, $\xi = e$ satisfy the criteria of Proposition 5. $\blacksquare$

Our next proposition gives a sufficient condition on an echelon space $\lambda$ so that not each continuous linear map is absolutely $\lambda$-summing.

**Proposition 6.** *Let $l$ be a step such that $(l^\times[\mathfrak{T}_b(l)])' = l \subset c_0$. Let $\lambda$ be an echelon space of order $l$ but not a Montel space. Then the identity map $T: l^\times \to l^\infty$ is continuous but not absolutely $\lambda$-summing.*

*Proof.* Since $\lambda$ is not a Montel space then ([2], p. 190) it follows that there exists a $k$ and a subsequence $J$ of $\mathbb{N}$ such that $\lambda_J = \frac{1}{a_J^k} l_J$. If $\mu = a^k \lambda$ then it suffices to show that $T$ is not absolutely $\mu$-summing. We have $\mu_J = l_J$. Define $(x_n)$ in $l^\times[\mathfrak{T}_b(l)]$ by

$$x_n = \begin{cases} e^n, & n \in J \\ 0, & n \notin J \end{cases}$$

and consider any $a \in (l^\times)' = l$. Then $((\langle x_n, a\rangle)_n)_J = a_J \in l_J = \mu_J$. Hence there exists a $\xi \in \mu$ with $\xi_J = a_J$ and we have $|\langle x_n, a\rangle| \leqq |\xi_n|$ for all $n \in \mathbb{N}$. So $(\langle x_n, a\rangle)_{n\in\mathbb{N}} \in \mu$ since $\mu$ is normal. However the sequence $(\|Tx_n\|)_{n\in\mathbb{N}}$ is equal to 1 for each $n \in J$ and 0 otherwise and so it cannot be in $\mu$ because $\mu_J = l_J \subset c_0$. The continuity of $T$ is clear. ∎

**Corollary 7.** *Let $l$ be a step such that $(l^\times[\mathfrak{T}_b(l)])' = l \neq l^\infty$. Then the identity map $T: l^\times \to l^\infty$ is not absolutely $l$-summing.*

*Proof.* In the proof of Proposition 6 we can take $a^k = ke$ and $\lambda = \bigcap_k \frac{1}{a^k} l = l$ so that $J = N$ and the same argument works. ∎

*Remark.* The hypothesis of Proposition 6 cannot be relaxed to, say, asserting only that $l \subset l^\infty$ as in Corollary 7. Indeed consider the step

$$l = \{(x_n) : (x_{2n+1})_n \in l^\infty \quad \text{and} \quad (x_{2n})_n \in l^2\}$$

so that $(l^\times[\mathfrak{T}_b(l)])' = l \subset l^\infty$ but $l \not\subset c_0$.

Choose $a^k$ such that $a^k_{2n+1} = k$ and $(a^k_{2n})_n \in (a^{k+1}_{2n})_n\, l^1$, so we can write $\lambda = \lambda_1 \oplus \lambda_2$ where $\lambda_1 = l^\infty$ and $\lambda_2$ is a nuclear echelon space. $\lambda$ contains a subspace isomorphic to $l^\infty$ so it cannot be a Montel space. However, by Proposition 4 it follows that every continuous linear map is absolutely $\lambda$-summing.

**Proposition 7.** *If $\lambda$ is a normal sequence space and $\mathfrak{T}$ is its normal topology and if $\lambda(\mathfrak{T})$ is complete but not nuclear then the identity map on $l^1$ is not absolutely $\lambda$-summing.*

*Proof.* Since $\lambda$ is not nuclear, we have ([5], p. 103)

$$l^1 \otimes_\varepsilon \lambda[\mathfrak{T}] \supsetneqq l^1 \otimes_\pi \lambda[\mathfrak{T}]$$

and therefore $\lambda(l^1) \supsetneqq \lambda[l^1]$ and this implies that the identity map on $l^1$ is not absolutely $\lambda$-summing. ∎

## 5. Power Series Spaces

In this section we first generalize the notion of power series spaces $\Lambda$ and apply the results of the previous section to discuss absolutely $\Lambda$-summing maps.

*Definition 2.* Let $l$ be a step, $0 < \varrho_0 \leqq \infty$ and $\alpha$, an increasing unbounded sequence of non-negative reals. We define the *power series space of order $l$, type $\varrho_0$ and power $\alpha$* to be the echelon space

$$\Lambda(\varrho_0, \alpha, l) = \Lambda(a, l)$$

where $a^k_n = \varrho_k^{\alpha_n}$ and $(\varrho_k)$ is any increasing sequence which converges to $\varrho_0$. (Clearly $\Lambda$ is independent of the choice of $(\varrho_k)$.)

*Remark.* If $0 < \varrho_1 < \infty$ then $\Pi_{\Lambda(\varrho_0,\alpha,l)} = \Pi_{\Lambda(\varrho_1\varrho_0,\alpha,l)}$. In particular we need consider only two cases, $\varrho_0 = 1$ and $\varrho_0 = \infty$ in which case we say that $\Lambda$ is of *finite*, respectively *infinite*, *type*. Indeed, for $\varrho_0 = \infty$ there is nothing to prove and for $\varrho_0 < \infty$ it follows that $\Lambda\ (\varrho_1\, \varrho_0,\ \alpha, l)$ is a diagonal transformation of $\Lambda\ (\varrho_0,\ \alpha, l)$ via the sequence $(\varrho_1^{\alpha_n})$ and therefore the spaces of absolutely summing maps agree.

**Proposition 8.** *If* $\alpha \in \beta \cdot l^\infty$ *then each absolutely* $\Lambda\ (\varrho_0,\ \alpha, l)$*-summing map is absolutely* $\Lambda\ (\varrho_0,\ \beta, l)$*-summing.*

*Proof.* We apply Proposition 2 with $a_n^k = \varrho_k^{\alpha_n}$, $b_n^k = \varrho_k^{\beta_n}$. We have $M > 0$ such that $\alpha \leqq M\beta$.

*Case 1.* $\varrho_0 = \infty$. We take $\varphi_k = k$. Given $j_0$ take $k_0 = 1$ and given $k \geqq 1$, take $j = k^M j_0$. Then

$$a_n^k\, b_n^j = k^{\alpha_n}\, j_0^{\beta_n} \leqq k^{M\beta_n}\, j_0^{\beta_n} = (k^M j_0)^{\beta_n} = k_0^{\alpha_n}\, j^{\beta_n} = a_n^{k_0}\, b_n^j,$$

so $b$ dominates $a$ and the result follows.

*Case 2.* $\varrho_0 = 1$. We take $\varrho_k = \dfrac{k}{k+1}$ for $\Lambda\ (1,\ \alpha, l)$ and $\varrho_k = \left(\dfrac{k}{k+1}\right)^M$ for $\Lambda\ (1,\ \beta, l)$.

Given $j_0$, take $k_0 = j_0$. Given $k \geqq k_0$ take $j = k$. Then

$$\frac{k\,(k_0+1)}{(k+1)k_0} = \frac{j\,(j_0+1)}{(j+1)j_0} \geqq 1$$

and therefore

$$\left(\frac{k(k_0+1)}{(k+1)k_0}\right)^{\alpha_n} \leqq \left(\frac{j(j_0+1)}{(j+1)j_0}\right)^{M\beta_n}.$$

Thus

$$\frac{a_n^k}{a_n^{k_0}} \leqq \frac{b_n^j}{b_n^{j_0}}$$

so $b$ dominates $a$ and the result follows. ∎

*Remark.* In Proposition 8, the containment need not be an equality; of course it is if both spaces are nuclear, but if $l = l^1$ and $\Lambda\ (\varrho_0, \alpha, l)$ is not nuclear but $\Lambda\ (\varrho_0, \beta, l)$ is nuclear then by Propositions 3 and 7 equality does not hold.

**Proposition 9.** *For arbitrary* $\alpha$ *and* $l$ *we have that each absolutely* $\Lambda\,(1, \alpha, l)$*-summing map is absolutely* $\Lambda\,(\infty, \alpha, l)$*-summing.*

*Proof.* We take, as before, $\varrho_k = k$ for $\Lambda\,(\infty, \alpha, l)$ and $\varrho_k = \dfrac{k}{k+1}$ for $\Lambda\,(1, \alpha, l)$. Then we apply Proposition 2. For any $j_0$, take $k_0 = 1$ and for any $k \geqq 1$ we take $j = 2j_0$. Then

$$\frac{a_n^k}{a_n^{k_0}} = \left(\frac{k(k_0+1)}{(k+1)k_0}\right)^{\alpha_n} = \left(\frac{k}{k+1}\right)^{\alpha_n} \cdot 2^{\alpha_n} \leqq 2^{\alpha_n} = \frac{j^{\alpha_n}}{j_0^{\alpha_n}} = \frac{b_n^j}{b_n^{j_0}}.$$ ∎

*Remark.* In the above proposition the containment need not be an equality. Indeed if we take $l=l^1$ and $\alpha_n=\log(n+1)$ then $\Lambda(\infty,\alpha,l)$ is nuclear but $\Lambda(1,\alpha,l)$ is not. So, again, by Propositions 3 and 7 the equality does not hold.

**Proposition 10.** *Let $\alpha$, $\beta$ be increasing unbounded sequences of non-negative numbers. Then the following statements are equivalent.*

i) $\alpha\in\beta\cdot l^\infty$,

ii) $\Lambda(\infty,\alpha,l)\supset\Lambda(\infty,\beta,l)$,

iii) $\Lambda(1,\alpha,l)\subset\Lambda(1,\beta,l)$.

*Proof.* i)$\Rightarrow$ii): Take $a_n^k=k^{\alpha_n}$ and $b_n^k=k^{\beta_n}$ and we have $\alpha_n\leqq M\beta_n$, so $a_n^k\leqq b_n^{k^M}$. *Hence*

$$\Lambda(\infty,\alpha,l)=\bigcap_{k=1}^{\infty}\frac{1}{a^k}\,l\supset\bigcap_{k=1}^{\infty}\frac{1}{b^{(k^M)}}\,l=\bigcap_{k=1}^{\infty}\frac{1}{b^k}\,l=\Lambda(\infty,\beta,l).$$

i)$\Rightarrow$iii): we have $\alpha_n\leqq M\beta_n$ and we take $a_n^k=\left(\frac{k}{k+1}\right)^{\alpha_n}$, $b_n^k=\left(\frac{k}{k+1}\right)^{\beta_n}$. Since $\left(\frac{k}{k+1}\right)^{\alpha_n}\geqq\left(\frac{k}{k+1}\right)^{M\beta_n}$, if we define $\bar{b}^k$ by $\bar{b}_n^k=\left(\frac{k}{k+1}\right)^{M\beta_n}$ it follows from the fact that $\left(\frac{k}{k+1}\right)^{M}$ increases with $k$ to 1 that

$$\Lambda(1,\alpha,l)=\bigcap_{k=1}^{\infty}\frac{1}{a^k}\,l\subset\bigcap_{k=1}^{\infty}\frac{1}{\bar{b}^k}\,l=\bigcap_{k=1}^{\infty}\frac{1}{b^k}\,l=\Lambda(1,\beta,l).$$

ii)$\Rightarrow$i): Let $a_n^k=k^{\alpha_n}$ and $b_n^k=k^{\beta_n}$; also by assumption, $\bigcap_k\frac{1}{a^k}\,l\supset\bigcap_k\frac{1}{b^k}\,l$. This implies that $e\in D(\Lambda(\infty,\beta,l),\Lambda(\infty,\alpha,l))$ and then, by Lemma 2, there exists $j$ such that $a^2\in b^j\,l^\infty$; thus we have $M>0$ with $2^{\alpha_n}=a_n^2\leqq M b_n^j=M j^{\beta_n}$ for each $n$. Thus if we choose $c>0$ such that $\log M\leqq c\beta_n$ for each $n$ (since $\lim\beta_n=\infty$), then $\alpha_n\log 2\leqq(\log j+c)\,\beta_n$ and $\alpha\in\beta\cdot l^\infty$.

iii)$\Rightarrow$i): Let $a_n^k=\left(\frac{k}{k+1}\right)^{\alpha_n}$ and $b_n^k=\left(\frac{k}{k+1}\right)^{\beta_n}$ and $\bigcap_k\frac{1}{a^k}\,l\subset\bigcap_k\frac{1}{b^k}\,l$.

As above, we conclude that there exists a $k$ with $b^1\in a^k\,l^\infty$ and so we have $M>0$ with

$$M\left(\frac{k}{k+1}\right)^{\alpha_n}=M\,a_n^k\geqq b_n^1=\left(\frac{1}{2}\right)^{\beta_n}.$$

Now choose $c>0$ such that

$$\frac{\log M}{-\log\left(\frac{k}{k+1}\right)}\leqq c\beta_n\ \text{for all } n$$

and we have

$$\alpha_n \log\left(\frac{k}{k+1}\right) + \log M \geqq -\beta_n \log 2 .$$

Therefore

$$\alpha_n \leqq \frac{\log 2}{\log\left(\frac{k}{k+1}\right)} \beta_n + \frac{\log M}{-\log\left(\frac{k}{k+1}\right)} \leqq \left( \frac{\log 2}{\log\left(\frac{k}{k+1}\right)} + c \right) \beta_n$$

and so $\alpha \in \beta \cdot l^\infty$. ∎

*Remark.* Referring now back to Definition 1 we see that, in view of Proposition 10 and the proof of Proposition 8, the condition "$b$ dominates $a$" is not related by implication to the condition $\lambda \subset \mu$.

Thus we may summarize our results by saying that if one is considering "absolute $p$-summability" or "absolute finite-type-power-series-summability" then $\lambda \subset \mu$ implies that each absolutely $\lambda$-summing map is absolutely $\mu$-summing. However, in view of Propositions 8 and 10 and the remark after Proposition 8 we have that this is not so for absolute summability corresponding to infinite type power series spaces, but instead $\lambda \subset \mu$ in this case implies that every absolutely $\mu$-summing map is absolutely $\lambda$-summing. Moreover we have that if $\lambda$ is a power series space of infinite type and $\mu$ is a power series space of finite type then in view of Proposition 9 and the remark following it, $\lambda \subset \mu$ and every absolutely $\mu$-summing map is absolutely $\lambda$-summing but not conversely.

## 6. Discontinuous Absolutely Summing Maps

We examine in this section whether there are discontinuous (linear maps) that are absolutely $\lambda$-summing and our result in Proposition 11 shows that for a normal sequence space $\lambda$ either each absolutely $\lambda$-summing map is continuous or else every linear map is absolutely $\lambda$-summing.

We start with the following lemma.

**Lemma 3.**[2] *Let $E$ be a Banach space and $(x_n)$ a sequence of non-zero elements of $E$. Then there exists an $a \in E'$ such that for each $n$, $\langle x_n, a\rangle \neq 0$.*

*Proof.* For each $n$, let $A_n = \{a \in E' : \langle x_n, a\rangle \neq 0\}$. Since the map $a \to \langle x_n, a\rangle$ is continuous, linear and non-zero on the Banach space $E'$ it follows that each $A_n$ is dense and open; so by the Baire category theorem there exists an element $a \in \bigcap_n A_n$ and this element obviously annihilates no $x_n$. ∎

**Proposition 11.** *If $\lambda$ is a normal sequence space then the following are equivalent:*

[2] This short proof was given by G. Maltese in a private conversation.

i) *every absolutely $\lambda$-summing map is continuous;*

ii) *on each infinite dimensional Banach space $E$ there exists a linear map $T$ which is not absolutely $\lambda$-summing;*

iii) *there exists a Banach space $E$ and a linear map $T$ on $E$ which is not absolutely $\lambda$-summing;*

iv) *there exists an infinite subsequence $J$ of $\mathbb{N}$ such that $\lambda$ has a sequence which does not vanish on $J$ but $\lambda_J \neq \omega$;*

v) *there exists $\xi \in \lambda$ and an infinite subsequence $J$ of $\mathbb{N}$ and $\eta \notin \lambda$ such that $\xi_n > 0$ for each $n \in J$ and $\eta_n = 0$ for each $n \notin J$.*

*Proof.* i)$\Rightarrow$ii): if ii) is false then there exists an infinite dimensional Banach space $E$ on which each linear map is absolutely $\lambda$-summing and hence a discontinuous linear map which is absolutely $\lambda$-summing so that i) is false.

ii)$\Rightarrow$iii) is obvious.

iii)$\Rightarrow$iv): suppose iv) is false.

Then for each infinite subsequence $J$ of $\mathbb{N}$, if $\lambda$ has a sequence which does not vanish on $J$ then $\lambda_J = \omega$. Take any Banach space $E$ and any linear map $T$ on $E$. We shall show that $T$ is absolutely $\lambda$-summing, thus iii) is false.

Indeed let $(x_n)$ be a sequence in $E$ such that $(\langle x_n, a\rangle)_n \in \lambda$ for each $a \in E'$. Let $J = \{n: x_n \neq 0\}$. If $J$ is finite, then since $T$ is linear, $(\|Tx_n\|)_n \in \phi \subset \lambda$; if $J$ is infinite, then by Lemma 3, we have $a \in E'$ with $\langle x_n, a\rangle \neq 0$ for all $n \in J$. Let $\xi = (\langle x_n, a\rangle)_{n\in\mathbb{N}} \in \lambda$.

Then $\xi$ does not vanish on $J$ and therefore $\lambda_J = \omega$; since $(\|Tx_n\|)_{n\in J} \in \omega$, there exists $\eta \in \lambda$ such that for $n \in J$, $\eta_n = \|Tx_n\|$ and since $\|Tx_n\| = 0$ for $n \notin J$ and since $\lambda$ is normal it follows that $(\|Tx_n\|)_{n\in\mathbb{N}} \in \lambda$. Thus $T$ is absolutely $\lambda$-summing.

iv)$\Rightarrow$v): We have $J$ and $\xi \in \lambda$ with $\xi_n \neq 0$ for all $n \in J$; since $\lambda$ is normal we may assume that $\xi_n > 0$ for all $n \in J$. Moreover since $\lambda_J \neq \omega$ we can find a sequence $(\bar\eta_n)_{n\in J} \in \omega \setminus \lambda_J$. Define $(\eta_n)_{n\in\mathbb{N}}$ by

$$\eta_n = \begin{cases} \bar\eta_n, & n \in J, \\ 0, & n \notin J. \end{cases}$$

Then $(\eta_n)_{n\in\mathbb{N}}$ is not in $\lambda$ for if it were, then $(\bar\eta_n)_{n\in J}$ would be in $\lambda_J$ which it is not; clearly $(\eta_n)_{n\in\mathbb{N}}$ satisfies v).

v)$\Rightarrow$i): Let $\xi$, $J$ and $\eta$ be as in v). Define $\bar\eta$ by

$$\eta_n = \begin{cases} \max\limits_{j\leq n} |\eta_j|, & n \in J \\ 0, & n \notin J \end{cases}$$

Then since $\lambda$ is normal, $\bar\eta \notin \lambda$ and for $n, k \in J$, $n < k$, we have $\bar\eta_n \leqq \bar\eta_k$. Now suppose that $T$ is a linear map which is absolutely $\lambda$-summing but not continuous. Then we have a sequence $(y_i)$ with

$$\lim_i \|y_i\| = 0, \quad \lim_i \|Ty_i\| = \infty;$$

so by passing to a subsequence we may assume $\|Ty_i\| \geqq \bar{\eta}_i$. Choose a subsequence $(i_j)_{j\in J}$, such that $\|y_{i_j}\| \leqq \xi_j$, $j \in J$. Then we have $\|Ty_{i_j}\| \geqq \bar{\eta}_{i_j} \geqq \bar{\eta}_j$. Define $(x_n)$ by

$$x_n = \begin{cases} y_{i_j}, & \text{for } n = i_j \text{ for some } j \in J, \\ 0, & \text{otherwise}. \end{cases}$$

Then $(\|x_n\|)_{n\in\mathbb{N}}$ is in $\lambda$, so for each $a \in E'$, $(\langle x_n, a\rangle)_n \in \lambda$ but $(\|Tx_n\|)_{n\in\mathbb{N}} \notin \lambda$. ∎

*Remark.* From the above proposition it follows that given a normal sequence space $\lambda$ then either every absolutely $\lambda$-summing map is continuous or every linear map is absolutely $\lambda$-summing and iv) and v) give precise conditions which determine which of the alternatives holds.

## *Examples*

6. If $\lambda$ is an echelon space then every absolutely $\lambda$-summing map is continuous.

Indeed, in this case, we take $J = \mathbb{N}$ and $\xi_n = \dfrac{1}{a_n^n}$.

7. If $l^1 \subset \lambda \subset l^\infty$ then every absolutely $\lambda$-summing map is continuous.

In this case we can take $J = \mathbb{N}$ and $\xi_n = \dfrac{1}{n^2}$.

8. If $\lambda = \phi, \omega$ then every linear map is absolutely $\lambda$-summing.

9. If $\lambda$ consists of elements which are infinite matrices each row of which is an element of $\omega$ but only finitely many rows are different from zero then every linear map is absolutely $\lambda$-summing.

Indeed if $J$ is as stated in Proposition 11, iv) then $J$ meets only finitely many rows, so $\lambda_J = \omega$.

10. If $\lambda$ consists of elements which are infinite matrices each row of which is in $\phi$ then every linear map is absolutely $\lambda$-summing. In fact, if $J$ is as in iv) of Proposition 11 then for each $k$, $\sup\{n : (k, n) \in J\} < \infty$. It follows then that $\lambda_J = \omega$.

*Remark.* By taking $\lambda = \omega$, $\mu = \phi$ we see that $\lambda$ and $\mu$ are perfect and every absolutely $\lambda$-summing map is absolutely $\mu$-summing (Example 8); however, by Example 4 the hypothesis of our theorem is not satisfied. Thus the converse of the theorem does not hold, even for perfect spaces.

## References


1. Crofts, G.: Concerning perfect Fréchet spaces and diagonal transformations. Math. Ann. **182**, 67—76 (1969).
2. Dubinsky, E.: Perfect Fréchet spaces. Math. Ann. **174**, 186—194 (1967).
3. Köthe, G.: Topological vector spaces, I. Berlin-Heidelberg-New York: Springer 1969.
4. Pietsch, A.: Absolut $p$-summierende Abbildungen in normierten Räumen. Studia Math. **28**, 333—353 (1967).
5. — Nukleare lokalkonvexe Räume. Berlin: Akademie-Verlag 1965.

6. Ramanujan, M. S.: Absolutely $\lambda$-summing operators, $\lambda$ a symmetric sequence space. Math. Z. **114**, 187—193 (1970).
7. — Power series spaces $\Lambda(\alpha)$ and associated $\Lambda(\alpha)$-nuclearity. Math. Ann. **189**, 161—168 (1970).
8. Rosier, R. C.: Generalized sequence spaces. Ph. D. Thesis, University of Maryland 1970.

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