Math. Ann. 225, 173—176 (1977)

Real Quadratic Fields with Large Class Number

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Let h, d, ε, and R be the class number, discriminant, fundamental unit, and regulator, respectively, of the real quadratic field $\mathbb{Q}(\sqrt{d})$. Let χ be the primitive quadratic character $(\bmod\, d)$, and let

$$L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)n^{-s}.$$

Then $R=\log\varepsilon$, and

$$L(1,\chi)=hRd^{-\frac{1}{2}}. \tag{1}$$

Since $L(1,\chi)\ll\log d$, $R>(\frac{1}{2}+o(1))\log d$, it follows that $h\ll\sqrt{d}$. Moreover, Littlewood [4] showed that if all nontrivial zeros of $L(s,\chi)$ lie on the critical line $\operatorname{Re}s=\frac{1}{2}$, then $L(1,\chi)<(2e^{\gamma}+o(1))\log\log d$. Hence

$$h<(4e^{\gamma}+o(1))d^{\frac{1}{2}}(\log d)^{-1}\log\log d \tag{2}$$

assuming the Generalized Riemann Hypothesis. In this paper we show that the hypothetical estimate (2) can not be improved upon, apart from the value of the constant.

Theorem. *There is an absolute constant* $c>0$ *such that*

$$h>cd^{\frac{1}{2}}(\log d)^{-1}\log\log d \tag{3}$$

for infinitely many real quadratic fields $\mathbb{Q}(\sqrt{d})$.

To prove the Theorem we construct d for which $R<\log d$, and $L(1,\chi)>c\log\log d$. Then (3) follows from (1). We consider only square-free d with $d\equiv 1\,(\bmod 4)$, $d=n^2+1$. Then $\varepsilon=n+\sqrt{d}<2\sqrt{d}<d$, and so $R<\log d$. To make $L(1,\chi)$ large we wish to have $\chi(p)=1$ for many small primes p. By quadratic reciprocity this amounts to having d lie in certain arithmetic progressions. Following an argument of Estermann [3], we show in Lemma 1 that such d exist. Then in Lemma 2 we relate $L(1,\chi)$ to $\chi(p)$ for small p. With these lemmas established, it is then a simple matter to complete the proof of the Theorem.

Lemma 1. *Let $D(x;q,a)$ denote the number of $d\leqq x$ such that d is square-free, $d=n^2+1$ for some integer n, and $n\equiv a\,(\bmod q)$. Suppose that $2|q$, and that $(a^2+1,q)=1$. Then*

$$D(x;q,a)=\frac{x^{\frac{1}{2}}}{q}\prod_{p\nmid q}(1-2p^{-2})+O(x^{\frac{1}{3}}\log x).$$

Proof. Clearly

$$\begin{aligned}D(x;q,a)&=\sum_{\substack{n\leqq(x-1)^{1/2}\\ n\equiv a(\bmod q)}}\ \sum_{r^2|(n^2+1)}\mu(r)\\ &=\sum_{r\leqq x}\mu(r)\sum_{\substack{n\leqq(x-1)^{1/2}\\ n\equiv a(\bmod q)\\ r^2|(n^2+1)}}1.\end{aligned}$$

If $(q,r)>1$ then the inner sum vanishes, since $(n^2+1,q)=1$ for $n\equiv a(\bmod q)$. Thus we may suppose that $(q,r)=1$. We consider $r\leqq y$, $y<r\leqq x$ separately. Writing $n^2+1=r^2s$, we see that

$$\sum_{y<r\leqq x}\mu(r)\sum_{\substack{n\leqq(x-1)^{1/2}\\ n\equiv a(\bmod q)\\ r^2|(n^2+1)}}1\ll\sum_{s\leqq xy^{-2}}\ \sum_{\substack{n,r\\ r^2s=n^2+1}}1.$$

From the theory of Pell's equation, the number of pairs u,v for which $u^2-sv^2=-1$, $1\leqq u\leqq U$, is $\ll\log U$, uniformly in s. Thus the inner sum above is $\ll\log x$, and so the contribution of $r>y$ is $\ll xy^{-2}\log x$. Thus

$$D(x;q,a)=\sum_{\substack{r\leqq y\\ (q,r)=1}}\mu(r)\sum_{\substack{n\leqq(x-1)^{1/2}\\ n\equiv a(\bmod q)\\ r^2|(n^2+1)}}1+O(xy^{-2}\log x).$$

For odd m the number of solutions n $(\bmod m)$ of the congruence

$$n^2+1\equiv 0\,(\bmod m)\quad\text{is}\quad\prod_{p|m}\left(1+\left(\frac{-1}{p}\right)\right)=c(m),\quad\text{say.}$$

But $2|q$ and $(q,r)=1$, so r is odd, and so the number of $n\,(\bmod qr^2)$ for which $r^2|(n^2+1)$, $n\equiv a\,(\bmod q)$, is $c(r^2)=c(r)$. Hence the inner sum above is $=c(r)(x^{\frac{1}{2}}q^{-1}r^{-1}+O(1))$. Now $c(r)\leqq d(r)$ so

$$\sum_{r\leqq y}c(r)\ll y\log y,\quad\text{and}\quad\sum_{r>y}c(r)r^{-2}\ll y^{-1}\log y.$$

Therefore

$$\begin{aligned}D(x;q,a)=\frac{x^{\frac{1}{2}}}{q}&\sum_{\substack{r=1\\ (q,r)=1}}^{\infty}\mu(r)c(r)r^{-2}+O(y\log y)+O(x^{\frac{1}{2}}q^{-1}y^{-1}\log y)\\ &+O(xy^{-2}\log x).\end{aligned}$$

Taking $y=x^{\frac{1}{3}}$, we obtain the result, since the sum over r is

$$=\prod_{p\nmid q}\left(1-\left(1+\left(\frac{-1}{p}\right)\right)p^{-2}\right).$$

Lemma 2. *Suppose that* $0<\delta<1$. *Then for* $(\log q)^\delta \leqq y \leqq \log q$, *and* χ *a primitive character* $(\mathrm{mod}\, q)$, $q>1$,

$$\log L(1,\chi)=\sum_{p\leqq y}\chi(p)p^{-1}+O_\delta(1)$$

unless χ *lies in an exceptional set* $\mathfrak{E}(\delta)$. *The set* $\mathfrak{E}(\delta)$ *contains* $\ll Q^\delta$ *primitive characters* χ *with conductor* $q\leqq Q$.

A more precise result of this sort has been given by Elliott [2]; for the sake of completeness we include a short proof of Lemma 2.

Proof. Clearly

$$\sum_{n\leqq x}\chi(n)\Lambda(n)(n\log n)^{-1}=\frac{1}{2\pi i}\int \log L(s,\chi)\frac{x^{s-1}}{s-1}ds,$$

where the contour is the straight line from $c-i\infty$ to $c+i\infty$, $c>1$. Let $\mathfrak{E}$ be the set of primitive characters χ for which $L(s,\chi)$ has at least one zero in the rectangle

$$1-\tfrac{1}{7}\delta\leqq\sigma\leqq 1,\qquad |t|\leqq(\log q)^2. \tag{4}$$

Suppose that $\chi\notin\mathfrak{E}$. Arguing in the usual manner (see Titchmarsh [8, Lemma 3.12]), we see that the portion of the above integral for which $|t|\geqq\log q$ contributes $\ll 1$, uniformly for $x\leqq q$. For $|t|\leqq\log q$ we take the contour to the abcissa $\sigma=1-\frac{1}{8}\delta$, passing the pole at $s=1$ (with residue $\log L(1,\chi)$). We may neglect higher powers of primes with error $\ll 1$, so

$$\sum_{p\leqq x}\chi(p)p^{-1}-\log L(1,\chi)\ll 1+\int_{-\log q}^{\log q}|\log L(1-\tfrac{1}{8}\delta+it,\chi)|\frac{dt}{\delta+|t|}.$$

But $L(s,\chi)\neq 0$ for s in the rectangle (4), which implies that $\log L(s,\chi)\ll_\delta \log q$ in the integrand above. Hence the above is $\ll 1+x^{-\frac{1}{8}\delta}(\log q)^2$. Taking $x=(\log q)^{16\delta^{-1}}$, we find that

$$\log L(1,\chi)-\sum_{p\leqq y}\chi(p)p^{-1}\ll_\delta 1+\sum_{y<p\leqq x}p^{-1}\ll_\delta 1,$$

since $y\geqq(\log q)^\delta$.

It remains now to estimate the number of characters lying in the set $\mathfrak{E}=\mathfrak{E}(\delta)$. Let $N(\sigma,T,\chi)$ denote the number of zeros $\varrho=\beta+i\gamma$ of $L(s,\chi)$ in the rectangle $\sigma\leqq\beta\leqq 1$, $|\gamma|\leqq T$. From a theorem of Montgomery [6] (see also [7, Theorem 12.2] or [1, Théorème 20]),

$$\sum_{q\leqq Q}\sum_{\chi}{}^{*}N(\sigma,T,\chi)\ll(Q^2T)^{3(1-\sigma)}(\log QT)^9.$$

Here $\sum_\chi^{*}$ denotes a sum over all primitive characters $\chi\,(\mathrm{mod}\, q)$. Thus the number of zeros in question is $\ll Q^\delta$, so Lemma 2 is established.

We now complete the proof of the Theorem. Let $y=\frac{1}{9}\log x$, $q=2\prod_{p\leqq y}p$, $a=0$. Then $q<x^{\frac{1}{8}}$, $2|q$, $(a^2+1,q)=1$, and if $d=n^2+1$, $n\equiv a\,(\mathrm{mod}\, q)$, then $\chi(p)=\left(\frac{d}{p}\right)=1$

for all $p \leqq y$. By Lemma 1 there are $\gg x^{\frac{1}{2}} q^{-1} \gg x^{\frac{3}{8}}$ such square-free $d \leqq x$. From Lemma 2, with $\delta < \frac{3}{8}$, we see that $L(1, \chi) > c_1 \log y > c \log \log d$ for almost all of these d. This completes the proof of the Theorem.

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Received July 6, 1976