

The distribution of bidegrees of smooth surfaces in $\text{Gr}(1, \mathbf{P}^3)$

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0 Introduction

The study of algebraic surfaces in $\text{Gr}(1, \mathbf{P}^3)$, the Grassmann variety parametrising lines in projective three-space, was a popular one for algebraic geometers of the late nineteenth and early twentieth centuries. Calling them line congruences, researchers such as Kummer, Fano, Roth and many others published many papers on the topic, classifying congruences and studying their invariants. Since that time, the field has lain dormant until very recently.

The classical geometers identified two numbers associated with a given congruence: the *order* and the *class*. Thinking of a congruence as a two dimensional family of lines, the order is the number of lines in the family passing through a general point in \mathbf{P}^3 , and the class the number of lines in the family contained in a general plane. Together, these two numbers make up the *bidegree* of the congruence. In modern terms, the bidegree gives the class of the congruence in the Chow ring of $\text{Gr}(1, \mathbf{P}^3)$.

In this paper, we consider the question: “for what values of a and b does there exist (or not exist) a smooth congruence of bidegree (a, b) ?” In particular, we try to find restrictions on the bidegree, using an approach suggested by Dolgachev and Reider in [8]. This approach is to study the restriction of the universal bundle \mathcal{E} of $\text{Gr}(1, \mathbf{P}^3)$, which appears in the exact sequence

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{O}_{\text{Gr}(1, \mathbf{P}^3)}^4 \rightarrow \mathcal{E}' \rightarrow 0,$$

to a surface $Y \subseteq \text{Gr}(1, \mathbf{P}^3)$. If $\mathcal{E}|_Y$ is semistable, we find by Bogomolov’s theorem that $c_1^2(\mathcal{E}|_Y) \leq 4c_2(\mathcal{E}|_Y)$ implies $a \leq 3b$, where (a, b) is the bidegree of Y .

We are unable to prove that \mathcal{E} restricted to any surface is semistable. Instead, we show that the hypothesis that $\mathcal{E}|_Y$ is unstable leads to a strong bound on the hyperplane section genus of Y , which in turn leads to a bound on a versus b . To summarize our results, we have

Theorem. *If $Y \subseteq \text{Gr}(1, \mathbf{P}^3)$ is a smooth surface of bidegree (a, b) , then*

- a) *if Y is not of general type, then $a \leq 3b$;*
- b) *if Y is of general type and $b \leq 19$, then $a \leq 3b$; in general $a \leq \mathcal{O}(\ell^{4/3})$.*

While these results are not as strong as one could hope for, it is a start. $a \leq 3b$ is certainly the best linear bound one can achieve, as there exist surfaces of bidegree $(3, 1)$ and $(6, 2)$. However, the asymptotically best examples for producing surfaces for which a differs widely from b (Remark 3.17) yield a sequence of surfaces such that a/b converges to 1. Note that complete intersections have $a = b$; surfaces with $|a - b|$ large compared to $a + b$ can be considered unusual.

These results immediately extend to give information about codimension 2 subvarieties of higher dimensional Grassmannians, augmenting results of [23] and [17] (Proposition 1.4).

In Sect. 1, we give standard preliminary notation and results on $\text{Gr}(1, \mathbf{P}^3)$.

In Sect. 2, we study surfaces $Y \subseteq \text{Gr}(1, \mathbf{P}^3)$ for which there is a curve $C \subseteq \mathbf{P}^3$ such that every line of Y passes through C . This is an important special case needed for our general results. We find

Theorem. *If $Y \subseteq \text{Gr}(1, \mathbf{P}^3)$ is a smooth surface of bidegree (a, b) all of whose lines pass through a curve $C \subseteq \mathbf{P}^3$, Y not contained in a hyperplane section of $\text{Gr}(1, \mathbf{P}^3)$, then either*

- a) *Y is given as the set of bisecants of a twisted cubic or an elliptic quartic in \mathbf{P}^3 , of bidegree $(1, 3)$ or $(2, 6)$ respectively; or*
- b) *C is a non-singular plane curve, and $a \leq b$.*

This is an extension of an unpublished result of Cossec, Dolgachev and Verra, [7].

1 Preliminaries

We will work over an algebraically closed ground field k , $\text{char } k = 0$ (we will need this especially to apply Bogomolov's theorem). Let $Q = \text{Gr}(1, \mathbf{P}^3)$, the Grassmannian of lines in projective three-space over k . We identify Q with the image of its Plücker embedding as a hyperquadric in \mathbf{P}^5 . Given a point $p \in Q$, we denote by l_p the corresponding line in \mathbf{P}^3 .

We state some well known facts about Q . (See, for example, [13, Chap. 1, Sect. 5 and Chap. 6, Sect. 1].) The Chow ring of Q is $A^1(Q) = \mathbf{Z}$, $A^2(Q) = \mathbf{Z} \oplus \mathbf{Z}$, $A^3(Q) = \mathbf{Z}$, $A^4(Q) = \mathbf{Z}$, generated by the Schubert cycles. The generators are: Z for $A^1(Q)$, representing a hyperplane section of Q . The Schubert cycle representing it is the locus of points in Q corresponding to the set of lines intersecting a fixed line in \mathbf{P}^3 . This corresponds to a singular hyperplane section of Q . If l is a fixed line of \mathbf{P}^3 , we denote by $Z(l)$ the corresponding Schubert cycle. A three-fold contained in Q is called a *complex*, and if its class is nZ in $A^1(Q)$, we call it a complex of degree n . A complex is always the complete intersection of Q and a four-fold of degree n in \mathbf{P}^5 .

The generators of $A^2(Q)$ are denoted by η and η' , and are the Schubert cycles consisting of lines through a given point of \mathbf{P}^3 , and lines contained in a given plane of \mathbf{P}^3 respectively. Both these cycles are isomorphic to \mathbf{P}^2 . Denote, for a given point $P \in \mathbf{P}^3$, the locus in Q of lines through P by $\eta(P)$, and for $H \subseteq \mathbf{P}^3$ a plane, the locus in Q of lines contained in H by $\eta'(H)$.

The generator of $A^3(Q)$ is denoted by l , and is represented by a pencil of lines in \mathbf{P}^3 . The generator of $A^4(Q)$ is a point in Q , denoted pt .

From this description, it is easy to see what the intersection pairing on $A^*(Q)$ is:

$$\begin{aligned} Z^2 &= \eta + \eta', & Z \cdot \eta &= Z \cdot \eta' = l, & Z \cdot l &= pt \\ \eta \cdot \eta &= \eta' \cdot \eta' = pt, & \eta \cdot \eta' &= 0. \end{aligned}$$

An arbitrary surface $Y \subseteq Q$ is represented by a cycle in $A^2(Q)$ which can be written as $a\eta + b\eta'$. In classical terminology, Y is called a *line congruence*, and a *ray* of the congruence is a line l_p in \mathbf{P}^3 with $p \in Y$. The pair (a, b) of integers is called the *bidegree* of Y . By the above pairing, $(a\eta + b\eta') \cdot \eta = a$, and $(a\eta + b\eta') \cdot \eta' = b$, so that a is the number of rays of Y through a general point in \mathbf{P}^3 , and b is the number of rays contained in a general plane of \mathbf{P}^3 .

Example 1.1. The locus of points in Q corresponding to lines intersecting two given, disjoint lines in \mathbf{P}^3 is a congruence of bidegree $(1, 1)$. The number of rays through a general point of \mathbf{P}^3 can be seen to be 1, by projecting the two given lines from the point. The ray in a general plane is the one connecting the two points of intersection of the two lines with the plane. It is easy to see that this locus is smooth, isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$.

This surface is an example of a complete intersection, being the intersection of two linear complexes. More generally, an intersection of a complex of degree n and a complex of degree m gives a congruence of bidegree (nm, nm) . Thus complete intersections lie along the diagonal of the (a, b) plane, and we would expect more special types of surfaces to lie off, but not too far from, the diagonal.

Example 1.2. The locus of points in Q corresponding to bisecants of a twisted cubic curve C in \mathbf{P}^3 forms a congruence of bidegree $(1, 3)$ and the locus of points in Q corresponding to bisecants of an elliptic quartic curve in \mathbf{P}^3 forms a congruence of bidegree $(2, 6)$, as can be checked as above. These are also examples of smooth congruences (see [3]).

By surface, we will mean, unless otherwise specified, a smooth, irreducible surface. We have the following standard proposition.

Proposition 1.3. *For a smooth surface $Y \subseteq Q$ with hyperplane section H , canonical divisor K , arithmetic genus p_a , hyperplane section genus π , second Chern class c_2 , and bidegree (a, b) , the following formula holds:*

$$a^2 + b^2 - 7(a + b) - K^2 - 4H \cdot K + c_2 = 0.$$

Proof. See, for example, [18]. \square

We prove the next result in order to demonstrate the significance of our results to higher dimensional Grassmannians.

Recall that $A^1(\text{Gr}(d, \mathbf{P}^n))$ is generated by the Schubert cycle x , which is represented by the set of d -planes intersecting a given $n - d - 1$ -plane, and $A^2(\text{Gr}(d, \mathbf{P}^n))$ is generated freely by x^2 and an additional Schubert cycle y , represented by the d -planes intersecting a given $n - d - 2$ -plane, if $0 < d < n - 1$. In [17] the authors showed that, in $\text{Gr}(2, \mathbf{P}^5)$, if the class $ax^2 + by$ is represented by a non-singular variety, then $b \equiv 0 \pmod{4}$. In addition, in [23], the author showed the same result for $\text{Gr}(2, \mathbf{P}^n)$, $n \geq 8$. To gain more information about these cases, we have

Proposition 1.4. *Suppose there is no smooth surface $Y \subseteq Q$ of class $aZ^2 + b\eta = (a + b)\eta + a\eta'$. Then there is no smooth subvariety of $\text{Gr}(d, \mathbf{P}^n)$ for $0 < d < n - 1$ represented by $ax^2 + by$.*

Proof. Assume $d < n/2$ —otherwise we can use the isomorphism $\text{Gr}(d, \mathbf{P}^n) \cong \text{Gr}(n - d - 1, \mathbf{P}^n)$. Embed $\text{Gr}(1, \mathbf{P}^3)$ into $\text{Gr}(d, \mathbf{P}^n)$ as follows. Choose a linear subspace L of \mathbf{P}^n of dimension $d - 2$ (empty if $d = 1$). Choose a $L' = \mathbf{P}^3 \subseteq \mathbf{P}^n$ which doesn't intersect L (which is possible since $d < n/2$). Then we get a map $\text{Gr}(1, \mathbf{P}^3) \rightarrow \text{Gr}(d, \mathbf{P}^n)$ by taking a line l in L' to the join of L and l . This is easily seen to be an embedding.

Now $PGL(n)$ acts on $\text{Gr}(d, \mathbf{P}^n)$, and if $Y \subseteq \text{Gr}(d, \mathbf{P}^n)$ is a non-singular, codimension 2 subvariety, we may apply Kleiman's theorem [16, III, Theorem 10.8] to see that with a suitable choice of L and L' , the image of the imbedding $\text{Gr}(1, \mathbf{P}^3) \rightarrow \text{Gr}(d, \mathbf{P}^n)$ intersects Y in a smooth surface. If Y has class $ax^2 + by$, it is enough to show that x restricts to Z and y to η to prove the theorem, for then $Y \cap \text{Gr}(1, \mathbf{P}^3)$ is a smooth surface having class $aZ^2 + b\eta$ in $\text{Gr}(1, \mathbf{P}^3)$, contradicting the hypothesis of the proposition.

The class x is described as the Schubert cycle of d -planes intersecting a given $n - d - 1$ -plane. Choose the $n - d - 1$ -plane M to intersect L' in a line m and intersect the join of L and L' only in the line m . Since the join of L and L' has dimension $d + 2$, this may be done. Then $l \subseteq L'$ intersects m if and only if the join of l and L intersects M . Thus this Schubert cycle restricts to $Z(m)$, which has class Z .

The class y is similarly described as the Schubert cycle of d -planes intersecting a given $n - d - 2$ -plane M . Again, choosing M to intersect L' in a point and to avoid the join of L and L' elsewhere, we see the restriction of y to $\text{Gr}(1, \mathbf{P}^3)$ is η . \square

Finally, recall that Q comes equipped with two vector bundles, the universal subbundle and the universal quotient bundle, appearing in the exact sequence

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{O}_Q^4 \rightarrow \mathcal{E}' \rightarrow 0.$$

(Here \mathcal{E}' is the dual of the bundle \mathcal{E} .) The fibre of \mathcal{E} at $p \in Q$ can be thought of as the two dimensional subspace of k^4 corresponding to $l_p \subseteq \mathbf{P}^3$. \mathcal{E}'_p is then the quotient space of this subspace. \mathcal{E} and \mathcal{E}' are interchangeable: the dual exact sequence

$$0 \rightarrow \mathcal{E}' \rightarrow \check{\mathcal{O}}_Q^4 \rightarrow \mathcal{E} \rightarrow 0$$

expresses Q as $\text{Gr}(1, \check{\mathbf{P}}^3)$. $\wedge^2 \mathcal{E}'$ induces the embedding of Q into \mathbf{P}^5 . The Chern classes of \mathcal{E} and \mathcal{E}' are well-known, for example by [13, p. 410]

$$\begin{aligned} c_1(\mathcal{E}) &= -1, & c_2(\mathcal{E}) &= (0, 1), \\ c_1(\mathcal{E}') &= -1, & c_2(\mathcal{E}') &= (1, 0). \end{aligned}$$

A surface in Q is called *degenerate* if it is contained in a hyperplane section of Q . We have the following well-known classification of such surfaces.

Theorem 1.5. *Any degenerate smooth surface in Q of even degree is a complete intersection and is thus of bidegree (n, n) for some n . Any degenerate smooth surface in Q of odd degree is contained in a cone over a non-singular quadric surface, and is linked to a plane, and thus is of bidegree $(n - 1, n)$ or $(n, n - 1)$ for some n .*

Remark 1.6. If $Y \subseteq Q$ is degenerate and is bidegree $(n - 1, n)$, we see $Y \subseteq Z(l)$ for some l . Furthermore, there are two pencils $|F|$ and $|F'|$, cut out by the two families of planes in $Z(l)$. We see that $F \subseteq \eta'(L) \cap Y$ for $L \ni l$, and $F' \subseteq \eta(P) \cap Y$, for $P \in l$, with $F \cdot H = n - 1$ and $F' \cdot H = n$. In addition, $F + F' = H$, and $|F|$ is base-point free. $|F'|$ is not base-point free, having one basepoint $x \in Y$ with $l_x = l$. It is not difficult to construct degenerate surface of each possible bidegree.

2 Singular points and planes

A *singular point* of a congruence Y was defined classically to be a point $P \in \mathbf{P}^3$ such that $\dim \eta(P) \cap Y = 1$, and a *singular plane* to be a plane $L \in \check{\mathbf{P}}^3$ such that $\dim \eta'(L) \cap Y = 1$. The *degree* of a singular point would be the degree of the one dimensional component of $\eta(P) \cap Y$. I shall use the term ‘singular point’ as it was universally used classically, and hope that it is not confused with the usual meaning of the term.

Much classical work (for example [10, 26]) ignores congruences with an infinite number of singular points. They seem to regard them as pathological cases which they exclude from consideration, much as we exclude surfaces which are not smooth. These congruences are very special, and we can obtain a satisfactory partial classification which will be necessary for our results of Sect. 3. Here we show that there are two types of such congruences. Congruences of one class are determined by the bisecants of a twisted cubic or an elliptic quartic. Congruences of the other type have all their rays passing through a non-singular plane curve. We also find tight restrictions on the bidegrees of such surfaces. Partial unpublished results have been obtained in this direction by Cossec, Dolgachev and Verra.

First, we shall study elementary properties of singular points and planes.

Definition-Lemma 2.1. *The incidence relation $I \subseteq \mathbf{P}^3 \times Q$ defined by $I = \{(P, q) \mid P \in l_q\}$ is isomorphic to $\mathbf{P}(\mathcal{E}(1))$ as a projective space bundle over Q , or to $\mathbf{P}(\Omega_{\mathbf{P}^3}(2))$ as a projective space bundle over \mathbf{P}^3 . Furthermore, if p_1 and p_2 are the projections onto \mathbf{P}^3 and Q respectively, then $p_2^*Z = c_1(\mathcal{O}_{\mathbf{P}(\Omega_{\mathbf{P}^3}(2))}(1))$ and $p_1^*L = c_1(\mathcal{O}_{\mathbf{P}(\mathcal{E}(1))}(1))$, where L is the class of a plane in \mathbf{P}^3 .*

Proof. I is the flag manifold $\{0 \subseteq V_1 \subseteq V_2 \subseteq k^4 \mid \dim V_i = i\}$, and it is well known (e.g. [11, 14.2.1]) this is isomorphic to $\mathbf{P}(\mathcal{E}(1))$ or $\mathbf{P}(\Omega_{\mathbf{P}^3}(2))$. We can verify the second statement by using $(p_1^*L)(p_2^*Z)^4 = 2$, where L is the class of a plane in \mathbf{P}^3 . \square

Thus we have a diagram

$$\begin{array}{ccc} \mathbf{P}^3 & \xleftarrow{p_1} & \mathbf{P}(\mathcal{E}(1)) \\ & & \downarrow p_2 \\ & & Q \end{array}$$

We may restrict p_2 to $p_2^{-1}(Y)$, to get a diagram

$$\begin{array}{ccc} \mathbf{P}^3 & \xleftarrow{p_1} & \mathbf{P}(\mathcal{E}(1)|_Y) \\ & & \downarrow p_2 \\ & & Y \end{array}$$

The map p_1 sends $p_2^{-1}(q)$ to the corresponding ray, l_q . A point $P \in \mathbf{P}^3$ is then a singular point of Y if and only if $\dim p_1^{-1}(P) \geq 1$.

Let L be a singular plane. As a section of $\mathcal{E}(1)$ vanishes on $\eta'(L)$, the corresponding section of $\mathcal{E}(1)|_Y$ vanishes on a one-dimensional scheme. This scheme can be decomposed into a divisor and a 0-dimensional residual component. To make this precise, we have

Proposition 2.2. *Let $L \subseteq \mathbf{P}^3$ be a plane and $Y \subseteq Q$ a smooth surface. Then we have a diagram*

$$\begin{array}{ccccc}
 & & Z & & \\
 & & \downarrow b & & \\
 C & \xrightarrow{a} & \eta'(L) \times_Q Y & \xrightarrow{j} & Y \\
 & & \downarrow j' & & \downarrow i' \\
 & & \eta'(L) & \xrightarrow{i} & Q
 \end{array}$$

where $\eta'(L) \times_Q Y$ is the scheme theoretic intersection of $\eta'(L)$ and Y , ja embeds C as an effective Cartier divisor on Y and $j'a$ embeds C as an effective Cartier divisor on $\eta'(L)$. Furthermore a, b are embeddings, Z has finite length, and

$$\mathcal{I}_{\eta'(L) \times_Q Y/Y} = \mathcal{I}_{C/Y} \cdot \mathcal{I}_{Z/Y}.$$

Furthermore, C and Z are the unique schemes with this property, and there is an exact sequence on Y

$$0 \rightarrow \mathcal{O}(C) \rightarrow \mathcal{E}(1)|_Y \rightarrow \mathcal{O}(H - C) \otimes \mathcal{I}_Z \rightarrow 0.$$

Proof. We show this locally. Let $\text{Spec } A = U \subseteq Q$ be an open affine subscheme such that $\mathcal{E}(1)|_U$ is free, coming from $M = A \oplus A$. Let $s \in H^0(\mathcal{E}(1))$ be the section which vanishes on $\eta'(L)$, and let $s|_U = (f, g) \in M$. Then the subscheme of U on which $s|_U$ vanishes is given by the ideal (f, g) . If $I \subseteq A$ is the ideal of $Y \cap U$ in U , then $s|_{Y \cap U}$ vanishes on the subscheme given by the ideal (\bar{f}, \bar{g}) , where \bar{f} and \bar{g} are the classes of f and g in $A' = A/I$. Thus it is clear that the zero-scheme of $s|_Y$ is $\eta'(L) \times_Q Y$.

Define C as the effective Weil divisor on Y locally on U as

$$\sum_{\substack{\mathfrak{p} \in \text{Spec } A' \\ \text{ht } \mathfrak{p} = 1}} \text{length}(A'_{\mathfrak{p}}/(\bar{f}, \bar{g})) \mathfrak{p}.$$

Let $\{(U_i, f_i)\}$ define the associated Cartier divisor with $f_i \in \Gamma(U_i, \mathcal{O}_Y)$. Then define Z locally on U_i by the ideal $(\bar{f}|_{f_i}, \bar{g}|_{f_i})$. It is clear that $\mathcal{I}_{C/Y} \cdot \mathcal{I}_{Z/Y} = \mathcal{I}_{\eta'(L) \times_Q Y/Y}$, and this is the only choice for C to ensure Z is a zero dimensional subscheme. Furthermore, C is a subscheme of $\eta'(L) \times_Q Y$, and thus a subscheme of $\eta'(L)$ and must be embedded as a divisor on $\eta'(L)$.

On U_i , we have the exact sequence

$$0 \rightarrow \mathcal{O}_{U_i} \rightarrow \mathcal{O}_{U_i} \oplus \mathcal{O}_{U_i} \rightarrow \mathcal{I}_{Z \cap U_i} \rightarrow 0$$

given by $1 \mapsto (\bar{f}|_{f_i}, \bar{g}|_{f_i})$ and $(x, y) \mapsto y\bar{f}|_{f_i} - x\bar{g}|_{f_i}$. This patches to give an exact sequence

$$0 \rightarrow \mathcal{O}_Y(C) \rightarrow \mathcal{E}(1)|_Y \rightarrow \mathcal{O}_Y(H - C) \otimes \mathcal{I}_Z \rightarrow 0,$$

as desired. \square

In the notation of the above proposition, the *degree* of the singular plane L is $C \cdot H$.

We can then apply the following well-known result:

Proposition 2.3. *Let \mathcal{F} be a rank 2 vector bundle on a smooth surface Y , and let $t = c_1(\mathcal{O}_{\mathbf{P}(\mathcal{F})}(1))$. Also, let C be a fixed Cartier divisor on Y . Then there is a one-to-one correspondence between exact sequences*

$$0 \rightarrow \mathcal{O}(C) \rightarrow \mathcal{F} \rightarrow \mathcal{I}_Z \otimes \mathcal{O}(C') \rightarrow 0,$$

where Z is a dimension 0 subscheme, and quasi-sections Y' of $\pi: \mathbf{P}(\mathcal{F}) \rightarrow Y$, with Y' of class $t - \pi^*C$. (A quasi-section Y' of π is an irreducible (but possibly singular) surface in $\mathbf{P}(\mathcal{F})$ which π maps birationally to Y . Alternatively, it is a surface $Y' \subseteq \mathbf{P}(\mathcal{F})$ for which the generic fibre intersects Y' once, or an irreducible surface whose class in $\text{Pic } \mathbf{P}(\mathcal{F})$ is $t - \pi^*C$ for some divisor C .) Here $C + C' = c_1(\mathcal{F})$, and $\pi: Y' \rightarrow Y$ fails to be an isomorphism exactly on the support of Z .

Proof. See [25, Proposition 4]. \square

Thus any plane $L \subseteq \mathbf{P}^3$ gives rise to an exact sequence

$$0 \rightarrow \mathcal{O}_Y(C) \rightarrow \mathcal{E}(1)|_Y \rightarrow \mathcal{O}_Y(H - C) \otimes \mathcal{I}_Z \rightarrow 0,$$

which in turn gives rise to a quasi-section Y' of $\mathbf{P}(\mathcal{E}(1)|_Y)$, of class $t - \pi^*C$, for some C effective, and $C = 0$ if and only if L is not a singular plane. We shall call the restriction of the map p_1 to Y' $p_{\mathcal{O}(C)}$, with the inclusion $\mathcal{O}(C) \rightarrow \mathcal{E}(1)|_Y$ understood. In this case, $p_{\mathcal{O}(C)}$ maps Y' to L , and geometrically, this map sends the general $p \in Y$ to $l_p \cap L$.

We say a subsheaf $\mathcal{O}(C)$ of $\mathcal{E}(1)|_Y$ is *saturated* if $\mathcal{E}(1)|_Y/\mathcal{O}(C)$ is torsion-free: i.e. we have an exact sequence

$$0 \rightarrow \mathcal{O}(C) \rightarrow \mathcal{E}(1)|_Y \rightarrow \mathcal{I}_Z \otimes \mathcal{O}(H - C) \rightarrow 0.$$

Any locally free subsheaf of $\mathcal{E}(1)|_Y$ is contained in a saturated subsheaf of $\mathcal{E}(1)|_Y$.

Finally, the intersection theory of $\mathbf{P}(\mathcal{E}(1)|_Y)$ is as follows. $A^*(\mathbf{P}(\mathcal{E}(1)|_Y))$ is generated by

$$t = c_1(\mathcal{O}_{\mathbf{P}(\mathcal{E}(1)|_Y)}(1))$$

and $p_2^* \text{Pic } Y$, and

$$t^2 = p_2^* Ht - b$$

where b indicates the pull-back of a zero-cycle of degree b on Y , and

$$\begin{aligned} t^3 &= p_2^* Ht^2 - bt \\ &= (H^2 - b)t \\ &= at. \end{aligned}$$

Thus we can consider the degree of a divisor in $\mathbf{P}(\mathcal{E}(1)|_Y)$ with respect to t (t need not be ample). If $D \subseteq \mathbf{P}(\mathcal{E}(1)|_Y)$ is a divisor, define $\text{deg}_t D = t^2 \cdot D$. If D is an irreducible surface in $\mathbf{P}(\mathcal{E}(1)|_Y)$, then $\text{deg}_t D$ is equal to the product of the degree of

$p_1(D)$ as a surface in \mathbf{P}^3 and the degree of the map from D to $p_1(D)$ (which is 0 if $\dim p_1(D) \leq 1$). In particular, given a quasi-section Y' of class $t - p_2^*C$, we have

$$\begin{aligned} \deg_t Y' &= t^2(t - p_2^*C) \\ &= a - C \cdot H. \end{aligned}$$

This number must always be non-negative.

All comments apply equally to the dual situation with $\mathbf{P}(\mathcal{E}'(1)|_Y)$.

Proposition 2.4. *Let $Y \subseteq Q$ be a smooth surface of bidegree (a, b) which is not a plane, and let C be a Cartier divisor, with $\mathcal{O}(C) \subseteq \mathcal{E}'(1)|_Y$. Then*

a) *if Y has a finite number of singular planes, then $C \cdot H < b$; if in addition Y is irrational, then $C \cdot H < b - 1$.*

b) *If $\mathcal{O}(C) \subseteq \mathcal{E}'(1)|_Y$ is saturated then the residual zero-dimensional subscheme defined by the quotient has length $a - C \cdot H + C^2$.*

Dually, if D is a Cartier divisor, with $\mathcal{O}(D) \subseteq \mathcal{E}(1)|_Y$, then

c) *if Y has a finite number of singular points, then $D \cdot H < a$; if in addition Y is irrational, then $D \cdot H < a - 1$.*

d) *If $\mathcal{O}(D) \subseteq \mathcal{E}(1)|_Y$ is saturated then the residual zero-dimensional subscheme defined by the quotient has length $b - D \cdot H + D^2$.*

Proof. a) One can assume $\mathcal{O}(C)$ is a saturated subbundle of $\mathcal{E}'(1)|_Y$; this will only increase $C \cdot H$. It then gives rise to a map $p_{\mathcal{O}(C)}: Y' \rightarrow \check{\mathbf{P}}^3$, where Y' is the quasi-section associated to $\mathcal{O}(C) \subseteq \mathcal{E}'(1)|_Y$. Since $\deg_t Y' = b - C \cdot H$, we must have $C \cdot H \leq b$. If $C \cdot H = b$, then Y' maps to a curve, and thus there are an infinite number of singular planes. If $C \cdot H = b - 1$, then $p_{\mathcal{O}(C)}$ is a birational map to a plane, so Y' , hence, Y , is rational.

b) We have

$$\begin{aligned} \deg Z &= c_2((\mathcal{E}'(1)|_Y)(-C)) \\ &= c_2(\mathcal{E}'(1)|_Y) - C \cdot H + C^2 \\ &= a - C \cdot H + C^2, \end{aligned}$$

as claimed.

c) and d) are dual. \square

We now consider congruences which have an infinite number of singular points or planes. The following theorem gives tight restrictions on such surfaces. This theorem was proved independently by Cossec, Dolgachev and Verra, but is unpublished.

Theorem 2.5. *Let $Y \subseteq Q$ be a non-degenerate smooth surface with an infinite number of singular points. Then either*

a) *Y is the locus of bisecants to a twisted cubic or elliptic quartic, being of bidegree $(1, 3)$ or $(2, 6)$, respectively; or*

b) *Each one dimensional component of the locus of singular points is a non-singular plane curve of degree d , each of whose points is a singular point of degree e , and Y has bidegree $(ed - D, ed)$, where e, d , and D satisfy the equation*

$$de(d - 1)(e - 1) + D[1 + 2e - 2de] + D^2 = 0.$$

Proof. Let C be a one dimensional, irreducible component of the set of singular points of Y . Then all the rays of Y intersect C . Indeed, the set of lines intersecting C

is a complex X' which must contain a two-dimensional component of Y , and hence Y itself. Let $p: \tilde{C} \rightarrow C$ be the normalization of C , and let $X = \mathbf{P}_{\tilde{C}}(p^*\Omega_{\mathbf{P}^3}(2))$. From the interpretation of $\mathbf{P}(\Omega_{\mathbf{P}^3}(2))$ in Lemma 2.1, we set there is a natural map $X \rightarrow I \rightarrow Q$, where the image of X is X' , and the fibers of $\pi: X \rightarrow \tilde{C}$ are mapped to the corresponding η -planes in Q . X is non-singular, and $X \rightarrow X'$ is generically 1-1, so $X \rightarrow Q$ gives a natural birational desingularization of X' .

There are now two cases. If Y is not contained in the singular locus of X' , let \tilde{Y} be the proper transform of Y . In this case \tilde{Y} is birational to Y . If Y is contained in the singular locus of X' , let \tilde{Y} be an irreducible component of the inverse image of Y in X which maps surjectively to Y . We have the following diagram:

$$\begin{array}{ccccc} \tilde{Y} \subseteq X & \rightarrow & \tilde{C} & \rightarrow & \mathbf{P}^3 \\ \downarrow & & \downarrow & & \\ Y \subseteq X' \subseteq Q & & & & \end{array}$$

Case 1. $\tilde{Y} \rightarrow Y$ is not birational. In this case, each ray of Y must intersect C at least twice, counted with multiplicities. $\tilde{Y} \rightarrow Y$ cannot be more than generically 2-1. For if the map is 3-1 or more, the rays of Y correspond to trisecants of C , and if C is not a plane curve, it cannot have a doubly infinite family of trisecants ([1, p. 110]). C cannot be a plane curve in this case, for then all rays of Y are contained in C 's plane. Thus Y is a component of the set of bisecants of C , and C is not a plane curve. Now there is a rational map $f: \tilde{C} \times \tilde{C} \rightarrow Y$ defined by taking $(x, y) \in \tilde{C} \times \tilde{C}$ to the point of Q corresponding to the line joining $p(x)$ and $p(y)$, if $p(x) \neq p(y)$. If $p(x)$ is a non-singular point, then $f(x, x)$ is the point of Q corresponding to the tangent line to C at $p(x)$. This map may be undefined for points $(x, y) \in \tilde{C} \times \tilde{C}$ such that $p(x) = p(y)$ is a singular point.

This rational map factors through the quotient of $\tilde{C} \times \tilde{C}$ by the involution interchanging the two factors, so we obtain a birational transformation $f': S^2\tilde{C} \rightarrow Y$. This map can be factored as a sequence of blowing-ups and blowing-downs: let Z be a smooth surface with birational morphisms $f_1: Z \rightarrow S^2\tilde{C}$ and $f_2: Z \rightarrow Y$, with $f' = f_2 \circ f_1^{-1}$.

As the transformation f' contracts no curves, the exceptional set of f_2 must be contained in the exceptional set of f_1 . Now let $C'_x \subseteq S^2\tilde{C}$ be the image of $\{x\} \times \tilde{C}$ or $\tilde{C} \times \{x\}$, $x \in \tilde{C}$. C'_x is smooth, isomorphic to \tilde{C} , and for general x , avoids the finite number of points blown-up by f_1 . Thus the proper transform of C'_x on Y must be smooth. Call this proper transform D_x .

It is clear that $D_x \subseteq \eta(p(x)) \cap Y$, and that D_x is then isomorphic to the projection of C to a plane from x , for the set of bisecants of C through x form a cone over this projection. But by Castelnuovo's bound on arithmetic genus for space curves, the only curves which can be projected to a smooth curve from a point on the curve into \mathbf{P}^2 is a twisted cubic or an elliptic quartic. We have already computed the bidegrees of these congruences in Example 1.2.

Case 2. $\tilde{Y} \rightarrow Y$ is birational. If $\tilde{Y} \rightarrow Y$ is not an isomorphism, it must contract a curve D on \tilde{Y} , by Zariski's main theorem. Since each fibre of the map $X \rightarrow \tilde{C}$ maps isomorphically to Q , D cannot be contained in one fibre, and so D maps surjectively to \tilde{C} . But then the ray corresponding to the contraction of D must pass through every point of C , thus C is a line, which contradicts $Y \not\subseteq Z(l)$ for any l . Thus $\tilde{Y} \rightarrow Y$ is an isomorphism. Thus in particular, \tilde{Y} is non-singular.

Finally, I claim C is actually smooth. First of all, suppose that $p: \tilde{C} \rightarrow C$ is not one-to-one. Then the map $f: X \rightarrow Q$ maps two planes of C to the same plane of Q ,

and thus two fibres of \tilde{Y} are mapped to plane curves which intersect, contradicting $\tilde{Y} \cong Y$.

Now suppose $p: \tilde{C} \rightarrow C$ is one-to-one, but not an isomorphism. The map $f: X \rightarrow Q$ factors through the natural map $X \rightarrow X''$ with $X'' = \mathbf{P}(\Omega_{\mathbf{P}^3}(2)|_C)$, with $\pi': X'' \rightarrow C$ the projection. Let $x \in C$ be a singular point of C . Then $\pi'^{-1}(x)$ is contained in the singular locus of X'' , and so $f(\pi'^{-1}(p^{-1}(x)))$ is contained in the singular locus of X' . Now consider the induced map on tangent bundles $f_*: \mathcal{T}_X \rightarrow f^* \mathcal{T}_Q$. For any point $y \in L = \pi'^{-1}(p^{-1}(x))$, the map on fibres $\mathcal{T}_{X,y} \rightarrow f^* \mathcal{T}_{Q,f(y)}$ cannot be injective; otherwise $f(y)$ is a smooth point of X' if y is general in L . Furthermore, since L is mapped isomorphically to a plane M in Q , the tangent space of X at y must be mapped to the tangent space of M at $f(y)$. Hence the image of $f_*|_L: \mathcal{T}_X|_L \rightarrow f^* \mathcal{T}_Q|_L$ is \mathcal{T}_M . Now $\mathcal{T}_X|_L \cong \mathcal{O}_L \oplus \mathcal{T}_L$, as can easily be seen from the exact sequence

$$0 \rightarrow \pi^* \Omega_{\tilde{C}} \rightarrow \Omega_X \rightarrow \Omega_{X/\tilde{C}} \rightarrow 0,$$

$\pi^* \Omega_{\tilde{C}}|_L \cong \mathcal{O}_L$, and $\Omega_{X/\tilde{C}}|_L \cong \Omega_L$. Thus we see that the kernel of $f_*|_L$ is \mathcal{O}_L .

Now we have $\tilde{Y} = Y \subseteq X$. Let $D = Y \cap L$, and let D' be an irreducible, reduced component of D . For any point $y \in D'$, the tangent space $\mathcal{T}_{Y,y} \subseteq \mathcal{T}_{X,y}$ cannot contain the kernel of $f_{*,y}$; otherwise $\mathcal{T}_{Y,y}$ is not mapped injectively into $f^* \mathcal{T}_{Q,f(y)}$, and f is not an embedding of Y at y . We have an exact sequence

$$0 \rightarrow \mathcal{T}_Y \rightarrow \mathcal{T}_X|_Y \rightarrow \mathcal{N}_{Y/X} \rightarrow 0,$$

and restricting to D' , we obtain a sequence

$$0 \rightarrow \mathcal{T}_Y|_{D'} \rightarrow \mathcal{O}_{D'} \oplus \mathcal{T}_L|_{D'} \rightarrow \mathcal{O}_{D'}(e) \rightarrow 0.$$

We now work with $\mathbf{P}(\mathcal{O}_{D'} \oplus \Omega_L|_{D'})$. One recalls a point of this space corresponds to a one-dimensional subspace of a fibre of $\mathcal{O}_{D'} \oplus \mathcal{T}_L|_{D'}$. Let α be the natural projection to D' . Now dualize and projectivize: the surjection

$$\mathcal{O}_{D'} \oplus \Omega_L|_{D'} \rightarrow \Omega_Y|_{D'} \rightarrow 0$$

determines an inclusion $T = \mathbf{P}(\Omega_Y|_{D'}) \subseteq \mathbf{P}(\mathcal{O}_{D'} \oplus \Omega_L|_{D'})$, and the surjection

$$\mathcal{O}_{D'} \oplus \Omega_L|_{D'} \rightarrow \mathcal{O}_{D'} \rightarrow 0$$

determined by the exact sequence

$$0 \longrightarrow \mathcal{O}_{D'} \longrightarrow \mathcal{T}_X|_{D'} \xrightarrow{f_*} \mathcal{T}_M|_{D'} \longrightarrow 0$$

determines a section $K \subseteq \mathbf{P}(\mathcal{O}_{D'} \oplus \Omega_L|_{D'})$, $K \cong D'$. If T and K intersect at a point over $y \in D'$, then $\mathcal{T}_{Y,y}$ contains the kernel of $f_{*,y}$. By construction, the restriction of $\mathcal{O}_{\mathbf{P}(\mathcal{O}_{D'} \oplus \Omega_L|_{D'})}(1)$ to K is \mathcal{O}_K , and the invertible sheaf associated to the divisor T is $\mathcal{O}_{\mathbf{P}(\mathcal{O}_{D'} \oplus \Omega_L|_{D'})}(1) \otimes \alpha^* \mathcal{O}_{D'}(e)$. Restricting this sheaf to K , we obtain an invertible sheaf of degree $e \deg D'$, which is positive, so $K \cap T \neq \emptyset$. Thus $f: Y \rightarrow Q$ is not an embedding.

Thus $\tilde{C} = C$ is non-singular.

Now let $g = p_a(C)$, $d = \deg C$. Suppose that the degree of the locus of rays passing through a general point of C is e . We wish to compute the invariants of \tilde{Y} and determine if \tilde{Y} can actually be embedded in Q . We compute the bidegree of Y , its Chern classes, and its hyperplane section genus.

Note that \tilde{Y} is a divisor of X , and $\text{Pic } X$ is generated by $t = c_1(\mathcal{O}_{\mathbf{P}(\mathbf{P}^* \Omega_{\mathbf{P}^3}(2))}(1))$ and $\text{Pic } C$. Since $c_1(\Omega_{\mathbf{P}^3}(2)) = 2L$, where L is the class of a plane in \mathbf{P}^3 , we have

$t^3 = 2\pi^*Lt^2$. Suppose the class of \tilde{Y} is $\alpha t - \pi^*D$, where D is a divisor on C . If $P \in C$, the fact that the rays of Y through P form a cone of degree e is translated into the formula

$$\deg(\alpha t - \pi^*D) \cdot \pi^*P \cdot t = e,$$

which gives $\alpha = e$.

Furthermore, the degree of Y is given by $\deg(et - \pi^*D) \cdot t^2 = \deg(2e\pi^*L - \pi^*D)t^2 = 2ed - \deg D$. Now a general plane intersects C in d points, from each of which radiate e rays, each group of rays distinct from every other group of rays. Thus the bidegree of Y is $(ed - \deg D, ed)$.

To compute c_1^2 and c_2 of Y , we first compute the Chern classes of \mathcal{F}_X , and then use the sequence

$$0 \rightarrow \mathcal{F}_Y \rightarrow \mathcal{F}_X|_Y \rightarrow \mathcal{N}_{Y/X} \rightarrow 0.$$

Ω_X appears in the exact sequence

$$0 \rightarrow \pi^*\Omega_C \rightarrow \Omega_X \rightarrow \Omega_{X/C} \rightarrow 0,$$

and the relative Euler sequence gives

$$0 \rightarrow \Omega_{X/C}(t) \rightarrow \pi^*p^*\Omega_{\mathbf{P}^3}(2) \rightarrow \mathcal{O}_X(t) \rightarrow 0,$$

where a twist by t means tensoring by $\mathcal{O}_{\mathbf{P}(p^*\Omega_{\mathbf{P}^3}(2))}(1)$.

We obtain

$$\begin{aligned} c_\alpha(\mathcal{F}_Y) &= c_\alpha(\mathcal{F}_X|_Y)/c_\alpha(\mathcal{N}_{Y/X}) \\ &= c_\alpha(\mathcal{F}_X|_Y)/(1 + (eH - \pi^*D)\alpha) \\ &= 1 - (\pi^*K_C + 2\pi^*L + (e-3)H - \pi^*D)\alpha \\ &\quad + [(2e-4)H\pi^*L + (e(e-3)+3)H^2 + (e-3)H\pi^*K_C \\ &\quad - (2e-3)H\pi^*D]\alpha^2. \end{aligned}$$

Thus we have

$$K_Y = 2\pi^*L + \pi^*K_C + (e-3)H - \pi^*D,$$

and

$$\begin{aligned} c_2 &= (2e-4)de + (e(e-3)+3)(2ed - \deg D) \\ &\quad + (e-3)e(2g-2) - (2e-3)e \deg D. \end{aligned}$$

We also have

$$H \cdot K_Y = 2ed + (2g-2)e + (e-3)(2ed - \deg D) - e \deg D$$

and

$$K_Y^2 = (e-3)^2(2ed - \deg D) + 2(e-3)[2ed + e(2g-2) - e \deg D].$$

Now take all of these formulas and plug them into the equation of Proposition 1.3. Then we get an equation

$$\begin{aligned} 2d^2e^2 - 4de^2 - 2e^2g - 2de - 2eg + 2e^2 + 2e \\ + (\deg D)[1 + 2e - 2de] + (\deg D)^2 = 0. \end{aligned} \quad (*)$$

We wish to know, for a given d , what values of g give a solution.

The discriminant of the above equation must be non-negative, yielding the equation

$$1 + (4d + 8g - 4)e + [8d + 8g - 4 - 4d^2]e^2 \geq 0.$$

Note that $8d + 8g - 4 - 4d^2 \geq 0$ implies $g \geq \frac{d^2}{2} - d + \frac{1}{2} = (d-1)^2/2 > (d-1) \cdot (d-2)/2$, so that the coefficient of e^2 is always negative. Thus for a given d and g , there are a finite number of possible e 's. Since $e > 0$, we can require the larger root of the inequality for e to be ≥ 1 . This gives

$$4d + 8g - 4 + \sqrt{(4d + 8g - 4)^2 - 4(8d + 8g - 4 - 4d^2)} \geq 2(4d^2 - 8d + 4 - 8g),$$

or

$$\sqrt{(4d + 8g - 4)^2 - 4(8d + 8g - 4 - 4d^2)} \geq 8d^2 - 20d + 12 - 24g.$$

If $g \leq \frac{d^2}{3} - \frac{5d}{6} + \frac{1}{2}$, then the righthand side is positive. Note that if C is not a plane curve, then $g \leq \frac{d^2}{4} - d + 1$, so that the righthand side is positive, and we can square both sides of the inequality without changing the relations. So assuming C is not a plane curve, we get an inequality quadratic in g :

$$\begin{aligned} -512g^2 + (384d^2 - 896d + 480)g \\ - 64d^4 + 320d^3 - 560d^2 + 416d - 112 \geq 0, \end{aligned}$$

which gives

$$\frac{4d^2 - 12d + 7}{16} \leq g \leq \frac{d^2 - 2d + 1}{2}.$$

The upper bound is trivial, while the lower bound is larger than the genus of any non-planar curve, for $d \geq 3$. Thus C is a plane curve. This gives the result of the theorem, and the equation is obtained by setting $g = (d-1)(d-2)/2$ in (*), and writing D instead of $\deg D$. \square

In the special case that $e = 1$, so that Y is a scroll, Goldstein has completed the classification in [12].

Corollary 2.6. *If $Y \subseteq Q$ is a non-degenerate smooth surface with an infinite number of singular points, of bidegree (a, b) , then $a \leq b$.*

Proof. For the inequality $a \leq b$, we need only observe that $D \geq 0$. Indeed, consider the function

$$f(D) = de(d-1)(e-1) + D(1+2e-2de) + D^2.$$

f obtains its minimum at

$$D = \frac{2de - 2e - 1}{2} > 0$$

for $e > 0$, $d > 1$, and so to check that any solution of $f(D) = 0$ is positive, it is sufficient to check that $f(0) \geq 0$. But $f(0) = de(d-1)(e-1) \geq 0$ for $e > 0$, $d > 1$. Thus $D \geq 0$. \square

This is all the information we need for our results of Sect. 3. However, we note the following conjecture.

Conjecture 2.7. *The only smooth non-degenerate surfaces with an infinite number of singular points are the scrolls of bidegree (2, 2) and (3, 3), a conic bundle of bidegree (3, 6), and the surfaces of bidegree (1, 3) and (2, 6).*

3 Limitations on bidegrees

We now consider the question: “for what values of a and b does there (or does there not) exist a smooth surface of bidegree (a, b) ?” The best general approach we have for this question is to consider the semi-stability of $\mathcal{E}(1)$, in the sense of Mumford and Takemoto (see [22]), restricted to a surface $Y \subseteq Q$. This was originally suggested to me by Dolgachev.

Definition. Let \mathcal{F} be a rank 2 vector bundle on a smooth surface Y , and let H be an ample divisor. Then \mathcal{F} is H -stable (H -semistable) if for every rank 1 subsheaf \mathcal{F}' of \mathcal{F} , $2c_1(\mathcal{F}') \cdot H < c_1(\mathcal{F}) \cdot H$ ($2c_1(\mathcal{F}') \cdot H \leq c_1(\mathcal{F}) \cdot H$).

Proposition 3.1. *If $Y \subseteq Q$ is a smooth surface of bidegree (a, b) , and $\mathcal{E}(1)|_Y$ is H -semistable, then $a \leq 3b$.*

Proof. We have $c_1(\mathcal{E}(1)|_Y) = H$ and $c_2(\mathcal{E}(1)|_Y) = b$; thus Bogomolov’s theorem [6] tells us that $a + b = H^2 \leq 4b$, or $a \leq 3b$. \square

Before we attack this problem, we give an example showing that additional hypotheses are needed.

Example 3.2. Let a smooth surface Y be contained in a singular linear complex $Z(l)$, with bidegree $(n + 1, n)$ (such a surface exists). By Remark 1.6, if $L \cong l$ is a plane containing the line l in \mathbf{P}^3 , then $\eta'(L)$ intersects Y in a curve of degree $n + 1$. Now there is a section s of $\mathcal{E}(1)$ vanishing on the plane $\eta'(L)$, and restricting s to Y , we get a section of $\mathcal{E}(1)|_Y$ vanishing on a curve C of degree $n + 1$. This tells us $\mathcal{O}(C)$ is a subsheaf of $\mathcal{E}(1)|_Y$, but $2C \cdot H = 2n + 2 \not\leq H^2 = 2n + 1$. Thus $\mathcal{E}(1)|_Y$ is H -unstable.

Thus we make the following two conjectures:

Conjecture 3.3. *If $Y \subseteq Q$ is a smooth, non-degenerate surface then $\mathcal{E}(1)|_Y$ is H -semistable.*

Conjecture 3.4. *If $Y \subseteq Q$ is a smooth surface of bidegree (a, b) , then $a \leq 3b$.*

Conjecture 3.4 follows from Conjecture 3.3, Proposition 3.1, and Theorem 1.5. We shall actually work primarily on Conjecture 3.4.

We use the setup and notation of Sect. 2, which is useful for studying any subbundle $\mathcal{O}(C) \subseteq \mathcal{E}(1)|_Y$. We preserve the meaning of p_1, p_2, t, Z , etc.

If we have a saturated subbundle $\mathcal{O}(C)$ of $\mathcal{E}(1)|_Y$, we have an exact sequence

$$0 \rightarrow \mathcal{O}(C) \rightarrow \mathcal{E}(1)|_Y \rightarrow \mathcal{F}_Z \otimes \mathcal{O}(C') \rightarrow 0$$

where Z is a scheme of finite length. We have $c_1(\mathcal{E}(1)|_Y) = H$, the hyperplane section, and in addition $c_2(\mathcal{E}(1)|_Y) = b$. Thus $H \sim C + C'$, and so $C' \sim H - C$. Furthermore,

$$\deg Z = b - C \cdot H + C^2.$$

(Proposition 2.4 d).

Furthermore, the exact sequence

$$0 \rightarrow \mathcal{O}(C) \rightarrow \mathcal{E}(1)|_Y \rightarrow \mathcal{I}_Z \otimes \mathcal{O}(H - C) \rightarrow 0$$

gives rise to a quasi-section Y' of $\mathbf{P}(\mathcal{E}(1)|_Y)$, birational to Y , and the class of Y' is (Proposition 2.3)

$$[Y'] = t - p_2^*C.$$

Restricting p_1 and p_2 to Y' , we obtain a diagram

$$\begin{array}{ccc} \mathbf{P}^3 & \xleftarrow{p_{\mathcal{O}(C)}} & Y' \\ & & \downarrow q_2 \\ & & Y \end{array}$$

where $p_{\mathcal{O}(C)}, q_2$ denote the restrictions. Y' is the blowing up of Y at Z .

Denote by S the image of Y' under $p_{\mathcal{O}(C)}$.

We can compute the degree of Y' relative to t as

$$\begin{aligned} \deg_t Y' &:= \deg t^2 \cdot (t - p_2^*C) \\ &= a - C \cdot H. \end{aligned}$$

This must be a non-negative number. Note also that

$$\begin{aligned} (H - C)^2 - \deg Z &= H^2 - 2C \cdot H + C^2 - (b - C \cdot H + C^2) \\ &= \deg_t Y'. \end{aligned}$$

Proposition 3.5. *If $a \leq b$, then $\mathcal{E}(1)|_Y$ is H -semistable.*

Proof. If $\mathcal{O}(C)$ is a saturated subsheaf of $\mathcal{E}(1)|_Y$, then $(a + b)/2 \geq a \geq C \cdot H$, so $\mathcal{E}(1)|_Y$ is semi-stable. \square

To proceed, we will need the following bound:

Theorem 3.6. *Let $Y \subseteq Q$ be a non-degenerate smooth surface with hyperplane section genus π and degree d . Then if $d \geq 9$,*

$$\pi \leq \pi_1(d) = \begin{cases} \frac{d^2 - 4d + 8}{8}, & d \equiv 0 \pmod{4}; \\ \frac{d^2 - 4d + 3}{8}, & d \equiv 1, 3 \pmod{4}; \\ \frac{d^2 - 4d + 4}{8}, & d \equiv 2 \pmod{4}. \end{cases}$$

Proof. By [9, Theorem 3.15], any non-degenerate curve in \mathbf{P}^4 whose genus falls above $\pi_1(d)$ must be contained in a surface of degree 3. Let $L \subseteq \mathbf{P}^5$ be a

hyperplane which intersects both Q and Y smoothly. If $\pi > \pi_1(d)$, then $L \cap Y \subseteq S$, a surface of degree 3. There are no surfaces of degree 3 contained in $L \cap Q$, a smooth quadric three-fold, so $L \cap Y \subseteq S \cap Q$, where $S \cap Q$ is a curve of degree 6. Thus the degree of Y is ≤ 6 , a contradiction. \square

Theorem 3.7. *Let $Y \subseteq Q$ be a non-degenerate surface of degree $d \geq 9$ with a finite number of singular points, and let $L \subseteq \mathbf{P}^3$ be a plane. Let C be the effective Cartier divisor of Proposition 2.2. Then $C \cdot H \leq d/2$. Thus $\mathcal{O}(C) \subseteq \mathcal{E}(1)|_Y$ is not a destabilising subsheaf of $\mathcal{E}(1)|_Y$.*

Proof. Let $c = \deg C = C \cdot H$. Because C is a divisor on the plane $\eta'(L)$, we must have $p_a(C) = (c-1)(c-2)/2$. We can write $Z(l) \cap Y = C + D = H$, for any line l contained in L , where D is effective. It is clear $\dim |D| \geq 2$ and $|D|$ has no fixed components.

Now we show $p_a(D) < 0$. For we have

$$\begin{aligned} 2p_a(D) - 2 &= (H - C) \cdot (H - C + K) \\ &= H^2 - 2c + C^2 + H \cdot K - C \cdot K \\ &= 2\pi - 2 + C^2 - C \cdot K - 2c \\ &= 2\pi - 2 - C^2 - C \cdot K + 2C^2 - 2c \\ &= 2\pi - 2 - c(c-3) + 2C^2 - 2c. \end{aligned}$$

We can use two estimates at this point. First, use the bound for π of Theorem 3.6:

$$\pi \leq \frac{d^2 - 4d + 8}{8} \quad \forall d \geq 9.$$

Also, we have $C^2 H^2 \leq (C \cdot H)^2 = c^2$, so $C^2 \leq c^2/d$. Thus we have

$$\begin{aligned} 2p_a(D) - 2 &\leq \frac{d^2 - 4d}{4} - c^2 + c + \frac{2c^2}{d} \\ &= \frac{d^2 - 4d}{4} - c^2(1 - 2/d) + c, \end{aligned}$$

and

$$p_a(D) \leq \frac{d^2 - 4d + 8}{8} - \frac{c^2}{2} \left(1 - \frac{2}{d}\right) + \frac{c}{2}.$$

Let c_0 be the larger real root (if it has one) of the equation

$$-\frac{c^2}{2} \left(1 - \frac{2}{d}\right) + \frac{c}{2} + \frac{d^2 - 4d + 8}{8} = 0.$$

Then if $c > c_0$, $p_a(D) < 0$.

Now

$$c_0 = \frac{d + d\sqrt{d^2 - 6d + 17 - 16/d}}{2d - 4},$$

and we wish to show that $c_0 < \frac{d+1}{2}$. One can reduce this inequality to one of degree 3:

$$0 < 8d^3 - 68d^2 + 96d + 16.$$

The largest root of this polynomial is less than 8, and so the desired result $p_a(D) < 0$ holds for $d \geq 9$. It is worthwhile to note, using a calculator, that c_0 is only very slightly less than $(d + 1)/2$, and we cannot achieve any better bound using this technique. This seems to suggest that there is something deeper going on, but I'm not sure what it is.

Now $\mathcal{O}_{\mathbf{P}(\mathcal{E}(1)|_Y)}(1)$ restricts to Y' to determine the linear system $\mathcal{D} \subseteq |D'|$, which induces the map $p_{\mathcal{O}(C)}: Y' \rightarrow \mathbf{P}^3$, with $q_{2*}\mathcal{O}(D') = \mathcal{I}_Z \otimes \mathcal{O}(D)$, where Z is determined by the exact sequence

$$0 \rightarrow \mathcal{O}(C) \rightarrow \mathcal{E}(1)|_Y \rightarrow \mathcal{I}_Z \otimes \mathcal{O}(D) \rightarrow 0.$$

If $D' \in \mathcal{D}$, then since $p_a(D) < 0$, $p_a(D') < 0$, and so the general member of $|D'|$ is reducible, and thus \mathcal{D} is composed of a pencil. Thus the image of the map induced by \mathcal{D} is a curve. This contradicts the assumption that there are a finite number of singular points. \square

We have, as a corollary of the proof, the following which is useful in other applications:

Corollary 3.8. *Let $Y \subseteq Q$ be a non-degenerate surface of degree d and hyperplane section genus π , with a finite number of singular points, and let $L \subseteq \mathbf{P}^3$, with the one-dimensional component of $\eta'(L) \cap Y$ the effective Cartier divisor C . Then*

$$0 \leq 2\pi - (C \cdot H)^2 \left(1 - \frac{2}{d}\right) + C \cdot H.$$

Proof. From the estimates in the proof of Theorem 3.7, we see

$$2p_a(D) - 2 \leq 2\pi - 2 - (C \cdot H)^2 \left(1 - \frac{2}{d}\right) + C \cdot H,$$

and furthermore, we must have $p_a(D) \geq 0$. \square

We will now explore the consequences of the possibility that $\mathcal{E}(1)|_Y$ is unstable. For the following sequence of results, we shall use

Hypothesis (*). *Let $Y \subseteq Q$ be a smooth, non-degenerate surface with a finite number of singular points, of bidegree (a, b) , and let $\mathcal{O}(C)$ be a saturated subbundle of $\mathcal{E}(1)|_Y$; i.e. we have an exact sequence*

$$0 \rightarrow \mathcal{O}(C) \rightarrow \mathcal{E}(1)|_Y \rightarrow \mathcal{I}_Z \otimes \mathcal{O}(H - C) \rightarrow 0.$$

Let Y' , $p_{\mathcal{O}(C)}$, q_2 , and S be as in the above discussion.

Proposition 3.9. *Given (*), suppose $\mathcal{O}(C) \subseteq \mathcal{E}(1)|_Y$ is destabilising. Then S is a surface of degree ≥ 2 .*

Proof. If $p \in Y$, then $l_p = p_1(p_2^{-1}(p))$, so $l_p \cap S \neq \emptyset$. Thus if S is a point, then Y is an η -plane, and if S is a curve, then Y has an infinite number of singular points, violating (*). Thus S is a surface. Suppose it is a plane.

Pulling back the plane S to $\mathbf{P}(\mathcal{E}(1)|_Y)$, we get a divisor $p_1^{-1}(S) \sim t$. If $y \in Y$, then either $l_y \subseteq S$, or otherwise $p_2^{-1}(y) \cap p_1^{-1}(S)$ consists of one point. Also, since Y' maps to \bar{S} , $p_1^{-1}(S)$ decomposes into Y' and $p_2^{-1}(D)$ for some curve $D \subseteq Y$. Since Y' has class $t - p_2^*C$, we must have $D \sim C$, and all the rays of D are contained in S . Thus, in the language of Sect. 2, S is a singular plane of degree $C \cdot H$. However, this cannot happen, by Theorem 3.7. \square

From this point on, we provide a series of estimates which will provide progress on Conjecture 3.4.

Theorem 3.10. *Given Hypothesis (*), with $\mathcal{O}(C) \subseteq \mathcal{E}(1)|_Y$ destabilising, suppose*

$$\deg_t Y' = a - C \cdot H \geq n.$$

Then

$$a \leq \max(3b, b^2/n + b + n).$$

Proof. Recall $a - C \cdot H = (H - C)^2 - \deg Z$, so we have

$$n \leq a - H \cdot C \leq (H - C)^2 \leq \frac{(H \cdot (H - C))^2}{H^2} = \frac{(a + b - H \cdot C)^2}{a + b}$$

and $C \cdot H > \frac{a + b}{2}$. Let $x = C \cdot H$. Then we have the two inequalities

$$a - n \geq x > \frac{a + b}{2}$$

and

$$(a + b)(a - x) \leq (a + b - x)^2,$$

or

$$f(x) = x^2 - (a + b)x + b(a + b) \geq 0.$$

Now suppose that

$$a \geq \max(3b, b^2/n + b + n).$$

First evaluate $f(x)$ at $x = \frac{a + b}{2}$, and we get a negative result:

$$\begin{aligned} \frac{(a + b)^2}{4} - \frac{(a + b)^2}{2} + b(a + b) &\geq 0 \Leftrightarrow -\frac{a + b}{4} + b \geq 0 \\ &\Leftrightarrow a \leq 3b, \end{aligned}$$

which we have ruled out.

Secondly, evaluate $f(x)$ at $x = a - n$, and again we get a negative result:

$$\begin{aligned} (a - n)^2 - (a + b)(a - n) + b(a + b) &\geq 0 \Leftrightarrow -an + bn + n^2 + b^2 \geq 0 \\ &\Leftrightarrow a \leq \frac{b^2}{n} + b + n. \end{aligned}$$

Thus we see that in the allowable range for x , $f(x)$ is always negative, and hence a contradiction. Thus we see that

$$a \leq \max(3b, b^2/n + b + n). \quad \square$$

Remark 3.11. Note that the above theorem still holds if Y is singular, by considering $f: Y' \rightarrow Q$ where Y' is a desingularization of Y . In this case, we study $f^* \mathcal{E}(1)$. In this more general setting, the theorem is tight: any surface of degree d in \mathbf{P}^3 can be mapped to Q by giving a surjective map $\mathcal{O}^4 \rightarrow \mathcal{O}(l) \oplus \mathcal{O}(k)$, for arbitrary $k, l \geq 1$. This gives a (singular) surface in Q of bidegree $((l + k)^2 d - lkd, lkd)$. If $l \leq k$, then $\mathcal{O}(k) \subseteq \mathcal{O}(l) \oplus \mathcal{O}(k)$ is destabilising, and with $n = l^2 d$ in Theorem 3.10, we see we obtain equality. However, these surfaces are never smooth.

Lemma 3.12. *Given Hypothesis (*), let $H \subseteq Y$ be a general hyperplane section, and let $H' = q_2^{-1}(H)$, $H' \cong H$. Then p_1 gives a birational map from H' to $p_1(H')$.*

Proof. H' cannot map to a point under p_1 ; otherwise every ray of H would pass through the point, and H would be a plane curve. Thus the map $H' \rightarrow p_1(H')$ is finite.

Let $\mathcal{D} = \mathbf{P}(H^0(\mathcal{O}_Q(1)))$ be the linear system of hyperplane sections (linear complexes) of Q , and let $W \subseteq \mathcal{D} \times Y$ be the universal divisor: i. e.

$$W = \{(Z, y) \mid y \in q_2^{-1}(Z \cap Y)\}.$$

Consider the map ϕ from W onto its image $W' \subseteq \mathcal{D} \times S$, defined by $\phi(Z, y) = (Z, p_{\mathcal{O}(C)}(y))$.

We will show ϕ is birational, and hence the general hyperplane section is mapped birationally to S . To show ϕ is birational, it is enough to show that there is a dense set on which it is one-to-one, and then apply generic smoothness.

Let $P \in S$ be a point which is not a singular point of Y . Then $p_{\mathcal{O}(C)}^{-1}(P)$ is a finite set, and $q_2(p_{\mathcal{O}(C)}^{-1}(P)) \subseteq Y \cap \eta(P)$. Since the general linear complex intersects $\eta(P)$ in a line, it is clear that

$$\{Z \in \mathcal{D} \mid Z \text{ contains at least two points of } q_2(p_{\mathcal{O}(C)}^{-1}(P))\}$$

is a closed subset of

$$\{Z \in \mathcal{D} \mid Z \text{ contains at least one point of } q_2(p_{\mathcal{O}(C)}^{-1}(P))\}.$$

This shows that the set of points of $(\mathcal{D} \times \{P\}) \cap W'$ on which ϕ is one-to-one is dense. Thus by generic smoothness for $\text{char } k = 0$, ϕ is birational. \square

Theorem 3.13. *Let $Y \subseteq Q$ be a smooth surface of bidegree (a, b) , not of general type. Then $a \leq 3b$.*

Proof. Suppose $a > 3b$. By Corollary 2.6, we can assume Y has a finite number of singular points. Then there exists a C for which Hypothesis (*) holds, and $\mathcal{O}(C) \subseteq \mathcal{E}(1)|_Y$ is destabilising. Now $a - C \cdot H < b$, for if $a - C \cdot H \geq b$, we have by Theorem 3.10 that $a \leq 3b$. The general hyperplane section H is mapped birationally to a curve in \mathbf{P}^3 of degree

$$\begin{aligned} \deg p_2^* H t \cdot (t - p_2^* C) &= H \cdot (H - C) \\ &= a + b - C \cdot H < 2b. \end{aligned}$$

Furthermore, it is contained in a surface S which is of degree between 2 and $a - C \cdot H$, so the curve $p_{\mathcal{O}(C)}(q_2^{-1}(H))$ is not contained in a plane. Hence Castelnuovo's bound tells us

$$\begin{aligned} \pi &\leq (H \cdot (H - C) / 2 - 1)^2 \\ &< (b - 1)^2. \end{aligned}$$

This is a very strong bound in the sense that it does not depend on a . This bound, along with the classification of surfaces, gives us our needed result.

First, for $p_a < 0$, we have $K^2 \leq 8p_a + 8$. Putting this into Proposition 1.3, along with $H \cdot K < 2(b-1)^2 - 2 - a - b$, we get

$$\begin{aligned} 0 &\leq a^2 + b^2 - 7a - 7b - 2(8p_a + 8) - 4(2(b-1)^2 - 2 - a - b) + 12 + 12p_a \\ &= a^2 + b^2 - 7a - 7b - 4p_a - 4 - 4(2(b-1)^2 - 2 - a - b) \\ &\geq a^2 + b^2 - 7a - 7b - 4(2(b-1)^2 - 2 - a - b) \\ &= a^2 - 7b^2 - 3a + 13b \\ &\geq (a-2)^2 - 7b^2 + 13b - 4, \end{aligned}$$

or $(a-2)^2 \leq 7b^2$, so $a \leq \sqrt{7}b + 2$, which is (for $b \geq 3$) a contradiction with the assumption $a > 3b$. Being more careful, we see this holds for $b = 2$, also. Thus $a \leq 3b$ for surfaces with negative arithmetic genus.

For rational surfaces, $p_a = 0$ and $K^2 < 9$, giving us similar results.

Finally, observe that for surfaces with $\kappa = 0$ or 1 , $p_a \geq -1$ and $K^2 \leq 0$, and again a similar computation completes this case. \square

Unfortunately, this bound isn't quite strong enough to handle surfaces of general type. To handle this case, the best we can do at the moment is

Theorem 3.14. *Let $Y \subseteq Q$ be a surface of general type with bidegree (a, b) . In the notation of Hypothesis (*), suppose $a - C \cdot H = n$. Then*

$$\begin{aligned} 0 &\leq \frac{n^4}{6} + \left(\frac{2b}{3} - \frac{4}{3}\right)n^3 + \left(b^2 + \frac{4a}{3} - \frac{8b}{3} + \frac{8}{3}\right)n^2 + \left(\frac{8ab}{3} - \frac{4b^2}{3} + \frac{2b^3}{3} - \frac{16a}{3}\right)n \\ &\quad + \frac{ab^2}{3} - a^2b + 2ab + \frac{11a^2}{3} - a^3 + b^2 - b^3 + \frac{b^4}{6}. \end{aligned}$$

Proof. Using Miyaoka's inequality $K^2 \leq 9\chi$, and from Proposition 1.3

$$12\chi = -a^2 - b^2 + 7(a+b) + 2K^2 + 4H \cdot K,$$

so

$$K^2 \leq 9\chi = \frac{3}{4}(-a^2 - b^2 + 7(a+b) + 2K^2 + 4H \cdot K)$$

or

$$0 \leq -a^2 - b^2 + 7(a+b) + \frac{2}{3}K^2 + 4H \cdot K. \quad (**)$$

Now we use the two estimates

$$H \cdot K \leq 2((b+n)/2 - 1)^2 - 2 - a - b,$$

obtained from Castelnuovo's bound for a curve of degree $H \cdot (H - C) = b + n$ in \mathbf{P}^3 , and

$$K^2 \leq (H \cdot K)^2 / H^2.$$

We obtain the given formula by substituting these estimates in (**) and multiplying through by H^2 . \square

Thus, we have two counteracting forces. If n is large, then Theorem 3.10 acts to eliminate that case, and if n is small, Theorem 3.14 acts best. Thus for a given b , in

order to find the largest a for which these constraints are satisfied, we have to find an n where the maximum of the two constraints is maximized.

Theorem 3.15. *If $Y \subseteq Q$ is a surface of bidegree (a, b) , and $b \leq 19$, then $a \leq 3b$. In general, a is bounded by a function $f(b) = \mathcal{O}(b^{4/3})$.*

Proof. Using MACSYMA, we can solve the equation of Theorem 3.14, and find the surprisingly simple solution

$$n \geq \sqrt{\sqrt{6} \sqrt{a+b} \sqrt{b^2 - b + a^2 - a - 4(a+b) + 4} - b + 2}.$$

Let $n_{\min}(a)$ be the minimum of this root and b . Then $a \leq b^2/n_{\min}(a) + b + n_{\min}(a)$, and so the largest possible a for which this equation holds gives us the upper bound for a . Using a computer, one can check that $a \leq 3b$ for $b \leq 19$.

To see that the general bound is $\mathcal{O}(b^{4/3})$, note that if there exists a surface of bidegree (a, b) with $a > 3b$, then $n_{\min}(a) < b$. The inequality

$$\sqrt{\sqrt{6} \sqrt{a+b} \sqrt{b^2 - b + a^2 - a - 4(a+b) + 4} - b + 2} < b$$

reduces to a polynomial inequality of degree 3 in a and degree 4 in b , and one can estimate the order of growth to be $a \leq \mathcal{O}(b^{4/3})$. \square

Remark 3.16. To give a feeling for the growth of the bound obtained, we note that if $b = 50$, then $a \leq 169$, and if $b = 100$, $a \leq 382$.

Remark 3.17. The values of bidegrees of smooth surfaces which we can currently construct which are closest to the bound $a \leq 3b$ are obtained by taking dependency loci of sections of $S^n(\mathcal{E}(1))$, obtaining a sequence of surfaces Y_n with ideal sheaf resolutions

$$0 \rightarrow \mathcal{O}_Q^{\oplus n} \rightarrow S^n(\mathcal{E}(1)) \rightarrow \mathcal{I}_Y(n(n+1)/2) \rightarrow 0.$$

For $n = 2, 3, 4$ and 5 , we obtain surfaces of bidegree $(2, 6)$, $(11, 21)$, $(35, 55)$, and $(85, 120)$ respectively. If Y_n has bidegree (a_n, b_n) , then we have

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 1.$$

If Y is a surface of bidegree $(3, n)$, it is known (see [15]) that $n \leq 7$. This suggests a better bound than $a \leq 3b$ is obtainable, but different methods must be applied.

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