Math. Ann. 306, 539–546 (1996)

Mathematische Annalen

An embedding of $\mathbb{C}$ in $\mathbb{C}^2$ with hyperbolic complement

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Received: 27 June 1995

Mathematics Subject Classification (1991): 32H02

1 Introduction

This paper was motivated by the question of whether there exists a proper holomorphic embedding $\Phi : \mathbb{C} \to \mathbb{C}^2$ such that $\mathbb{C}^2 - \Phi(\mathbb{C})$ is Kobayashi hyperbolic.

For the case of a polynomial embedding $P : \mathbb{C} \to \mathbb{C}^2$, Abhyankar and Moh [A-M] and Suzuki [S], proved that there exists a polynomial automorphism F of $\mathbb{C}^2$ such that $(F \circ P)(\mathbb{C}) = \mathbb{C} \times \{0\}$. In contrast, Forstneric, Globevnik and Rosay [F-G-R] have shown that there exists a proper holomorphic embedding $\Phi : \mathbb{C} \to \mathbb{C}^2$ such that for no automorphism H of $\mathbb{C}^2$ is it true that $(H \circ \Phi)(\mathbb{C}) = \mathbb{C} \times \{0\}$. This leaves open the question of whether the complement of an embedded copy of $\mathbb{C}$ can be Kobayashi hyperbolic. We answer this question in the following theorem.

Theorem 1.1 *There exists a proper holomorphic embedding $\Phi : \mathbb{C} \to \mathbb{C}^2$ such that $\mathbb{C}^2 - \Phi(\mathbb{C})$ is Kobayashi hyperbolic.*

This theorem is a corollary of Theorem 4.1. As additional corollaries, we obtain the following theorems.

Theorem 1.2 *There exists a proper holomorphic embedding of $\mathbb{C}$ into $\mathbb{C}^2$ such that any nonconstant holomorphic image of $\mathbb{C}$ intersects this embedding infinitely many times.*

Theorem 1.3 *Let X be any closed, 1-dimensional, complex subvariety of $\mathbb{C}^2$. Then there exists a Fatou-Bieberbach domain Ω containing X with $\overline{\Omega} \neq \mathbb{C}^2$ and $\Omega - X$ Kobayashi hyperbolic.*

Research at MSRI is supported in part by NSF grant DMS-9022140

By taking X to be the union of any finite set of affine complex lines, this last theorem answers a question of Rosay and Rudin [R-R], who constructed a Fatou-Bieberbach domain containing the coordinate axes and asked if a Fatou-Bieberbach domain can contain more than 2 complex lines.

2 Controlling images of the unit disk

We first prove a lemma showing that certain collections of linear disks give control on certain (nonlinear) maps from the unit disk to $\mathbb{C}^2$.

Given two concentric balls in $\mathbb{C}^2$, we find a collection of linear disks near the boundary of the inner ball with the following property: any holomorphic map from the unit disk into the larger ball which maps 0 near the center of the two balls and which avoids the linear disks must map most of the disk into the smaller ball. We make this more precise in the following lemma. The proof uses a normal families argument and is similar in spirit to the construction of non-tame sets in [R-R].

For notation, let π_j denote projection to the jth coordinate, $j = 1, 2$, let $\Delta(0, r)$ be the (open) disk of radius r centered at 0 in $\mathbb{C}$, and let $\mathbb{B}(0, r)$ denote the (open) ball of radius r centered at the origin in $\mathbb{C}^2$.

Lemma 2.1 *Let $k \in \mathbb{Z}^+$, $0 \leq n_1 < n_2 < n_3$. Let X be a closed, 1-dimensional, complex subvariety of $\mathbb{C}^2$. Then there exist finitely many affine complex linear maps $L_j : \mathbb{C} \to \mathbb{C}^2$ with $\overline{\Delta_j} := L_j(\overline{\Delta(0,1)}) \subseteq \mathbb{B}(0, n_2) - (\overline{\mathbb{B}(0, n_1)} \cup X)$ having pairwise disjoint closures such that if*

$$\phi : \Delta(0,1) \to \mathbb{B}(0, n_3) - \cup_j \Delta_j,$$

$$\phi(0) \in \overline{\mathbb{B}(0, n_1/2)},$$

with $\operatorname{dist}(\phi(0), X) \geq 1/k$, *then* $\phi(\Delta(0, 1 - 1/2^k)) \subseteq \mathbb{B}(0, n_2)$.

In fact we prove a stronger result allowing small perturbations in place of the linear disks obtained above.

Lemma 2.2 *With the hypotheses of Lemma 2.1, there exist disks $\Delta_j = L_j(\Delta(0, 1))$ as in that lemma, plus $\delta > 0$, such that if Δ_j' is the image of a holomorphic map which is within δ of L_j on $\Delta(0, 1)$ for all j, then for any*

$$\phi : \Delta(0,1) \to \mathbb{B}(0, n_3) - \cup_j \Delta_j',$$

$$\phi(0) \in \overline{\mathbb{B}(0, n_1/2)},$$

with $\operatorname{dist}(\phi(0), X) \geq 1/k$, *we have* $\phi(\Delta(0, 1 - 1/2^k)) \subseteq \mathbb{B}(0, n_2)$.

Proof of Lemma 2.2: Let $n_1 < r < n_2$, and let $\{a_j\}$ and $\{b_j\}$ be countable dense subsets of $\mathbb{B}(0,r) - \overline{\mathbb{B}(0,n_1)}$ and $\mathbb{B}(0,n_2) - \overline{\mathbb{B}(0,r)}$, respectively, such that $\pi_l a_j \neq \pi_l a_m$ if $j \neq m$, $l = 1, 2$, and likewise for b_j.

For each j, let $A_j(z) = (\alpha_j z, 0) + a_j$ and $B_j(w) = (0, \beta_j w) + b_j$, where $\alpha_j > 0$ is chosen maximal such that $A_j(\Delta(0,2)) \subseteq \mathbb{B}(0,r) - (\overline{\mathbb{B}(0,n_1)} \cup X)$, and analogously for β_j with disks in the outer shell.

For notation, we let $A'_{j,m}$ and $B'_{j,m}$ denote functions holomorphic and within $1/m$ of A_j and B_j on $\Delta(0,1)$. For such functions, let

$$D_m := \cup_{j=1}^m (A'_{j,m}(\Delta(0,1)) \cup B'_{j,m}(\Delta(0,1))).$$

To reach a contradiction, assume that for each m there exists D_m as above and $\phi_m : \Delta(0,1) \to \mathbb{B}(0,n_3) - D_m$, $\phi_m(0) \in \overline{\mathbb{B}(0,n_1/2)}$ with $\text{dist}(\phi_m(0), X) \geq 1/k$, and $\phi(\Delta(0, 1 - 1/2^k)) \not\subseteq \mathbb{B}(0,n_2)$.

The set $\{\phi_m\}$ is a normal family since the image of each ϕ_m lies inside a fixed ball, so we may assume that some subsequence converges to a map $\phi : \Delta(0,1) \to \mathbb{B}(0,n_3)$ with $\phi(0) \in \overline{\mathbb{B}(0,n_1/2)}$, $\text{dist}(\phi(0), X) \geq 1/k$, and $\phi(\Delta(0,1)) \not\subseteq \mathbb{B}(0,n_2)$. Then there exist nonempty open sets $\Omega_1, \Omega_2 \subseteq \Delta(0,1)$ such that $\phi(\Omega_1) \subseteq \mathbb{B}(0,r) - \overline{\mathbb{B}(0,n_1)}$ and $\phi(\Omega_2) \subseteq \mathbb{B}(0,n_2) - \overline{\mathbb{B}(0,r)}$.

We claim that $\pi_2\phi$ is constant. If not, then since $\phi(0) \notin X$, there exists $z_1 \in \Omega_1$ such that $\phi(z_1) \notin X$ and $(\pi_2\phi)'(z_1) \neq 0$, and there exists a subsequence $a_{j_l} \to \phi(z_1)$. Moreover, there exists $c > 0$ such that $\alpha_{j_l} \geq c$ for all l.

By continuity and the open mapping theorem, there exists a neighborhood V of z_1 such that $|\pi_1\phi(z) - \pi_1\phi(z_1)| < c/2$ for all $z \in V$ and such that $\pi_2\phi(V)$ is a neighborhood of $\pi_2\phi(z_1)$. For l large, $\pi_2 a_{j_l} \in \pi_2\phi(V)$, and hence by choice of V there exists $z_1' \in V$ such that $\phi(z_1') \in A_{j_l}(\Delta(0,1))$. But then $\phi(z) - A_{j_l}(\zeta)$ has an isolated zero in $\Delta^2(0,1)$, which persists under small perturbations. Hence ϕ_m must intersect D_m for large m, a contradiction, so $\pi_2\phi \equiv \mathit{const}$. A similar argument for $\pi_1\phi$ shows that $\phi \equiv \mathit{const}$, which contradicts $\phi(0) \in \overline{\mathbb{B}(0,n_1/2)}$ and $\phi(\Delta(0,1)) \not\subseteq \mathbb{B}(0,n_2)$.

Thus, no such sequence $\{\phi_m\}$ exists, so we obtain the lemma by taking $\{L_j\}$ to be some finite subset of $\{A_j\} \cup \{B_j\}$ and δ sufficiently small. □

3 Approximation of linear disks by complex subvarieties

In the following lemma, we use the notation $reg(X)$ to denote the set of regular points of a subvariety.

Lemma 3.1 *Let $K \subseteq \mathbb{C}^2$ be polynomially convex and let X be a closed, 1-dimensional, complex subvariety of $\mathbb{C}^2$. Let $L \subseteq X$ be compact such that for each $p \in \text{reg}(X) - (K \cup L)$ there exists a curve $\eta : [0,1) \to \text{reg}(X) - (K \cup L)$ with $\eta(0) = p$ and $\lim_{t\to 1} \|\eta(t)\| = \infty$.*

Then $K \cup L$ is polynomially convex.

Proof: Although the ideas in the proof are standard, we include it for convenience.

Let $p \in \mathbb{C}^2 - (K \cup L)$. If $p \notin X$, then there exists an entire g such that $g \equiv 0$ on X but $g(p) \neq 0$. Also, there exists f entire such that $|f| < 1$ on K but $|f(p)| > 1$. Then for some m, we have $|f^m g(p)| > 1$ and $|f^m g| < 1$ on $K \cup L$, and we can approximate $f^m g$ by a polynomial with the same properties.

If $p \in X$, then the preceding argument shows that if $K_p := K \cup L \cup \{p\}$, then $\hat{K}_p \subseteq K \cup X$, where $\hat{K}_p$ is the polynomial hull of K_p. The hypotheses on L imply that $X \cap K_p$ is Runge in X, so we can approximate functions holomorphic on $X \cap K_p$ by functions holomorphic on X.

Let $f(p) = 2$ and $f \equiv 1/2$ on $(X \cap K_p) - \{p\}$, and approximate f by a function g holomorphic on X such that $|f - g| < 1/4$ on the domain of f. By [G-R, Theorem 18, ch. VIII], we may extend g to be holomorphic on $\mathbb{C}^2$, and then we may restrict g to a neighborhood V of X such that $|g| < 1$ on $V \cap K$. Composing with a convex function σ, we obtain a psh function $\sigma \circ |g|$ which is 0 on a neighborhood of $(X \cap K_p) - \{p\}$ and larger than 1 at p. We can extend this function by 0 so that it is psh in a neighborhood of $K \cup X$, 0 on $K \cup L$, and larger than 1 at p, then restrict it to a Runge neighborhood of $\hat{K}_p$, which exists since $\hat{K}_p$ is polynomially convex.

Using standard $\overline{\partial}$ techniques, e.g. [H, 4.3.3, 4.3.4], it follows that there exists a polynomial P on $\mathbb{C}^2$ such that $|P(p)| > 1$ and $|P| \leq 1$ on $K \cup L$, so $K \cup L$ is polynomially convex. □

In the following lemma, we start with a closed, 1-dimensional, complex subvariety X of $\mathbb{C}^2$, an automorphism Ψ, and some linear disks contained outside $\Psi(X)$ and outside the ball of radius n. We construct an automorphism of $\mathbb{C}^2$ such that the image of $\Psi(X)$ under this automorphism contains pieces approximating each of the linear disks, and such that this automorphism is near the identity on $\mathbb{B}(0,n)$ and on some large piece of $\Psi(X)$. We do this in such a way that given some polynomially convex set contained in a ball disjoint from $\Psi(X)$ and $\mathbb{B}(0,n+1)$, the image of this set is disjoint from $\mathbb{B}(0,n+2)$.

For notation, let $\epsilon > 0$, $R, n \geq 0$, and let Ψ be an automorphism of $\mathbb{C}^2$. Let X be a closed, 1-dimensional, complex subvariety of $\mathbb{C}^2$, and let $L_j : \mathbb{C} \to \mathbb{C}^2$, $j = 1, \ldots, N$ be finitely many affine complex linear maps with $\Delta_j := L_j(\Delta(0,1))$ having pairwise disjoint closures such that $\overline{\Delta_j} \cap \Psi(X) = \emptyset$ and $\overline{\Delta_j} \cap \overline{\mathbb{B}(0,n)} = \emptyset$ for all j. Let $\mathbb{B}_0 = \mathbb{B}(p_0, r_0)$ be a ball with $\overline{\mathbb{B}_0} \cap (\overline{\mathbb{B}(0,n+1)} \cup \Psi(X) \cup \cup_j \overline{\Delta_j}) = \emptyset$, and let $K_0 \subseteq \mathbb{B}_0$ be polynomially convex and compact.

Lemma 3.2 *There exists $H : \mathbb{C}^2 \to \mathbb{C}^2$ an automorphism such that $\|H - I\| < \epsilon$ on $\mathbb{B}(0,n) \cup \Psi(\mathbb{B}(0,R) \cap X)$ and such that for each j, there is a relatively open subset σ_j of $H\Psi(X)$ such that the orthogonal projection of σ_j onto Δ_j is a diffeomorphism and such that σ_j is near Δ_j in the sense of Lemma 2.2. Moreover, there exists a ball $\mathbb{B}_0'$ with $\overline{\mathbb{B}_0'} \cap (\overline{\mathbb{B}(0,n+2)} \cup H\Psi(X)) = \emptyset$ and $H(K_0) \subseteq \mathbb{B}_0'$.*

Proof: Choose $p_1, \ldots, p_N \in \Psi(reg(X) - \overline{\mathbb{B}(0,R)}) - \overline{\mathbb{B}(0,n)}$ and choose pairwise disjoint C^2 curves $\gamma_j : [0,1] \to \mathbb{C}^2$, $j = 0, \ldots, N$ which are disjoint from $\Psi(X \cap \overline{\mathbb{B}(0,R)}) \cup \overline{\mathbb{B}(0,n)}$ such that $\gamma_j(0) = p_j$ and such that if $j \geq 1$, then $\gamma_j(1) = L_j(0)$, while $\|\gamma_0(1)\| > n+2$. Let W_j be a neighborhood of γ_j such that the W_j have pairwise disjoint closures which are also disjoint from $\Psi(X \cap \overline{\mathbb{B}(0,R)}) \cup \overline{\mathbb{B}(0,n)}$, and such that $\overline{\Delta_j} \subseteq W_j$ if $j \geq 1$ and $\overline{\mathbb{B}(p_0, r_0)} \subseteq W_0$. For $j \geq 1$, let $r_j > 0$ such that $\Psi(X) \cap \overline{\mathbb{B}(p_j, r_j)} \subseteq W_j$ and such that there is an embedding $g_j : \Delta(0,2) \to \mathbb{C}^2$ with $g_j(\overline{\Delta(0,1)}) = \Psi(X) \cap \overline{\mathbb{B}(p_j, r_j)}$.

Let $K = \Psi(X \cap \overline{\mathbb{B}(0,R)}) \cup \overline{\mathbb{B}(0,n)} \cup \overline{\mathbb{B}_0} \cup (\cup_{j=1}^N g_j(\overline{\Delta(0,1)}))$. In order to apply the approximation result in [F-R, Theorem 2.1], we will construct a family Φ_t of maps which are biholomorphic in a neighborhood of K and C^2 in t with $\Phi_0 = I$. We will construct Φ_t so that Φ_1 is the identity on $\mathbb{B}(0,n) \cup \Psi(X \cap \mathbb{B}(0,R))$ and maps each $g_j(\Delta(0,1))$ to a disk near Δ_j, and such that $\Phi_t(K)$ is polynomially convex for all t, then approximate Φ_1 by a global automorphism.

We show how to construct Φ_t around a neighborhood of p_1. For simplicity, assume $p_1 = 0$ and $g_1(0) = p_1$. Now, given any complex linear map L such that $L(\overline{\Delta(0,1)}) \subseteq W_1$ is tangent to $g_1(\Delta(0,1))$ at p_1, we can first use the family $g_1(\chi_1(t) g_1^{-1}(p))$ with $\chi_1(0) = 1$, $\chi_1(1)$ small and positive, to shrink $g_1(\Delta(0,1))$ to a nearly linear disk tangent to $L(\Delta(0,1))$, then use the family $A_1(t) g_1(\chi_1(1) g_1^{-1}(p))$ to expand the small disk to approximate $L(\Delta(0,1))$.

Note that by choosing $\chi_1(1)$ very small, we can make the diameter of $g_1(\chi_1(1)\Delta(0,1))$ as small as we like. In particular, we can translate this disk along γ_1 and use a one-parameter family of rotations to make the image tangent to Δ_j, then expand as before to make the new image approximate Δ_j. Note that all of this can be done within W_j, and that at each stage, the image of $\Delta(0,1)$ is contained in an affine linear image of X. Moreover, we can make the family C^2 in t. Reparametrizing, and using a similar construction for each p_j, we can define Φ_t on $\cup_{j=1}^N g_1(\overline{\Delta(z_j, r_j)})$. We can use a similar procedure to shrink $\overline{\mathbb{B}_0}$ and move it along γ_0 inside W_0 so that the image is $\overline{\mathbb{B}_0'}$ as in the statement.

Finally, each $g_j, j \geq 1$ can be extended to be a biholomorphic embedding of $\overline{\Delta^2(0,1)}$ into W_j, so we can use the same argument to define Φ_t on a neighborhood of $\cup_{j=1}^N g_j(\overline{\Delta(0,1)})$, and similarly extend it to a neighborhood of $\overline{\mathbb{B}_0}$. Define $\Phi_t \equiv I$ on a neighborhood of $\Psi(X \cap \overline{\mathbb{B}(0,R)}) \cup \overline{\mathbb{B}(0,n)}$ for all t.

Since the union of two disjoint closed balls is polynomially convex, and since the remaining hypotheses of the previous lemma are satisfied for all t, we see that $\Phi_t(K)$ is polynomially convex for each t. Hence by [F-R, Theorem 2.1], there is a neighborhood V of K such that Φ_1 can be approximated uniformly on V by automorphisms H of $\mathbb{C}^2$.

Choosing Φ_t such that $\Phi_1 g_j(\Delta(0,1))$ is close to Δ_j for all j and $\Phi_1(K_0) \subseteq \mathbb{B}_0'$, then choosing an automorphism H close to Φ_1, we obtain the lemma. □

4 Main theorem

Theorem 4.1 *Let X be a closed, 1-dimensional, complex subvariety of $\mathbb{C}^2$ and $\mathbb{B}_0$ a ball with $\overline{\mathbb{B}_0} \cap X = \emptyset$. Then there exists a domain $\Omega \subseteq \mathbb{C}^2 - \overline{\mathbb{B}_0}$ containing X and a biholomorphic map Φ from Ω onto $\mathbb{C}^2$ such that $\mathbb{C}^2 - \Phi(X)$ is Kobayashi hyperbolic. Moreover, all nonconstant images of $\mathbb{C}$ in $\mathbb{C}^2$ intersect $\Phi(X)$ in infinitely many points.*

Remark: Since Φ in the statement of the theorem is biholomorphic, it is necessarily a proper holomorphic embedding of X into $\mathbb{C}^2$.

Proof: The first step is to construct Φ as the limit of automorphisms of $\mathbb{C}^2$ which are constructed inductively. We will define a sequence Φ_m of automorphisms, numbers $R_m \nearrow \infty$, and finitely many affine complex linear disks Δ_j^m such that

(1_m) $\|\Phi_m(p)\| \geq m+1$ if $p \in X$ and $\|p\| \geq R_m$,
(2_m) $\|\Phi_m(p)\| \geq m+1$ if $p \in \overline{\mathbb{B}_0}$,
(3_m) $\|\Phi_{m+1} - \Phi_m\| \leq 1/2^m$ on $X \cap \overline{\mathbb{B}(0,R_m)}$,
(4_m) $\|\Phi_{m+1} \circ \Phi_m^{-1} - I\| \leq 1/2^m$ on $\overline{\mathbb{B}(0,m)}$ and
(5_m) if $l \leq m$ and $\phi : \Delta(0,1) \to \mathbb{B}(0,l+2) - \cup_j(\Delta_j^l)'$ as in Lemma 2.2 with $\phi(0) \in \overline{\mathbb{B}(0,l/2)}$ and $\text{dist}(\phi(0),\Phi_l(X)) \geq 1/l$, then $\phi(\Delta(0,1-1/2^l)) \subseteq \mathbb{B}(0,l+1)$.

Changing coordinates by a translation, we may assume $\overline{\mathbb{B}_0} \cap \overline{\mathbb{B}(0,1)} = \emptyset$, so we can take $\Phi_0 = I$, $R_0 = 1$, and choose a ball $\mathbb{B}_0^0$ such that $\overline{\mathbb{B}_0} \subseteq \mathbb{B}_0^0$ and $\overline{\mathbb{B}_0^0} \cap (\overline{\mathbb{B}(0,1)} \cup X) = \emptyset$.

For the inductive construction, suppose we have $m \geq 0$, Φ_l and R_l for $l \leq m$ and linear discs Δ_j^l for $l < m$ satisfying (1_l) and (2_l) for $l \leq m$ and (3_l), (4_l) and (5_l) for $l < m$. Suppose also that there exist balls $\mathbb{B}_0^l$, $l \leq m$, with $\Phi_l(\overline{\mathbb{B}_0}) \subseteq \mathbb{B}_0^l$ and $\overline{\mathbb{B}_0^l} \cap (\overline{\mathbb{B}(0,l+1)} \cup \Phi_l(X)) = \emptyset$, $l \leq m$. Note that $\Phi_m(\overline{\mathbb{B}_0})$ is polynomially convex since Φ_m is a global automorphism. Suppose also that $\Phi_m(X \cap \mathbb{B}(0,R_m))$ has nonsingular, relatively open subsets approximating each Δ_j^l as in Lemma 2.2.

We use Lemma 2.2 with m, $m+1$ and $m+2$ in place of n_1, n_2 and n_3, respectively, with $\Phi_m(X)$ in place of X, and with $k=m$. This gives finitely many affine complex linear $L_j^m : \mathbb{C} \to \mathbb{C}^2$ with $\overline{\Delta_j^m} := L_j^m(\overline{\Delta(0,1)}) \subset \mathbb{B}(0,m+1) - \overline{\mathbb{B}(0,m)}$ having pairwise disjoint closures such that if $\phi : \Delta(0,1) \to \mathbb{B}(0,m+2) - \cup_j(\Delta_j^m)'$ as in Lemma 2.2, and $\phi(0) \in \overline{\mathbb{B}(0,m/2)}$ with $\text{dist}(\phi(0),\Phi_m(X)) \geq 1/m$, then $\phi(\Delta(0,1-1/2^m)) \subseteq \mathbb{B}(0,m+1)$.

We then apply Lemma 3.2 with $R = R_m$, $n = m$, $\Psi = \Phi_m$, X unchanged, $\{L_j^m\}$ in place of $\{L_j\}$, $\mathbb{B}_0^m$ in place of $\mathbb{B}_0$, and $K_0 = \Phi_m(\overline{\mathbb{B}_0})$. This gives an automorphism $H : \mathbb{C}^2 \to \mathbb{C}^2$ such that $\|H - I\| \leq \epsilon$ on $\overline{\mathbb{B}(0,m)} \cup \Phi_m(X \cap \overline{\mathbb{B}(0,R_m)})$ and such that there are submanifolds σ_j^m of $H\Phi_m(X)$ which are close enough to the Δ_j^m for Lemma 2.2 and such that there is a ball $\mathbb{B}_0^{m+1}$ with $\overline{\mathbb{B}_0^{m+1}} \cap (\overline{\mathbb{B}(0,m+2)} \cup H\Phi_m(X)) = \emptyset$ and $H\Phi_m(\overline{\mathbb{B}_0}) \subseteq \mathbb{B}_0^{m+1}$.

By taking ϵ sufficiently small, (1_m) together with the fact that $\overline{\Delta_j^l} \subseteq \mathbb{B}(0,m)$ for $l < m$ implies that we can make $H\Phi_m(X)$ pass close to all of these disks, in the sense of Lemma 2.2. Thus, taking $\Phi_{m+1} = H\Phi_m$ and $R_{m+1} \geq R_m + 1$ large, we obtain the inductive condition on submanifolds of $\Phi_{m+1}(X)$, as well as $(1_{m+1}), (2_{m+1}), (3_m), (4_m)$ and (5_m).

To show that $\{\Phi_m\}$ converges to give a map $\Phi : \Omega \to \mathbb{C}^2$, let $\Omega_m := \Phi_m^{-1}(\mathbb{B}(0,m))$. If $p \in \Omega_m$, then $\|\Phi_{m+1}(p) - \Phi_m(p)\| = \|(\Phi_{m+1} \circ \Phi_m^{-1})(\Phi_m(p)) - I(\Phi_m(p))\| < \frac{1}{2^m}$. Hence $\|\Phi_{m+1}(p)\| < m + 1/2^m < m+1$, so $p \in \Omega_{m+1}$. It follows that the sequence $\{\Phi_m\}$ converges locally uniformly on $\Omega := \cup\Omega_m$ to a map Φ.

Suppose $p \in \Omega_{m+1} - \Omega_m$. Then

$$\begin{aligned} \|\Phi(p) - \Phi_{m+1}(p)\| &= \|\lim_{k\to\infty}(\Phi_k(p) - \Phi_{m+1}(p))\| \\ &\leq \sum_{k\geq m+1} \|\Phi_{k+1}(p) - \Phi_k(p)\| \\ &< \frac{1}{2^m} \end{aligned}$$

Hence $\Phi(p) \in \mathbb{B}(0, m+1+1/2^m) - \mathbb{B}(0, m-1/2^m)$. It follows that Φ is a proper map from Ω to $\mathbb{C}^2$. In particular, Φ must have maximal rank 2. Since Φ is a limit of automorphisms, it follows that Φ has nonvanishing Jacobian everywhere in Ω. A standard argument implies that Φ is injective on Ω, and (4_m) implies that $\mathbb{B}(0, m-1) \subseteq \Phi_k(\Omega_m)$ for $k \geq m$, so that Φ is surjective. Hence Φ is a biholomorphism from Ω onto $\mathbb{C}^2$.

From (2_m) we see that $\overline{\mathbb{B}_0} \cap \Omega_m = \emptyset$ for all m, hence $\overline{\mathbb{B}_0} \cap \Omega = \emptyset$.

If $p \in X$, then $p \in X \cap \overline{\mathbb{B}(0, R_m)}$ for some m. It follows from (3_m) that $\|\Phi_k(p)\| \leq \|\Phi_m(p)\| + \sum_{j=m}^{k-1} 1/2^j < \|\Phi_m(p)\| + 1$ for all $k > m$. Hence if k is large enough, $\|\Phi_k(p)\| < k$ and so $p \in \Omega_k$. Therefore $X \subset \Omega$.

Before continuing, note that the approximation of Δ_j^k by each $\Phi_m(X \cap \mathbb{B}(0, R_m))$ implies that $\Phi(X)$ has submanifolds which approximate each Δ_j^k in the sense of Lemma 2.2.

We show next that $\mathbb{C}^2 - \Phi(X)$ is Kobayashi hyperbolic. If not, there is a point p in $\mathbb{C}^2 - \Phi(X)$, a nonzero tangent vector $\xi \in T_p\mathbb{C}^2$, and a sequence of holomorphic maps $\phi_k : \Delta(0, k+1) \to \mathbb{C}^2 - \Phi(X)$ such that $\phi_k(0) = p_k$, $\phi_k'(0) = \xi_k$, and $(p_k, \xi_k) \to (p, \xi)$.

Since $p \notin \Phi(X)$, there is an integer $m_0 > 1$ such that $p \in \mathbb{B}(0, m_0/2)$ and $\operatorname{dist}(p, \Phi(X)) > 1/m_0$. From (1_m) and (3_m), there exist $k_1 > 1$, $m_1 > m_0$ such that $\operatorname{dist}(p_k, \Phi_m(X)) \geq 1/m_0$ for $k \geq k_1$, $m \geq m_1$.

Fixing $k \geq k_1$, we see that $\phi_k(\Delta(0, k)) \subseteq \mathbb{B}(0, m+2)$ for some $m \geq m_1$ large. Since ϕ_k misses $\Phi(X)$, we see that $\psi_k(z) := \phi_k(kz)$ maps $\Delta(0, 1)$ into $\mathbb{B}(0, m+2) - \cup_j(\Delta_j^m)'$, $\psi_k(0) = p_k$ and $\operatorname{dist}(\psi_k(0), \Phi_m(X)) \geq 1/m$. Thus $\psi_k(\Delta(0, 1 - 1/2^m)) \subset \mathbb{B}(0, m+1)$ by (5_m). By induction, we see that $\psi_k(\Delta(0, \Pi_{m\geq m_1}(1 - 1/2^m))) \subset \mathbb{B}(0, m_1+1)$. Hence there exists $r > 0$ independent of k such that $\|\psi_k(z)\| < m_1+1$ for all $|z| < r$. This implies that $\|\phi_k(z)\| < m_1 + 1$ for all $|z| < kr$ and hence that $\phi_k'(0) \to 0$, a contradiction. Thus, $\mathbb{C}^2 - \Phi(X)$ is Kobayashi hyperbolic.

Finally, if $g : \mathbb{C} \to \mathbb{C}^2$ is a nonconstant holomorphic map which intersects $\Phi(X)$ only finitely many times, then these points of intersection are contained in some large ball. By reparametrizing, we may assume that $g(0) \notin \Phi(X)$, then use an argument like the one just given to show that the image of g must be contained in some large ball, a contradiction. □

The theorems in Sect. 1 now follow immediately by taking X to be the z-axis for the first two theorems and to be any finite collection of complex lines for the third theorem. In Theorem 1.3 one can replace X by any countable union of closed subvarieties of $\mathbb{C}^2$ avoiding a fixed ball. The proof of Theorem 4.1 still goes through. Moreover, if X is a dense, countable union of closed subvarieties of $\mathbb{C}^2$ and S is any discrete set of points in the complement of X, then there exists a Fatou-Bieberbach domain Ω, $X \subset \Omega \subset \mathbb{C}^2 \setminus S$.

Note that Theorem 4.1 can be applied to any Riemann surface which admits a proper holomorphic embedding into $\mathbb{C}^2$, so in particular, there is an embedding of the disk whose complement is Kobayashi hyperbolic.

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