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On Finite Affine Planes of Rank 3

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1. Introduction. LÜNEBURG has proposed the problem of determining the finite affine planes admitting rank 3 groups of collineations – i.e., groups of collineations which act as rank 3 permutation groups on the points – observing that the members of his new family of planes [3], Section 12, have this property. In the present note we take the first step towards a solution of this problem by proving the following

Theorem. *If a finite affine plane Π admits a rank 3 group G of collineations then it is a translation plane and G contains the group T of all translations.*

As a consequence we prove

Corollary. *If Π admits a rank 3 group G of collineations such that G induces a regular group of permutations of the points at infinity then Π is a desarguesian plane of odd order.*

The proof of the Theorem is reduced to an application of a theorem of WAGNER [4]. We conjecture that if a rank 3 group induces a doubly transitive group on the points at infinity then the plane is a Lüneburg plane.

Throughout the paper Π denotes a finite affine plane of order n and G denotes a rank 3 group of collineations of Π. When convenient we regard Π as being obtained from a projective plane Π' by specializing a line l_∞ to be the line at infinity, and then we regard G as a group of collineations of Π'. But the terms "point" and "line" will always refer to the points and lines of Π, the points of l_∞ being referred to as "points at infinity".

2. Proof of the Theorem. We use the notation of [1] for rank 3 permutation groups, denoting by $\Delta(P)$ and $\Gamma(P)$ the two G_P-orbits $\neq \{P\}$ of points, for a given point P, with $\Delta(P)^g = \Delta(P^g)$ and $\Gamma(P)^g = \Gamma(P^g)$ for all $g \in G$. By the theorem of WAGNER [4] it suffices to prove that G is flag-transitive, which we now do.

If G has odd order then for P a point, $|\Delta(P)| = |\Gamma(P)| = (n^2-1)/2$, and this number divides the order of G and so must be an odd integer, which is impossible. Hence the order of G is even and therefore Δ and Γ are self-paired.

Now suppose that no line through P meets both $\Delta(P)$ and $\Gamma(P)$. Then the lines through P fall into two classes, Δ_P and Γ_P, such that

$$X \in \Delta(P) \Leftrightarrow XP \in \Delta_P,$$

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and

$$Y \in \Gamma(P) \Leftrightarrow YP \in \Gamma_P.$$

If $|\Delta_P| = t$ then $|\Delta(P)| = t(n-1)$, $|\Gamma_P| = n-t+1$ and $|\Gamma(P)| = (n-t+1)(n-1)$. Now

$$XP \in \Delta_P \Leftrightarrow X \in \Delta(P) \Leftrightarrow P \in \Delta(X) \Leftrightarrow XP \in \Delta_X$$

and hence

$$\lambda = |\Delta(P) \cap \Delta(X)| = (t-1)^2 + n - 2.$$

Similarly, if $Y \in \Gamma(P)$ then YP is in neither Δ_P nor Δ_Y so $\mu = |\Delta(P) \cap \Delta(Y)| = t^2$. By [1], Lemma 5, we have

$$(t-1)(n-t)\,t(n-1) = t^2(n-t+1)(n-1)$$

giving $n=0$, which is impossible. Hence there is a line l through P which meets both $\Delta(P)$ and $\Gamma(P)$, and it follows that G_P is transitive on the set of lines through P. Hence G is flag-transitive, completing the proof of the Theorem.

3. Proof of the Corollary. Some preparation is needed before we introduce the assumption about the action of G on the points at infinity. By the Theorem, Π is a translation plane and G contains the group T of all translations of Π. So we must have $n = q$, a prime power, and $G = TG_P$ for any point P.

Let P, l be an incident point-line pair in Π and let P_∞ be the point on l_∞ incident with l. Then $G_{P_\infty} \geqq T$ so $G_{P_\infty} = TG_{P,l}$, and $G_l = T_l G_{P,l}$.

Now we determine the G_l-orbits of lines in Π. Since G_P has four point orbits in Π' it has four line orbits in Π' and therefore three line orbits in Π. Hence G_l has three point orbits in Π. Let ρ be the number of G_l-orbits of points in l_∞. Then G_l has $3+\rho$ point orbits in Π' and hence $2+\rho$ line orbits in Π. On the other hand, G_l stabilizes the following $1+\rho$ sets of lines in Π:

$\{l\}$, the set of lines $\neq l$ parallel to l, and the $\rho-1$ sets of lines meeting l_∞ in the $\rho-1$ point orbits $\neq \{P_\infty\}$ of G in l_∞.

Each G_l-orbit of lines $\neq l$ through P_∞ is a $G_{P,l}$-orbit since $G_l = T_l G_{P,l}$ and T_l fixes every line through P_∞. Suppose that there is just one such orbit. Then every line $\neq l$ parallel to l carries the same number, say a, of points of $\Delta(P)$. Hence

$$|\Delta(P)| = (q-1)\,a + b$$

where $b = |l \cap \Delta(P)|$. On the other hand, since each line through P meets $\Delta(P)$ in b points,

$$|\Delta(P)| = (q+1)\,b$$

so $b = q-1$, which is impossible. It follows that the G_l-orbits of lines in Π are the listed sets, except that the set of lines $\neq l$ parallel to l is the union of two orbits $A(l)$ and $B(l)$. As we remarked above, $A(l)$ and $B(l)$ are $G_{P,l}$-orbits.

Now assume that G induces a regular group of permutations on l_∞. Then the same is true of G_P, and the elements $\neq 1$ of $G_{P,l}$ are (P, l_∞)-homologies. Hence for $m \in A(l)$ and $n \in B(l)$,

$$G_{P,l,m} = G_{P,l,n} = 1, \quad \text{so} \quad |A(l)| = |B(l)| = G_{P,l},$$

and therefore

$$|G_{P,l}|=(q-1)/2.$$

The existence of a group of (P, l_∞)-homologies of this order implies that Π is desarguesian (cf. [3], Section 11), proving the Corollary.

4. Remarks. (1) In the case of the Corollary it can be seen that G_P is cyclic of order $(q^2-1)/2$.

(2) Since G is flag-transitive by the Theorem, it is primitive on the points of Π by [2], Proposition 3, and T is its unique minimal normal subgroup.

(3) It is easily seen that a group of collineations of a finite affine plane Π has rank 3 on the lines of Π if and only if it is doubly transitive on the points Π and on the points at infinity.

References

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