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On the Hyperinvariant Subspaces for Isometries

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Let $\mathscr{H}$ be a complex Hilbert space. A (bounded linear) operator V on $\mathscr{H}$ is an isometry if $\|Vx\|=\|x\|$ for x in $\mathscr{H}$. In this note we determine the hyperinvariant subspaces for a general isometry. Recall that a subspace $\mathscr{M}$ of $\mathscr{H}$ is said to be *hyperinvariant* for an operator T on $\mathscr{H}$ if $\mathscr{M}$ is invariant for every operator that commutes with T. This notion was introduced in [5] and [2], where the hyperinvariant subspaces for various classes of operators were determined. In particular, it was shown in [2] that the hyperinvariant subspaces for a unitary operator (in fact, any normal operator) are the spectral subspaces and in [5] (without proof) and [2] that the hyperinvariant subspaces for the unilateral shift U_+ on $\mathsf{H}_{\mathscr{D}}$ are the subspaces $q\,\mathsf{H}_{\mathscr{D}}$, where q is a scalar inner function.

Before stating the characterization for general isometries we need the following facts about isometries and unitary operators. For a Hilbert space $\mathscr{D}$ we let $\mathsf{H}_{\mathscr{D}}$ denote the space of functions f from the non negative integers Z^+ to $\mathscr{D}$ so that $\sum_{n=0}^{\infty}\|f(n)\|^2<\infty$. The space $\mathsf{H}_{\mathscr{D}}$ is a Hilbert space with respect to pointwise addition and scalar multiplication and the inner product $\langle f,g\rangle=\sum_{n=0}^{\infty}(f(n),g(n))$. The unilateral shift U_+ is defined on $\mathsf{H}_{\mathscr{D}}$ so that

$$(U_+ f)(n)=\begin{cases}0, & n=0\\ f(n-1), & n>0\end{cases}, \qquad \text{for } f \text{ in } \mathsf{H}_{\mathscr{D}}.$$

The operator U_+ is an isometry and its adjoint, the backward shift, satisfies $(U_+^* f)(n)=f(n+1)$ for f in $\mathsf{H}_{\mathscr{D}}$. The sequence $\{U_+^{*n}\}$ converges strongly to 0. The minimal unitary extension U of U_+ is the bilateral shift defined on $\mathsf{L}_{\mathscr{D}}$, where $\mathsf{L}_{\mathscr{D}}$ is the space of function f from the integers Z to $\mathscr{D}$ so that $\sum_{n=-\infty}^{\infty}\|f(n)\|^2<\infty$ and U is defined $(Uf)(n)=f(n-1)$ for f in $\mathsf{L}_{\mathscr{D}}$. It is easily verified that U is unitary and if we identify $\mathsf{H}_{\mathscr{D}}$ as a subspace of $\mathsf{L}_{\mathscr{D}}$ in the obvious way, then $U_+=U|\mathsf{H}_{\mathscr{D}}$.

A result due to von Neumann [4] states that every isometry is of the form $U_+\oplus W$ on $\mathsf{H}_{\mathscr{D}}\oplus\mathscr{K}$, where U_+ is the unilateral shift on $\mathsf{H}_{\mathscr{D}}$ and W is a unitary operator on $\mathscr{K}$.

A further decomposition of the unitary part into its absolutely continuous and singular parts will be of interest. If W is a unitary operator on $\mathscr{K}$ with spectral measure $E(\delta)$, then W is said to be *absolutely continuous [singular]* if the measure $\mu(\delta)=(E(\delta)x,x)$ is absolutely continuous [singular] for each vector x in $\mathscr{K}$. If W is a unitary operator on $\mathscr{K}$, then $\mathscr{K}=\mathscr{K}_a \oplus \mathscr{K}_s$, where $\mathscr{K}_a$ and $\mathscr{K}_s$ are reducing subspaces for W so that $W/\mathscr{K}_a$ is absolutely continuous while $W/\mathscr{K}_s$ is singular. The operator $W/\mathscr{K}_a$ is said to be the absolutely continuous part of W. (See [3] for details and proofs.)

Theorem. *Let V be an isometry on $\mathscr{H}$ and let $\mathscr{H}=\mathsf{H}_{\mathscr{D}} \oplus \mathscr{K}_1 \oplus \mathscr{K}_2$ be the unique decomposition of $\mathscr{H}$ into reducing subspaces for V so that $U_+=V/\mathsf{H}_{\mathscr{D}}$ is the unilateral shift on $\mathsf{H}_{\mathscr{D}}$, $W_1=V/\mathscr{K}_1$ is an absolutely continuous unitary operator and $W_2=V/\mathscr{K}_2$ is a singular unitary operator. The hyperinvariant subspaces for V are of the form $\mathscr{M} \oplus \mathscr{K}_1 \oplus F\mathscr{K}_2$ or $(0) \oplus E\mathscr{K}_1 \oplus F\mathscr{K}_2$, where $\mathscr{M}$ is hyperinvariant for U_+, E is a spectral-projection for W_1, and F is a spectral projection for W_2.*

Proof. We begin by observing that the hyperinvariant subspaces for a direct sum $T_1 \oplus T_2 \oplus T_3$ are always of the form $\mathscr{M}_1 \oplus \mathscr{M}_2 \oplus \mathscr{M}_3$, where each of $\mathscr{M}_1$, $\mathscr{M}_2$, and $\mathscr{M}_3$ is hyperinvariant for the respective operator. This is because the projections $I \oplus 0 \oplus 0$, $0 \oplus I \oplus 0$, and $0 \oplus 0 \oplus I$ and the operators $S_1 \oplus 0 \oplus 0$, $0 \oplus S_2 \oplus 0$, and $0 \oplus 0 \oplus S_3$ commute with $T_1 \oplus T_2 \oplus T_3$ where S_i is any operator which commutes with T_i. Thus we may assume that a hyperinvariant subspace for V is of the form $\mathscr{M} \oplus E\mathscr{K}_1 \oplus F\mathscr{K}_2$, where $\mathscr{M}$ is hyperinvariant for U_+ and E and F are spectral projections for W_1 and W_2, respectively. Our task is now to decide which of these subspaces is hyperinvariant for V.

To this end we need to determine the commutant for V. The unitary extension of V is $U \oplus W_1 \oplus W_2$ on $\mathsf{L}_{\mathscr{D}} \oplus \mathscr{K}_1 \oplus \mathscr{K}_2$, where U is the bilateral shift on $\mathsf{L}_{\mathscr{D}}$. From Corollary 5.4 of [1] (cf. [6]) it follows that if A_+ commutes with V, then $A_+=A/\mathscr{H}$ where A is an operator on $\mathsf{L}_{\mathscr{D}} \oplus \mathscr{K}_1 \oplus \mathscr{K}_2$ which commutes with $U \oplus W_1 \oplus W_2$. Since a bilateral shift is an absolutely continuous unitary operator (its spectral measure is equivalent to Lebesque measure), from Lemma 4.1 of [1] it follows that the matrix for A relative to the decomposition $\mathsf{L}_{\mathscr{D}} \oplus \mathscr{K}_1 \oplus \mathscr{K}_2$ is of the form

$$\begin{pmatrix} A_1 & A_2 & 0 \\ A_3 & A_4 & 0 \\ 0 & 0 & A_5 \end{pmatrix}.$$

Thus it is clear that the spectral projection F is independent of $\mathscr{M}$ and E. Thus we can confine our attention to the subspace $\mathsf{H}_{\mathscr{D}} \oplus \mathscr{K}_1$ and the matrix

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}.$$

If this matrix is to represent an operator which commutes with

$$\begin{pmatrix} U & 0 \\ 0 & W_1 \end{pmatrix},$$

then A_1 must commute with U, A_4 must commute with W_1, and A_2 and A_3 must satisfy $A_2 W_1 = U A_2$ and $A_3 U = W_1 A_3$. Further, if this is to be the extension of an operator which commutes with $U_+ \oplus W_1$, then it must leave $\mathsf{H}_{\mathscr{D}} \oplus \mathscr{K}_1$

invariant. This implies upon computing that $A_2\mathscr{K}_1$ must be contained in $\mathsf{H}_{\mathscr{D}}$. But by Lemma 4.1 of [1] the closure of $A_2\mathscr{K}_1$ is a reducing subspace for U. Since the only reducing subspace for U contained in $\mathsf{H}_{\mathscr{D}}$ is (0) it follows that $A_2=0$. Lastly, we have that $\mathsf{H}_{\mathscr{D}}$ is an invariant subspace for A_1.

The preceding argument is reversible to obtain that the commutant of $U_+ \oplus W_1$ consists of the restriction of operators with matrix

$\begin{pmatrix} A_1 & 0 \\ A_3 & A_4 \end{pmatrix}$, where $A_1 U = U A_1$, $A_1 \mathsf{H}_{\mathscr{D}} \subset \mathsf{H}_{\mathscr{D}}$, $A_3 U = W_1 A_3$ and $A_4 W_1 = W_1 A_4$.

We return now to the problem of determining which subspaces of the form $\mathscr{M} \oplus E\mathscr{K}_1$ are invariant under these operators. If $f \oplus g$ is in $\mathscr{M} \oplus E\mathscr{K}_1$, then

$$\begin{pmatrix} A_1 & 0 \\ A_3 & A_4 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}=\begin{pmatrix} A_1 f \\ A_3 f + A_4 g \end{pmatrix}$$

so that $\mathscr{M}$ can be any hyperinvariant subspace for U_+. If $\mathscr{M}=(0)$, then $E\mathscr{K}_1$ can be any spectral subspace for W_1. The proof will be complete if we show for $\mathscr{M} \neq (0)$ that the set of vectors of the form $A_3 f$, where A_3 satisfies $A_3 U = W_1 A_3$ and f is in $\mathscr{M}$, is dense in $\mathscr{K}_1$.

Choose x in $\mathscr{K}_1$ and let $\mathscr{G}$ be the minimal reducing subspace for W_1 containing x. Since W_1 is absolutely continuous, there exists a Borel set γ contained in the unit circle so that if we define $\nu(\delta)=m(\delta \cap \gamma)$ where m is normalized Lebesgue measure, then $W_1/\mathscr{G}$ is unitarily equivalent to the operator L_z defined as multiplication by z on $L^2(\nu)$. Let C be an isometry from $\mathscr{G}$ onto $L^2(\nu)$ so that $L_z C = C W_1/\mathscr{G}$.

If $\mathscr{M} \neq (0)$, then $\mathscr{M} \neq U\mathscr{M}$ and if f is a unit vector in $\mathscr{M} \ominus U\mathscr{M}$, then $\{U^n f\}_{n=-\infty}^{\infty}$ is an orthonormal subset of $\mathscr{K}_1$. The subspace $\mathscr{F}$ of $\mathscr{K}_1$ spanned by the vectors $\{U^n f\}_{n=-\infty}^{\infty}$ reduces W_1. For k a bounded function in $L^2(\nu)$, we define an operator D_k from $\mathsf{L}_{\mathscr{D}}$ to $\mathscr{K}_1$ as follows: for h orthogonal to $\mathscr{F}$ we set $D_k h=0$; for $h=\sum_{n=-\infty}^{\infty} \alpha_n U^n f$, we set $D_k h = C^*\left(k \sum_{n=-\infty}^{\infty} \alpha_n e^{int}\right)$. To show that D_k is bounded observe that

$$\begin{aligned}
\|D_k h\|^2 &= \left\| C^*\left(k \sum_{n=-\infty}^{\infty} \alpha_n e^{int}\right)\right\|^2 = \left\| k\left(\sum_{n=-\infty}^{\infty} \alpha_n e^{int}\right)\right\|^2 = \frac{1}{2\pi}\int_{\gamma} \left| k\left(\sum_{n=-\infty}^{\infty} \alpha^n e^{int}\right)\right|^2 dm \\
&\leqq \|k\|_\infty^2 \frac{1}{2\pi}\int_{\gamma} \left|\sum_{n=-\infty}^{\infty} \alpha_n e^{int}\right|^2 dm \leqq \|k\|_\infty^2 \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{n=-\infty}^{\infty} \alpha_n e^{int}\right|^2 dm \\
&= \|k\|_\infty^2 \left(\sum_{n=-\infty}^{\infty} |\alpha_n|^2\right) = \|k\|_\infty^2 \left\|\sum_{n=-\infty}^{\infty} \alpha_n U^n f\right\|^2 = \|k\|_\infty^2 \, \|h\|^2.
\end{aligned}$$

An easy computation shows that $D_k U = W_1 D_k$ so that the operator on $\mathsf{H}_{\mathscr{D}} \oplus \mathscr{K}_1$ with matrix

$$\begin{pmatrix} 0 & 0 \\ D_k & 0 \end{pmatrix}$$

commutes with $U_+ \oplus W_1$. Moreover

$$\begin{pmatrix} 0 & 0 \\ D_k & 0 \end{pmatrix} \begin{pmatrix} f \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ C^* k \end{pmatrix}$$

so that the set of vectors of the form $A_3(f \oplus 0)$, where A_3 commutes with $U_+ \oplus W_1$ and f is in $\mathscr{M}$, is dense in $\mathscr{G}$. Since x was arbitrary, the set is dense in $\mathscr{K}_1$ and the proof is complete.

The preceding argument could be used to prove the "same result" for those subnormal operators having the property that every commuting map "lifts" to the minimal normal extensions. For a general subnormal the above argument shows that every hyperinvariant subspace is of this form but the possibility of further operators in the commutant obviates the argument showing that each of these subspaces is hyperinvariant.

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