Math. Z. 156, 85–91 (1977)

Mathematische Zeitschrift

Strongly Annular Functions with Small Taylor Coefficients

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1. Introduction

A holomorphic function f in the unit disk D is *strongly annular* provided there exists a sequence $\{r_m\}$ such that $r_m \nearrow 1$ and

$$\lim_{m\to\infty} \min_{|z|=r_m} |f(z)| = \infty.$$

The simplest examples of strongly annular functions have the form

$$f(z) = \sum_{m=1}^{\infty} c_m z^{k_m}, \tag{1}$$

where the sequence of coefficients c_m increases fast enough so that

$$|c_1| + \cdots + |c_{m-1}| = o(c_m),$$

and where the sequence of exponents k_m increases so rapidly that on the circle $|z| = 1 - 1/k_m$ the m^{th} term of the series in (1) is much larger than the sum of the later terms. (See [4, Chapter 2, Section 13].)

It is natural to ask whether unboundedness of the sequence of Taylor coefficients is necessary for strong annularity, and whether the Taylor series of a strongly annular function must be almost lacunary in an appropriate sense (see [1, p. 59, Question 6.9]).

The answer to both questions is negative. For instance, Howell [3] has recently shown that if we impose the topology of uniform convergence on compact subsets of D on the space of functions $\sum_0^{\infty} (\pm 1) z^n$, then the strongly annular functions in that space constitute a residual subset of the space.

We shall exhibit an explicit counterexample for both questions. Our auxiliary computations yield new information about the Taylor coefficients of the k^{th} power b^k of a univalent function b from D onto D.

If in (1) we replace z with its Möbius transform

$$b(z)=\frac{1/2-z}{1-z/2},$$

our function f takes the form

$$f(z)=\sum_{m=1}^{\infty} c_m[b(z)]^{k_m}=\sum_{n=0}^{\infty} a_n z^n.$$

We shall use the interchangeable symbols $\alpha(k,n)$ and α_{kn} to denote the n^{th} Taylor coefficient (at the origin) of the function $[b(z)]^k$. With this notation, we can write the Taylor coefficients of our new function f in the form

$$a_n=\sum_{m=1}^{\infty} c_m\,\alpha(k_m,n). \tag{2}$$

In Section 2, we analyze the sequences $\{\alpha_{kn}\}_{n=0}^{\infty}$ $(k=1,2,\ldots)$. We find that, with the notation

$$A(k)=\max_n |\alpha_{kn}|,$$

the order of magnitude of $A(k)$ is $k^{-1/3}$. The coefficients α_{kn} whose order of magnitude is $k^{-1/3}$ occur in long blocks, and their indices n have the order of magnitude k.

In Section 3, we choose the sequences $\{c_m\}$ and $\{k_m\}$ that produce our example. Indeed, if $c_m=k_m^{1/3}m^{-1/2}$ and $k_m\to\infty$ fast enough, then $a_n\to 0$, but $a_n=\Omega(m^{-1/2})$ for many indices n near $3\,k_m$. It follows that f is not the sum of a bounded function and a function whose Taylor series at the origin is even mildly lacunary.

2. Bounds on the Taylor Coefficients

Theorem 1. *There exist positive numbers A_1 and A_2 such that for all k and n the coefficients α_{kn} in* (2) *satisfy the inequality*

$$|\alpha_{kn}|<A_1 k^{-1/3}, \tag{3}$$

and such that, for each nonnegative integer j,

$$\liminf_{k\to\infty} k^{1/3}|\alpha(k,3k+j)|>A_2. \tag{4}$$

In addition to estimates on the largest coefficients in the Taylor series for b^k, we need upper bounds on the coefficients outside of certain blocks.

Theorem 2. *The coefficients α_{kn} in* (2) *satisfy the inequalities*

$$|\alpha_{kn}|\leqq 6/(k-3n)\pi \qquad (n<k/3), \tag{5}$$

$$|\alpha_{kn}|\leqq 2/(n-3k)\pi \qquad (n>3k). \tag{6}$$

We devote the remainder of this section to the proofs of Theorems 1 and 2, and we suggest that readers who wish to avoid computations proceed to Section 3, where we apply the theorems to the construction of our example.

It is convenient to begin with the proof of Theorem 2. By Cauchy's formula,

$$\alpha_{kn}=\frac{1}{2\pi}\int_{-\pi}^{\pi}[b(e^{i\theta})]^k e^{-in\theta}d\theta.$$

To put the integrand into manageable form, we observe that because $|b(e^{i\theta})|=1$ and $b(-1)=1$, we may write $b(e^{i\theta})=\exp[i\psi(\theta)]$, where ψ is a continuously differentiable function and $\psi(-\pi)=0$. Since

$$b'(z)=\frac{-3/4}{(1-z/2)^2} \quad \text{and} \quad \psi'(\theta)=|b'(e^{i\theta})|,$$

we see that

$$\psi'(\theta)=\frac{3}{4|1-e^{i\theta}/2|^2}=\frac{3}{5-4\cos\theta}.$$

Because the coefficients α_{kn} are real, Cauchy's formula implies that

$$\alpha_{kn}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos[k\psi(\theta)-n\theta]d\theta. \tag{7}$$

Letting $g(\theta)$ denote the expression in brackets in the integrand, and observing that $g(-\pi)=n\pi$, we now have the formulas

$$\alpha_{kn}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos g(\theta)\,d\theta$$

and

$$\begin{aligned} g(\theta) &= \int_{-\pi}^{\theta}\left(\frac{3k}{5-4\cos t}-n\right)dt+n\pi,\\ g'(\theta) &= \frac{3k}{5-4\cos\theta}-n,\\ g''(\theta) &= \frac{-12k\sin\theta}{(5-4\cos\theta)^2}. \end{aligned} \tag{8}$$

Since $g''(\theta)$ and $\sin\theta$ have opposite signs, the maximum and minimum values of g' on $[-\pi,\pi]$ are

$$g'(0)=3k-n \quad \text{and} \quad g'(\pi)=g'(-\pi)=\frac{k}{3}-n.$$

We note that g' is an even function and that it decreases in $[0,\pi]$.

To obtain the bounds (5) and (6), we shall use the following proposition from calculus.

Lemma. *Suppose that g is a real-valued function on the interval* $[a,b]$, *that* g' *exists and is monotone on* $[a,b]$, *and that* $|g'(x)| \geqq B > 0$ *on* $[a,b]$. *Then*

$$\left|\int_a^b \cos g(x)\,dx\right| \leqq 2/B.$$

In proving the lemma, we can obviously restrict ourselves to the case where g' is increasing and $g'(a) \geqq B$. Making the change of variable $t = g(x)$ and applying Bonnet's form of the second mean-value theorem [2, p. 311], we obtain the relations

$$\int_a^b \cos g(x)\,dx = \int_\alpha^\beta \frac{\cos t\,dt}{g'(x(t))} = \frac{1}{g'(a)} \int_\alpha^\xi \cos t\,dt,$$

where ξ denotes an appropriate point between α and β. The lemma now follows immediately.

To establish the bound (5) in Theorem 2, we apply the lemma to our function g on the intervals $[-\pi, 0]$ and $[0, \pi]$, with $B = k/3 - n$. To establish (6), we proceed similarly, using the value $B = n - 3k$. This completes the proof of Theorem 2.

In the proof of Theorem 2, we were able to use our lemma because for $n < k/3$ and $n > 3k$, the derivative g' has no zeros on the interval $[-\pi, \pi]$. In the proof of Theorem 1, we may have to cope with one or two zeros of g'. Moreover, even if g' has no zeros, the minimum value of $|g'(\theta)|$ may be so small that the lemma does not give the inequality (3), which for some values of n is much stronger than (5) and (6).

Let θ_0 denote the point in $[0, \pi]$ where $|g'(\theta)|$ takes its minimum value. Clearly, the contribution from the interval $[\theta_0 - k^{-1/3}, \theta_0 + k^{-1/3}]$ to the integral in (7) is less than $2k^{-1/3}$.

In the set $[0, \pi] \setminus [\theta_0 - k^{-1/3}, \theta_0 + k^{-1/3}]$, the value of $|g'(\theta)|$ is

$$|g'(\theta_0)| + \left|\int_{\theta_0}^{\theta} g''(t)\,dt\right|.$$

By virtue of the third formula in (8), the second term in this expression is greater than

$$\left|\int_{\theta_0}^{\theta_0 \pm k^{-1/3}} \frac{12k \sin t\,dt}{81}\right|,$$

where the sign is chosen so that the point $\theta_0 \pm k^{-1/3}$ lies in $[0, \pi]$. Thus, with an obvious choice of the ambiguous sign, we obtain for all θ in the interval the inequalities

$$\begin{aligned} |g'(\theta)| &> \left|\int_{\theta_0}^{\theta_0 \pm k^{-1/3}} g''(t)\,dt\right| \geqq \left|\int_{\theta_0}^{\theta_0 \pm k^{-1/3}} \frac{12k \sin t\,dt}{81}\right| \\ &\geqq \int_0^{k^{-1/3}} \frac{12k}{81}\,\frac{2t}{\pi}\,dt = 4k^{1/3}/27\pi. \end{aligned}$$

Applying the lemma and using similar considerations for the interval $[-\pi, 0]$, we arrive at the estimate (3).

The proof of the second part of Theorem 1 is more delicate. Before giving its details, we point out that if $g'(\theta)$ is fairly small at a point where $g''(\theta)=0$, then near that point the integrand in (7) changes so slowly that α_{kn} may be relatively large.

We let j denote a nonnegative integer, and we consider the coefficient $\alpha(k,n) =\alpha(k,3k+j)$, where k is much larger than j. It is convenient to introduce the function h defined by the equation

$$g(\theta)=k\pi-h(\theta).$$

We can easily verify that $h(0)=0$; since h' is an even function, it follows that h is odd. By virtue of the second formula in (8), we can write

$$h'(\theta)=j+3k\left[1-\frac{1}{1+4(1-\cos\theta)}\right]. \tag{9}$$

Let θ_1 and θ_2 denote the first two points in $[0,\pi]$ where $\cos h(\theta)=0$. Then

$$\int_0^\pi \cos h(\theta)\,d\theta=\left(\int_0^{\theta_1}+\int_{\theta_1}^{\theta_2}+\int_{\theta_2}^{\pi}\right)\cos h(\theta)\,d\theta.$$

Because h' is an increasing function in $[0,\pi]$, the first integral on the right is greater than $2\theta_1/\pi$. The second integral is negative, and its absolute value is less than $\theta_2-\theta_1$. With the change of variable $t=h(\theta)$, the third integral takes the form

$$\int_{h(\theta_2)}^{h(\pi)}\frac{\cos t\,dt}{h'(\theta(t))}.$$

Because the denominator in the integrand is an increasing function of t, we can regard this integral as a finite series with decreasing terms of alternate signs, the first term being positive. In other words, the third integral is positive; we shall ignore it.

To obtain a positive lower bound on the sum of the first two of our integrals, we need estimates on θ_1 and θ_2. By (9), the function h' has a Taylor series

$$h'(\theta)=j+3k(2\theta^2+\cdots),$$

and since $h(0)=0$, it follows that

$$h(\theta)=j\theta+2k(\theta^3+\cdots).$$

Consequently, in the particular case where $j=0$,

$$\theta_1=(\pi/4k)^{1/3}[1+o(1)],$$
$$\theta_2=3^{1/3}\theta_1[1+o(1)].$$

Moreover, these formulas hold also for positive values of j, provided $j=o(k^{1/3})$. Therefore

$$|\alpha(k,3k+j)|>\frac{1}{2\pi}\left[\frac{2\theta_1}{\pi}-(\theta_2-\theta_1)\right]\sim\frac{\theta_1}{2\pi}\left[\frac{2}{\pi}-(3^{1/3}-1)\right].$$

Because the last expression in brackets is positive and $\theta_1\sim(\pi/4k)^{1/3}$, this establishes the relation (4), and the proof of Theorem 1 is complete.

Remarks. 1. We have shown that the Taylor series of b^k has a long block of relatively large coefficients with index near $3k$. Obviously, a similar block occurs near $n=k/3$. It is fairly easy to verify that the two blocks extend to both sides of $3k$ and $k/3$, respectively.

2. A careful inspection of our proof shows that we can make the second assertion in Theorem 1 slightly stronger: There exists a positive constant η such that $|\alpha(k, 3k+j)| > A_2 k^{-1/3}$ whenever k is large enough and $0 \leqq j \leqq \eta k^{1/3}$.

3. Theorems 1 and 2 have obvious analogues for the more general functions

$$\left(\frac{w-z}{1-\bar{w}z}\right)^k = \sum_{n=0}^{\infty} \alpha(k, n, w) z^n,$$

where w denotes any point in the unit disk ($w \neq 0$). Naturally, some of the parameters in the analogues depend on $|w|$; but the exponents $\pm 1/3$ survive.

3. The Example

Theorem 3. *If $\{k_m\}$ is a sequence of positive integers and $k_m \to \infty$ rapidly enough, then the function f defined by the formula*

$$f(z) = \sum_{m=1}^{\infty} \frac{k_m^{1/3}}{m^{1/2}} [b(z)]^{k_m} = \sum_{n=0}^{\infty} a_n z^n \tag{10}$$

is strongly annular and has the additional properties

(i) $\lim_{n \to \infty} a_n = 0$,

(ii) *in every decomposition $f = f_1 + f_2$, where $f_1(z) = \sum d_n z^{\lambda_n}$ and $\lambda_{n+1} - \lambda_n \geqq 2$ for all n, the component f_2 fails to belong to the Hardy class H^2.*

It is easy to choose the exponents k_m so that f is strongly annular. Indeed, on the circle $|z|=1$ the m^{th} term in the series (10) has modulus $k_m^{1/3} m^{-1/2} = B_m$, and therefore there exists a circle $|z| = r_m$ ($r_m < 1$) on which the term has modulus greater than $B_m/2$. If k_m is large enough, this is greater than $4(B_1 + B_2 + \cdots + B_{m-1})$. If the exponents $k_{m+1}, k_{m+2}, \ldots$ are large enough, then the sum of the corresponding terms is smaller than $B_m/8$ on the circle $|z| = r_m$, and therefore $|f(z)| > B_m/4$ whenever $|z| = r_m$.

To make certain that $a_n \to 0$, we require first that no two of the intervals $[k_m/6, 4k_m]$ overlap. If the index n does not lie in the m^{th} of these intervals, then the bounds in Theorem 2 give the inequality

$$k_m^{1/3} m^{-1/2} |\alpha_{kn}| < k_m^{1/3} m^{-1/2} \frac{1}{\pi k_m/6} = \frac{6}{\pi m^{1/2} k^{2/3}}.$$

If n lies in the m^{th} interval, we invoke the bound (3) and obtain the inequality

$$k_m^{1/3} m^{-1/2} |\alpha_{kn}| < A_1 m^{-1/2}.$$

Now we subject the sequence $\{k_m\}$ to the additional requirement that $\sum k_m^{-2/3} < \infty$, and we observe that for each integer k the sequence $\{\alpha_{kn}\}_{n=0}^{\infty}$ converges to 0. Clearly, f has property (i).

Finally, suppose that the equation $f=f_1+f_2$ represents a decomposition of f such that one of every pair of consecutive coefficients in the power series of f_1 is zero. By the second part of Theorem 1, the two coefficients with index $3k_m$ and $3k_m+1$ in the power series of the m^{th} term in the middle member of (10) have modulus greater than $A_2/m^{1/2}$. If the sequence $\{k_m\}$ increases fast enough, then one of the two corresponding Taylor coefficients of f_2 has modulus at least $A_2/2m^{1/2}$. Since $\sum 1/m=\infty$, we see at once that $f_2 \notin H^2$. Therefore f has the property (ii), and the proof of Theorem 3 is complete.

The reader may feel that while the function (10) does not have an essentially lacunary power series, we have merely hidden the lacunarity by means of a Möbius transformation. To carry the obliteration of lacunarity a step further, we could replace each function $[b(z)]^{k_m}$ in (10) with the corresponding function

$$\left(\frac{e^{mi}/2-z}{1-ze^{-mi}/2}\right)^{k_m}.$$

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Received January 3, 1977