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Nonvanishing Univalent Functions*

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The class S of functions $g(z)=z+c_2 z^2+c_3 z^3+\dots$ analytic and univalent in the unit disk $|z|<1$ has been thoroughly studied, and its properties are well known. Our purpose is to investigate another class of functions which, by contrast, seems to have been rather neglected. This is the class S_0 of functions $f(z)=1+a_1 z+a_2 z^2+\dots$ analytic, univalent, and nonvanishing in the unit disk, normalized by the condition $f(0)=1$. It will become apparent that S_0 is closely related to the more familiar class S and is in some ways easier to handle.

After making a few preliminary observations, we adapt the elementary method of Brickman [2] to obtain information about the extreme points and support points of S_0. We then use Schiffer's method of boundary variation to consider a wide class of extremal problems and to study the support points of S_0 in greater depth. Whereas the geometry of the arcs omitted by extreme points and support points of S is related to the families of linear rays and circles centered at the origin, it turns out that the corresponding geometry for S_0 is related to the families of ellipses and hyperbolas with foci at 0 and 1. The paper concludes with a detailed study of the specific linear extremal problem $\min \operatorname{Re}\{f(\zeta)\}$, which provides an interesting family of support points in S_0.

§1. Elementary Observations

Although S_0 is a normal family, it is not compact. The constant function $f(z)\equiv 1$ may occur as the uniform limit of functions in S_0. For example, the functions $h_\alpha(z)=1+\alpha z$ are in S_0 for $0<|\alpha|\leqq 1$, and $h_\alpha(z)\to 1$ uniformly as $\alpha\to 0$. However, the enlarged family $\hat{S}_0=S_0\cup\{1\}$ is normal and compact, and so every real-valued continuous functional attains a maximum on $\hat{S}_0$. As usual, "continuous" refers to the topology of uniform convergence on compact subsets of the disk.

An important function in S_0 is

$$k_0(z)=\left(\frac{1+z}{1-z}\right)^2=1+4\sum_{n=1}^{\infty} n z^n,$$

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which maps the disk onto the complement of the nonpositive real axis. It plays a role in S_0 analogous to that of the Koebe function

$$k(z)=\frac{z}{(1-z)^2}=\sum_{n=1}^{\infty} n z^n$$

in S. Among other things, it suggests the conjecture that the coefficients of each function

$$f(z)=1+a_1 z+a_2 z^2+\ldots$$

in S_0 satisfy $|a_n|\leqq 4n$ for all n. Another form of the conjecture, apparently stronger, is that $|a_n|\leqq|a_1|n$ for all n.

We shall now show that the first of these conjectures is equivalent to the Littlewood conjecture for the class S, and the second is equivalent to the Bieberbach conjecture. Littlewood's conjecture asserts that if

$$g(z)=z+c_2 z^2+c_3 z^3+\ldots$$

belongs to S and $g(z)\neq\omega$, then $|c_n|\leqq 4|\omega|n$. Bieberbach's conjecture says that for each $g\in S$, $|c_n|\leqq n$ for all n. By the Koebe one-quarter theorem, Bieberbach's conjecture implies Littlewood's. (See [6] for a fuller discussion.)

Given $f\in S_0$, construct

$$g(z)=\frac{f(z)-1}{a_1}=z+\sum_{n=2}^{\infty}\frac{a_n}{a_1}z^n.$$

Then $g\in S$, and Bieberbach's conjecture would imply $|a_n|\leqq|a_1|n$. Since $f(z)\neq 0$, it follows that $g(z)\neq -1/a_1$, and so the Koebe one-quarter theorem gives $|a_1|\leqq 4$. Thus the inequality $|a_n|\leqq|a_1|n$ would imply $|a_n|\leqq 4n$. The latter inequality would also follow directly from Littlewood's conjecture, since $g(z)\neq -1/a_1$.

Conversely, given $g\in S$ with $g(z)\neq\omega$, let

$$f(z)=\frac{g(z)-\omega}{-\omega}=1-\frac{1}{\omega}(z+c_2 z^2+\ldots).$$

Then $f\in S_0$, and the inequality $|a_n|\leqq 4n$ would imply Littlewood's conjecture $|c_n|\leqq 4|\omega|n$; while the inequality $|a_n|\leqq|a_1|n$ would imply Bieberbach's conjecture $|c_n|\leqq n$.

Since the Bieberbach conjecture has been proved for $n\leqq 6$, the preceding argument shows for each $f\in S_0$ that $|a_n|\leqq|a_1|n$ for $n\leqq 6$. For higher n, the FitzGerald-Horowitz estimate [9, 15] gives

$$|a_n|<1.07|a_1|n.$$

It also follows from well-known results for S that $|a_n|\leqq|a_1|n$ if f has real coefficients, or if its range is starlike.

The class S_0 is invariant under the rotation $f\to f_\theta$, where $f_\theta(z)=f(e^{i\theta}z)$. The problem of maximizing $|a_n|$ in S_0 is therefore equivalent to the *linear* problem of maximizing $\mathrm{Re}\{a_n\}$. A general discussion of linear problems will appear in Sect. 5.

The sharp bounds for $|f(z)|$ and $|f'(z)|$ at a fixed point z can be found by passing from $f \in S_0$ to the function $g \in S$ defined as above by

$$g(z) = \frac{f(z)-1}{f'(0)}.$$

The well-known estimates [12, p. 4] for $|g(z)|$ and $|g'(z)|$ give

$$\frac{r}{(1+r)^2}|f'(0)| \leqq |f(z)-1| \leqq \frac{r}{(1-r)^2}|f'(0)|$$

and

$$\frac{1-r}{(1+r)^3}|f'(0)| \leqq |f'(z)| \leqq \frac{1+r}{(1-r)^3}|f'(0)|$$

for each $f \in S_0$, where $r = |z|$. Since $|f'(0)| \leqq 4$, these inequalities lead to the sharp bounds

$$\left(\frac{1-r}{1+r}\right)^2 \leqq |f(z)| \leqq \left(\frac{1+r}{1-r}\right)^2$$

and

$$0 < |f'(z)| \leqq 4\frac{1+r}{(1-r)^3}.$$

The lower bound for $|f(z)|$ results from the obvious fact that $1/f \in S_0$ whenever $f \in S_0$. The functions $1+\alpha z$ show that 0 is the sharp lower bound for $|f'(z)|$.

In fact, it is easily seen that every $g \in S$ is the image of some $f \in S_0$ under the transformation above. This remark allows other known results for the class S to be translated to S_0. For example, the rotation theorem [10, p. 115] for S gives the sharp estimates for $\arg\{f'(z)/f'(0)\}$; and for each fixed z the region of values of

$$\log\left\{\frac{f(z)-1}{zf'(0)}\right\}, \qquad f \in S_0,$$

is a circular disk (see [19, p. 196] or [8]).

For the integral means

$$I_p(r,f) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta, \qquad -\infty < p < \infty,$$

the method of Baernstein [1] shows that $I_p(r,f) \leqq I_p(r,k_0)$ for all $f \in S_0$ and for all $r < 1$.

§2. Extreme Points and Support Points

A function $f \in S_0$ is called an *extreme point* of S_0 if it has no representation of the form

$$f = tf_1 + (1-t)f_2, \qquad 0 < t < 1, \tag{1}$$

as a proper convex combination of two distinct functions f_1 and f_2 in S_0. A *support point* of S_0 is a function $f \in S_0$ which maximizes $\operatorname{Re}\{L\}$ for some complex-valued functional L which is continuous and linear over the space of all analytic functions in the disk, and which is not constant on S_0.

Brickman [2] and Brickman and Wilken [3] showed in an elementary fashion that the extreme points and support points of S map the disk onto the complement of an arc which goes to infinity with increasing modulus; that is, monotonically with respect to the family of circles centered at the origin. (See also [7].) We shall now adapt their method to prove a theorem which gives corresponding results for S_0.

Theorem 1. *If a function $f \in S_0$ omits two values on the same (possibly degenerate) ellipse with foci at 0 and 1, then it is a proper convex combination of two distinct functions in S_0 which omit nonempty open sets.*

Proof. Suppose $f \in S_0$, $f(z) \neq \alpha$, and $f(z) \neq \beta$, where $\alpha \neq \beta$. Let D be the range of f. Since D is simply connected, some branch of the function

$$\psi(w) = \{(w-\alpha)(w-\beta)\}^{1/2}$$

is analytic and single-valued in D. We claim that the two functions $w \pm \psi(w)$ are univalent in D and have disjoint ranges. Indeed, if

$$\psi(w_1) \pm \psi(w_2) = \pm(w_1 - w_2)$$

for some pair of points w_1 and w_2 in D, we could square both sides to obtain

$$\pm 2\psi(w_1)\,\psi(w_2) = (w_1-\alpha)(w_1-\beta) + (w_2-\alpha)(w_2-\beta) - (w_1-w_2)^2.$$

Squaring again, we could then deduce (after a lengthy calculation) that

$$(\alpha-\beta)^2 (w_1-w_2)^2 = 0,$$

which implies that $w_1 = w_2$. This shows that $w \pm \psi(w)$ are univalent in D and have disjoint ranges; in particular, both functions omit open sets.

We now normalize these functions by defining

$$\psi_1(w) = \frac{w + \psi(w) - \psi(0)}{1 + \psi(1) - \psi(0)}; \qquad \psi_2(w) = \frac{w - \psi(w) + \psi(0)}{1 - \psi(1) + \psi(0)}.$$

Then $\psi_1(0) = \psi_2(0) = 0$ and $\psi_1(1) = \psi_2(1) = 1$. The composite functions $f_1 = \psi_1 \circ f$ and $f_2 = \psi_2 \circ f$ therefore belong to S_0 and omit nonempty open sets. A simple calculation shows

$$f = tf_1 + (1-t)f_2,$$

where

$$t=\tfrac{1}{2}[1+\psi(1)-\psi(0)].$$

A final calculation (cf. [21, p. 69]) shows that $0<t<1$ if α and β lie on the same ellipse with foci at 0 and 1. This is easy to show if α and β lie on the open real segment joining 0 and 1 (a degenerate ellipse). Thus the omitted set of an extreme point is an arc with endpoints 0 and ∞.

Corollary. *If $f\in S_0$ is an extreme point or a support point of S_0, then it maps the disk onto the complement of a continuous arc extending from* 0 *to* ∞ *monotonically with respect to the family of ellipses with common foci at* 0 *and* 1; *that is, intersecting each such ellipse exactly once. In particular,* 0 *is an endpoint of the arc.*

Proof. If $f\in S_0$ omits two distinct points on the same ellipse with foci at 0 and 1, the theorem tells us it must have the form (1), where f_1 and f_2 belong to S_0 and omit open sets. This shows that f is not an extreme point of S_0.

In order to show that f is not a support point, we appeal to the fact that every support point has dense range. This can be proved by adapting an elementary argument due to Marty (see [21], p. 90), but it will follow also from the results of Sect. 4, which will be derived by a variational method. In particular, neither f_1 nor f_2 is a support point, since both functions omit open sets. Therefore, if f were to maximize the real part of some continuous linear functional L, it would follow that

$$\operatorname{Re}\{L(f)\}=t\operatorname{Re}\{L(f_1)\}+(1-t)\operatorname{Re}\{L(f_2)\}<\operatorname{Re}\{L(f)\}.$$

This contradiction shows that f is not a support point.

We remark that the proof extends easily to (real-valued) convex functionals.

§3. Extremal Problems and Elementary Transformations

Let ϕ be a continuous complex-valued functional on the compact set $\hat{S}_0=S_0\cup\{1\}$. In this section and the next we shall consider the problem of finding the maximum of $\operatorname{Re}\{\phi\}$ over S_0.

Since ϕ is continuous and $\hat{S}_0$ is compact, there exists a function $f\in\hat{S}_0$ where $\operatorname{Re}\{\phi\}$ attains its maximum. If f is the constant function 1, nothing more can be said. We shall assume from now on that f belongs to S_0.

We shall also assume that the functional ϕ has a *Fréchet differential* at f. This is a continuous linear functional $\ell(\cdot;f)$ on the space of all analytic functions in the disk such that

$$\phi(f^*)=\phi(f)+\varepsilon\ell(h;f)+o(\varepsilon),\qquad \varepsilon\to 0,$$

whenever $f^*=f+\varepsilon h+o(\varepsilon)$ belongs to S_0. Since $\operatorname{Re}\{\phi(f^*)\}\leqq\operatorname{Re}\{\phi(f)\}$, we may obtain information about f by constructing comparison functions f^* within the family S_0.

Some information comes from the various elementary transformations which preserve the family S_0. Here is a partial list.

(i) *Conjugation.* If $f \in S_0$ and $f^*(z) = \overline{f(\bar{z})}$, then $f^* \in S_0$.

(ii) *Contraction.* If $f \in S_0$, then $f^* = f \circ \varphi \in S_0$ for any univalent mapping φ of the unit disk into itself with $\varphi(0) = 0$. The following are examples.

(iia) *Rotation.* If $f \in S_0$ and $f^*(z) = f(e^{i\theta} z)$, then $f^* \in S_0$ for $-\pi < \theta \leqq \pi$. Note that

$$f^*(z) = f(z) + i\theta z f'(z) + o(\theta) \quad \text{as} \quad \theta \to 0.$$

(iib) *Dilation.* If $f \in S_0$ and $f^*(z) = f(rz)$, then $f^* \in S_0$ for $0 < r < 1$. Note that

$$f^*(z) = f(z) - (1-r) z f'(z) + o(1-r) \quad \text{as} \quad r \to 1.$$

(iic) *Incision.* If $f \in S_0$ and $\kappa(z) = z(1 + e^{-i\theta} z)^{-2}$, then $f^* = f \circ \kappa^{-1} \circ r\kappa \in S_0$ for $-\pi < \theta \leqq \pi$ and $0 < r < 1$. Note that

$$f^*(z) = f(z) - (1-r) z f'(z) \frac{e^{i\theta} + z}{e^{i\theta} - z} + o(1-r) \qquad \text{as } r \to 1.$$

(iii) *Generalized contraction.* If $f \in S_0$ and $f^*(z) = f \circ \varphi(z) / f \circ \varphi(0)$, then $f^* \in S_0$ for any univalent mapping φ of the unit disk into itself. The most important example is as follows.

(iiia) *Marty transformation.* If $f \in S_0$ and $f^*(z) = f\left(\dfrac{z+\zeta}{1+\bar{\zeta} z}\right) \Big/ f(\zeta)$, where $|\zeta| < 1$, then $f^* \in S_0$. Note that

$$f^*(z) = f(z) + \zeta[f'(z) - f'(0) f(z)] - \bar{\zeta} z^2 f'(z) + o(\zeta) \qquad \text{as } \zeta \to 0.$$

(iv) *Range transformation.* If $f \in S_0$, then $f^* = \Phi \circ f \in S_0$ for every function Φ analytic, univalent, and nonvanishing on the range of f, with $\Phi(1) = 1$. The following are examples.

(iva) *Power transformation.* If $f \in S_0$, then $f^* = f^p \in S_0$ for $-1 \leqq p < 1$, $p \neq 0$. This includes *inversion:* if $f \in S_0$, then $1/f \in S_0$. Note that

$$f^*(z) = f(z) - (1-p) f(z) \log f(z) + o(1-p) \qquad \text{as } p \to 1.$$

(ivb) *Omitted-value transformations.* If $f \in S_0$ and $f(z) \neq \omega$ for some $\omega \neq 0$, then the functions

$$f^* = \frac{f - \omega}{1 - \omega} \quad \text{and} \quad f^* = \frac{f - \omega}{(1-\omega) f}$$

and their reciprocals belong to S_0.

From the asymptotic expansions of the transformations (iiabc), (iiia), and (iva), one easily obtains some necessary conditions for an extremal function. These are collected in the following theorem. The remaining elementary transformations are valuable for linear problems and will be used later.

Theorem 2. *Let ϕ be a continuous complex-valued functional defined on $\hat{S}_0$, and suppose that* $\operatorname{Re}\{\phi\}$ *attains its maximum at $f \in S_0$. If ϕ has a Fréchet differential $\ell(\cdot\,;f)$ at f, then*

(a) $\operatorname{Im}\{\ell(zf';f)\}=0$;

(b) $\operatorname{Re}\{\ell(zf';f)\}\geqq 0$;

(c) $\operatorname{Re}\left\{\ell\left(zf'\frac{e^{i\theta}+z}{e^{i\theta}-z};f\right)\right\}\geqq 0, \qquad -\pi<\theta\leqq\pi;$

(d) $\ell(f';f)-f'(0)\ell(f;f)=\overline{\ell(z^2f';f)}$;

(e) $\operatorname{Re}\{\ell(f\log f;f)\}\geqq 0$.

Clearly, (a) and (b) imply $\ell(zf';f)\geqq 0$. In addition, (c) and the Herglotz formula imply that $\operatorname{Re}\{\ell(zf'\psi;f)\}\geqq 0$ for every analytic function ψ with positve real part in the unit disk and with $\psi(0)=1$.

§4. Boundary Variation

We now adapt Schiffer's method of boundary variation [20, 16, 21, 8] to the solution of extremal problems in S_0.

As in Section 3, suppose that $\operatorname{Re}\{\phi\}$ attains its maximum at $f\in S_0$ and that ϕ has a Fréchet differential $\ell(\cdot\,;f)$ at f. Let D be the range of f, and let $\Gamma=\mathbb{C}-D$ be the set omitted by f. Fix a point $w_0\in\Gamma$, $w_0\neq 0$, and let D_r be the domain complementary to the component of

$$\Gamma\cap\{w\in\mathbb{C}: |w-w_0|\leqq r\}$$

which contains w_0. The method of boundary variation involves a one-parameter family of functions

$$F_r(w)=w+\lambda_r(w-w_0)^{-1}+O(r^2), \qquad r\to 0,$$

analytic and univalent in D_r, with $\lambda_r=O(r)$. Since $w_0\neq 0$, it is clear that $0\in D_r$ for all r sufficiently small, and we can modify F_r by forming

$$G_r(w)=\frac{F_r(w)-F_r(0)}{F_r(1)-F_r(0)}=w+\lambda_r\frac{w(1-w)}{w_0(1-w_0)(w-w_0)}+O(r^2),$$

which preserves both 0 and 1: $G_r(0)=0$ and $G_r(1)=1$. Thus $f^*=G_r\circ f\in S_0$ for all r sufficiently small.

Since $\operatorname{Re}\{\phi\}$ attains a maximum at f, the relation $\operatorname{Re}\{\phi(f^*)\}\leqq\operatorname{Re}\{\phi(f)\}$ implies

$$\operatorname{Re}\left\{\frac{\lambda_r}{w_0(1-w_0)}\ell\left(\frac{f(1-f)}{f-w_0};f\right)+o(r)\right\}\leqq 0.$$

We may now appeal to Schiffer's fundamental lemma, which asserts that if $s(w)\not\equiv 0$ is a single-valued analytic function defined in a neighborhood of a nondegenerate continuum Γ and if

$$\operatorname{Re}\{\lambda_r s(w_0)+o(r)\}\leqq 0$$

for each $w_0\in\Gamma$, then Γ is the union of analytic arcs $w=w(t)$ satisfying the differential equation

$$s(w(t))\left(\frac{dw}{dt}\right)^2>0.$$

Since a continuous linear functional can be represented as an integral with respect to a measure supported on a compact subset of the unit disk (*cf.* [21], Ch. 4), the function s defined by

$$s(w)=\frac{1}{w(1-w)}\ell\left(\frac{f(1-f)}{f-w};f\right)$$

is analytic in a neighborhood of the omitted set Γ, except possibly for a simple pole at the origin. If s is not the zero-function, it follows that Γ is the union of analytic arcs satisfying

$$\frac{1}{w(1-w)}\ell\left(\frac{f(1-f)}{f-w};f\right)dw^2>0. \tag{2}$$

It is clear that $0\in\Gamma$, by the definition of S_0. The quadratic differential has a simple pole at the origin if $\ell(1-f;f)\neq 0$, which implies [17] that the origin is an endpoint of Γ. It is clear also that $\infty\in\Gamma$. Besides 0 and ∞, the only possible points where Γ can branch or fail to be analytic are the zeros of $\ell\left(\frac{f(1-f)}{f-w};f\right)$. This function is analytic at ∞ and therefore can have at most a finite number of zeros on Γ, since we assume it does not vanish identically. It cannot vanish identically unless ℓ is a scalar multiple of point-evaluation at the origin (see Section 5). Thus Γ consists of finitely many analytic arcs if the Fréchet differential of ϕ at f does not have the form $\ell(g;f)=\alpha g(0)$.

The differential equation defined by (2) is a functional differential equation for Γ insofar as it involves also the unknown extremal function f. In particular cases one attempts, nevertheless, to determine Γ from it. Here the *a priori* knowledge of two boundary points, 0 and ∞, and the abundance of elementary transformations (Sect. 3) are great advantages. Unfortunately, as we shall see, the differential equation (2) leads quickly to elliptic and hyperelliptic integrals.

§5. Linear Problems and Support Points

We shall now specialize the general extremal problem of Sect. 4 to a *linear* problem, by choosing ϕ to be a continuous linear functional L. In this case the differential equation (2) takes the form

$$\frac{1}{w(1-w)} L\left(\frac{f(1-f)}{f-w}\right) dw^2 > 0. \tag{3}$$

Unless L has the form $L(g)=\alpha g(0)$ for some constant α (in which case L is constant on S_0), the analytic function

$$s(w)=\frac{1}{w(1-w)} L\left(\frac{f(1-f)}{f-w}\right) \tag{4}$$

does not vanish identically on Γ. To prove this, consider the linear functional M defined by

$$M(g)=L(g)-L(1)\, g(0).$$

If $s(w)\equiv 0$ on Γ, then $L(1-f)=0$, since $0\in\Gamma$. Thus

$$\begin{aligned} M\left(\frac{1}{f-w}\right) &= L\left(\frac{1}{f-w}\right)-\frac{1}{1-w}L(1)=\frac{1}{1-w}L\left(\frac{1-f}{f-w}\right) \\ &= s(w)-\frac{1}{w(1-w)}L(1-f)=0 \end{aligned}$$

for all $w\in\Gamma$. This implies that M is the zero-functional (see [21], Lemma 4.5). Hence $L(g)=\alpha g(0)$ for all g, where $\alpha=L(1)$.

Some further information about support points is contained in the following theorem.

Theorem 3. *Let f be a support point of S_0, maximizing the real part of a continuous linear functional L which is not constant on S_0. Then*

(a) $L(zf')\geqq 0$;

(b) $\operatorname{Re}\{L(zf'\psi)\}\geqq 0$ *for every analytic function ψ with positive real part in the unit disk and with $\psi(0)=1$;*

(c) $L(f')-f'(0)L(f)=\overline{L(z^2 f')}$;

(d) $\operatorname{Re}\{L(f\log f)\}\geqq 0$.

Furthermore, for each point w ($w\neq 0,\infty$) on the arc Γ omitted by f,

(e) $\operatorname{Re}\left\{\frac{w}{1-w}L(1-f)\right\}>0$;

(f) $\operatorname{Re}\left\{L\left((1-f)\left[1+\frac{1}{f-w}\right]\right)\right\}<0$;

(g) $\operatorname{Re}\left\{L\left((1-f)\left[1-\frac{w}{(1-w)f}\right]\right)\right\}<0$;

(h) $\operatorname{Re}\left\{L\left(\frac{f(1-f)}{f-w}\right)\right\}<0$.

Proof. The relations (a), (b), (c), and (d) are consequences of Theorem 2. To prove the remaining assertions, fix an interior point $w \in \Gamma$ and let f^* be any of the four omitted-value transformations given in Section 3 (ivb). Since f^* omits two points on some (possibly degenerate) ellipse with foci at 0 and 1, it follows from the corollary to Theorem 1 that f^* is not a support point. Thus $\operatorname{Re}\{L(f^*)\} < \operatorname{Re}\{L(f)\}$, which gives the four properties (e), (f), (g), and (h).

We showed previously (corollary to Theorem 1) that each support point of S_0 maps the disk onto the complement of an arc Γ which is monotonic with respect to the family of ellipses with foci at 0 and 1. We can now give a much better description of the omitted arc Γ.

Theorem 4. *Each support point f of S_0 maps the disk onto the complement of a single analytic arc Γ which extends from 0 to ∞ monotonically with respect to the family of ellipses with foci at 0 and 1. The arc Γ is analytic even at the endpoints 0 and ∞; in particular, it has asymptotic directions there. At each interior point w on Γ $(w \neq 0, \infty)$, there is an angle of less than $\pi/4$ between Γ and the hyperbola with foci at 0 and 1 which passes through w. At the origin there is an angle of at most $\pi/2$ between Γ and the negative real axis. Finally, if f maximizes the real part of a linear functional L, nonconstant on S_0, then Γ satisfies the differential equation* (3).

Proof. We have already shown that Γ consists of analytic arcs satisfying the differential equation (3). The remaining assertions follow from the inequality

$$\operatorname{Re}\left\{L\left(\frac{f(1-f)}{f-w}\right)\right\} < 0, \qquad w \in \Gamma, \qquad w \neq 0, \infty, \tag{5}$$

which was proved in Theorem 3(h). This inequality implies that the function s defined by (4) never vanishes on Γ. In other words, every interior point of Γ is a regular point of the quadratic differential (3), and so the trajectory Γ is an analytic arc (without corners). A second implication of (5), together with (3), is that

$$\operatorname{Re}\left\{\frac{dw^2}{w(w-1)}\right\} > 0, \qquad w \in \Gamma, \qquad w \neq 0, \infty. \tag{6}$$

This is equivalent to the $\pi/4$-property (cf. [21], p. 99). In fact, it is easily seen in general that for distinct complex numbers α and β the trajectories of

$$\frac{dw^2}{(w-\alpha)(w-\beta)} > 0$$

are the hyperbolas with foci at α and β.

The points 0 and ∞ are at worst simple poles of the quadratic differential (3). More precisely, at each of these points the quadratic differential has a simple pole, a regular point, or a zero of finite order. In any of these cases the general theory of the local trajectory structure (see [17]) guarantees that Γ is analytic at 0 and ∞ and has a limiting direction at each of these endpoints.

It follows that at the origin, Γ makes an angle of at most $\pi/2$ with the negative real axis. Otherwise, because its tangent direction is continuous, Γ would make near the origin an angle larger than $\pi/4$ with some nondegenerate hyperbola with foci at 0 and 1.

It should be remarked that the quadratic differential actually has a simple pole at the origin, since $L(1-f)\neq 0$ by Theorem 3(e). It has a simple pole at infinity if $L(f(1-f))\neq 0$, but this question is left open.

As in Hengartner-Schober [14], the $\pi/4$-property in Theorem 4 gives information about the coefficients of support points.

Theorem 5. *If f is a support point of S_0 and $e^{i\beta}$ is the point that f maps to ∞, then*

$$P(z)=\frac{z(e^{i\beta}-z)^2[f'(z)]^2}{e^{i\beta}f(z)[f(z)-1]}$$

is analytic in $|z|\leqq 1$ and has positive real part in $|z|<1$.

Corollary. *If $f(z)=1+a_1z+a_2z^2+\ldots$ is a support point of S_0 and $f(e^{i\beta})=\infty$, then*

(a) $\operatorname{Re}\{e^{i\beta}a_1\}>0$;

(b) $|e^{i\beta}(3a_2-a_1^2)-2a_1|<2\operatorname{Re}\{e^{i\beta}a_1\}$;

(c) $\operatorname{Re}\{a_2/a_1^2\}>\frac{1}{3}$;

(d) $|a_1|^2<3|a_2|$.

Proof of Theorem. Let $e^{i\alpha}$ be the point which f maps to the tip of the slit at 0. The only possible singularities of P in $|z|\leqq 1$ are at 0, $e^{i\alpha}$, and $e^{i\beta}$. The origin is obviously a removable singularity. Since f has a double zero at $e^{i\alpha}$ and a pole of second order at $e^{i\beta}$, these points are also removable singularities. Parametrizing the slit Γ by $w(t)=f(e^{it})$ and appealing to (6), we find

$$\operatorname{Re}\{P(e^{it})\}=-4\sin^2\left(\frac{\beta-t}{2}\right)\operatorname{Re}\left\{\frac{(dw/dt)^2}{w(1-w)}\right\}>0$$

for $e^{it}\neq e^{i\alpha}$, $e^{i\beta}$. Thus $\operatorname{Re}\{P(z)\}>0$ for $|z|<1$, by the minimum principle for harmonic functions.

Proof of Corollary. The inequalities (a) and (b) are simply the estimates $\operatorname{Re}\{P(0)\}>0$ and $|P'(0)|<2\operatorname{Re}P(0)$. The strict inequality reflects the fact that P is bounded. By rewriting the inequality (b) as

$$\left|\left(\frac{3a_2}{a_1^2}-1\right)-\frac{2e^{-i\beta}}{a_1}\right|<\operatorname{Re}\left\{\frac{2e^{-i\beta}}{a_1}\right\},$$

one easily obtains (c) and (d).

§6. Minimum Real Part

As a specific application of the variational method, we shall now consider the problem of minimizing $\operatorname{Re}\{f(\zeta)\}$ for $f\in S_0$, where ζ is a fixed point in the unit disk. Because S_0 is preserved under rotation, we may assume without loss of generality that $0<\zeta<1$. The corresponding maximum real part problem is equivalent to the maximum modulus problem (see Sect. 1).

For small ζ, the minimum real part problem is also equivalent to the minimum modulus problem, as the following elementary theorem shows.

Theorem 6. *For $0 \leqq |z| \leqq 3-\sqrt{8}=0.171\ldots$ and for each $f \in S_0$,*

$$\operatorname{Re}\{f(z)\} \geqq k_0(-r)=\left(\frac{1-r}{1+r}\right)^2, \qquad r=|z|.$$

Strict inequality holds unless f is a suitable rotation of k_0.

Proof. We observed in Sect. 1 that

$$|f(z)-1| \leqq \frac{r}{(1-r)^2}|f'(0)|, \qquad f \in S_0,$$

with equality only for a suitable rotation of k_0. Replacing f by $1/f$ and recalling that $|f'(0)| \leqq 4$, we obtain

$$|f(z)-1| \leqq \frac{4r}{(1-r)^2}|f(z)|.$$

If $r<3-\sqrt{8}$, then $\rho=4r/(1-r)^2<1$, and the above inequality asserts that the point $w=f(z)$ lies in the disk bounded by the circle of Apollonius $|w-1|=\rho|w|$, with inverse points 0 and 1. It is geometrically clear that the point w in this disk with minimum real part is real and satisfies $1-w=\rho w$; thus $w=\left(\frac{1-r}{1+r}\right)^2$. If $r=3-\sqrt{8}$, then $\rho=1$ and the disk degenerates to the half-plane $\operatorname{Re}\{w\} \geqq \frac{1}{2}$. This leads to the same conclusion, and completes the proof.

For each fixed ζ, $0<\zeta<1$, there is a function $f \in S_0$ where $\operatorname{Re}\{f(\zeta)\}$ attains a minimum. According to Theorem 4, this function f maps the disk onto the complement of an analytic arc Γ extending from 0 to ∞ and satisfying

$$\frac{B(B-1)}{w(w-1)(w-B)}dw^2>0, \qquad B=f(\zeta). \tag{7}$$

One sees by inspection that the negative real axis is a solution to (7) if and only if B is real and $0<B<1$. Thus f is a rotation of k_0 only if $0<B<1$. Theorem 6 shows that this is true if $\zeta \leqq 3-\sqrt{8}$. On the other hand, for ζ beyond the radius of convexity of S_0; that is, for $2-\sqrt{3}<\zeta<1$, one verifies either by geometric considerations or by direct calculation that the minimum value of $\operatorname{Re}\{k_0(z)\}$ on the circle $|z|=\zeta$ occurs at a pair of (complex conjugate) points where $B=k_0(z)$ is nonreal. Since the negative real axis is not then a trajectory of (7), this shows that the extremal function f cannot be a rotation of k_0 if $\zeta>2-\sqrt{3}=0.267\ldots$.

Another consequence of the differential equation (7) for the omitted arc Γ is that different values of ζ must give rise to different extremal functions, unless Γ is the negative real axis in both cases, so that both extremal funtions are $k_0(-z)$. Indeed, let $0<\zeta_1<\zeta_2<1$ and suppose $B_1=f(\zeta_1)$ and $B_2=f(\zeta_2)$ are the corresponding values of the same extremal function f. Then the omitted arc Γ must satisfy

$$\frac{B_n(B_n-1)}{w(w-1)(w-B_n)}dw^2>0, \qquad n=1,2.$$

Dividing one differential equation by the other, we find

$$\frac{B_2(B_2-1)}{B_1(B_1-1)}\cdot\frac{w-B_1}{w-B_2}>0, \qquad w\in\Gamma.$$

Thus $\arg\{(w-B_1)/(w-B_2)\}$ is constant on Γ. Sending w to 0 and to ∞ along Γ, we conclude that $\arg\{B_1/B_2\}=0$. In other words, $B_2=\lambda B_1$ for some $\lambda>0$ with $\lambda\neq 1$. Since $(w-B_1)/(w-\lambda B_1)$ is positive, Γ is the ray $w=-tB_1, t\geqq 0$. It is easily seen, however, that the ray $w=-tB$ can satisfy (7) only if $0<B<1$. (This argument is due to Brown; compare [5].)

It is difficult to determine B precisely as a function of ζ, but a few observations can be made which limit its possible range. First of all, B cannot be real and negative. If $B<0$, then the differential equation (7) would imply that Γ is the linear segment from 0 to B, which is impossible. Since S_0 is preserved under conjugation, we may assume that $\operatorname{Im}\{B\}\geqq 0$. It is clear that $\operatorname{Re}\{B\}\leqq k_0(-\zeta)<1$, because $k_0(-z)$ belongs to S_0. Another simple observation is that $\operatorname{Re}\{B\}\leqq\operatorname{Re}\{1/B\}$, since S_0 is preserved under inversion. This implies that $|B|\leqq 1$ if $\operatorname{Re}\{B\}>0$, while $|B|\geqq 1$ if $\operatorname{Re}\{B\}<0$. It follows that B must lie in the shaded region in Fig. 1.

As ζ increases from 0 to 1, the corresponding point B moves from 1 to ∞ with continuously decreasing real part. According to Theorem 6, the point B moves initially along the real axis from 1 toward 0. Since it is plausible that B must move in a continuous path, Figure 1 suggests that $B=i$ for some ζ. Later we shall show that this conclusion is actually valid. Taking it for granted and setting $B=i$ in the differential equation (7), one makes the surprising discovery that the corresponding omitted arc Γ is simply the radial line $w=-(1+i)t$, $t\geqq 0$.

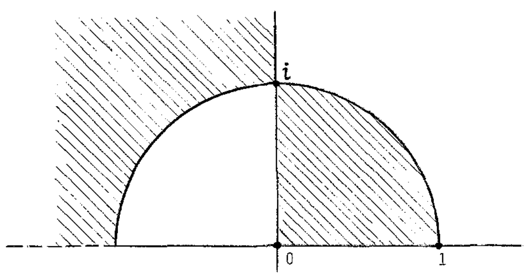

Fig. 1. *Region containing* $B=f(\zeta)$, $\operatorname{Im}\{B\}\geqq 0$

The extremal function f is therefore some rotation of h^2, where

$$h(z)=\frac{1+cz}{1-z}, \qquad c=\sqrt{i}=\frac{1+i}{\sqrt{2}}.$$

The appropriate rotation and the associated value of ζ are determined by the requirement that $f(\zeta)=i$. The equation $h(z)=\sqrt{i}$ has the unique solution

$$z=\tfrac{1}{4}[(2-\sqrt{2})+\sqrt{2}i]=\varepsilon\zeta,$$

where

$$\zeta=|z|=\tfrac{1}{2}\sqrt{2-\sqrt{2}}$$

and

$$\varepsilon=\zeta+\frac{\sqrt{2}}{4\zeta}i; \qquad |\varepsilon|=1.$$

These results are summarized in the following theorem, which is contingent only upon the proof (given later as a consequence of Theorem 9) that $B=i$ actually occurs as the value of an extremal function for some ζ, and that B can have no other value on the positive imaginary axis.

Theorem 7. *For each $f\in S_0$, the inequality $\operatorname{Re}\{f(z)\}\geqq 0$ holds in the disk $|z|\leqq\frac{1}{2}\sqrt{2-\sqrt{2}}=0.382\ldots$. This bound is sharp. If $z=\zeta=\frac{1}{2}\sqrt{2-\sqrt{2}}$, then $\operatorname{Re}\{f(\zeta)\}=0$ if and only if f is either the function*

$$F(z)=\left(\frac{1+c\varepsilon z}{1-\varepsilon z}\right)^2, \qquad c=\frac{1+i}{\sqrt{2}}, \qquad \varepsilon=\zeta+\frac{\sqrt{2}}{4\zeta}i,$$

with $F(\zeta)=i$, or its conjugation $\overline{F(\bar{z})}$.

§7. Geometry of the Omitted Arc

For the problem of minimum real part, we shall now explore the geometric properties of the arc Γ omitted by the extremal function f. A straightforward analysis of the quadratic differential (7) shows that Γ is tangent to the ray

$$w=(\bar{B}-1)t, \qquad t\geqq 0 \tag{8}$$

at the origin, and that Γ is asymptotic to the half-line

$$w=\tfrac{1}{3}(B+1)+B(B-1)t, \qquad t\geqq 0 \tag{9}$$

near infinity.

The differential equation (7) has a simple geometric interpretation. The trajectories of

$$\frac{dw^2}{w(w-1)}>0$$

are the hyperbolas with foci at 0 and 1. Thus Γ and all other trajectories of (7) are determined by the requirement that they meet each of these hyperbolas at an angle

$$\tfrac{1}{2}\arg\left\{\frac{w-B}{B(B-1)}\right\},$$

where w is the point of intersection. In particular, at each point of the half-line

$$w=B+B(B-1)t, \qquad t\geqq 0, \tag{10}$$

which is parallel to the asymptotic half-line (9), all trajectories of (7) are orthogonal to the ellipses with foci at 0 and 1. More generally, on each half-line

$$w=B+Ct, \qquad t\geqq 0$$

the trajectories of (7) intersect these ellipses at a constant angle.

Specializing the inequality (5) to the minimum real-part problem, one finds that Γ lies in the half-plane

$$\mathrm{Re}\left\{\frac{w-B}{B(B-1)}\right\}>0 \tag{11}$$

bounded by the line through B perpendicular to the asymptotic half-line (9).

Suppose first that $\mathrm{Re}\{B\}\leqq 0$, and assume without loss of generality that $\mathrm{Im}\{B\}>0$. Then

$$\pi<\arg\{\bar{B}-1\}<\frac{3\pi}{2}, \qquad \pi<\arg\{B(B-1)\}<2\pi, \tag{12}$$

and

$$0<\arg\left\{\frac{-B}{B(B-1)}\right)<\frac{\pi}{2}. \tag{13}$$

It follows that the asymptotic half-line (9) lies (except for a finite segment) in the sector

$$\arg\{B(B-1)\}<\arg\{w-B\}<\arg\{-B\} \tag{14}$$

bounded by the half-lines (10) and $w=B-Bt$, $t\geqq 0$. The typical situation is illustrated in Fig. 2.

We claim now that $\arg\{w-B\}$ decreases as w leaves the origin and moves along Γ to infinity. It is clearly decreasing near the origin. If it were to increase somewhere, then Γ would cross some half-line $w=B+Ct$ twice, once in each direction. But at each point on this line, Γ makes the same angle with the intersecting ellipse with foci at 0 and 1. Thus in one of these crossings, Γ would violate its monotonicity with respect to this family of ellipses (Corollary to Theorem 1).

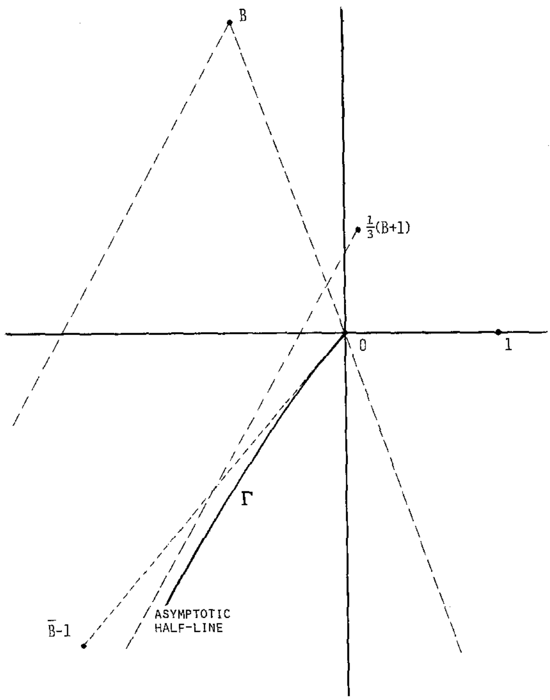

Fig. 2. *The omitted arc Γ, with* $\mathrm{Re}\{B\} \leqq 0$

It follows that Γ is confined to the sector (14). Indeed, since Γ is confined to the half-plane (11), it cannot intersect the bounding half-line (10) without violating the monotonicity of $\arg\{w-B\}$ in returning to the asymptotic half-line (9).

It is also clear that Γ must stay in the lower half-plane $\mathrm{Im}\{w\} \leqq 0$, for it cannot cross the segment of the real axis lying in the sector (14) without violating the monotonicity of Γ with respect to the confocal ellipses.

Next suppose that $\mathrm{Re}\{B\} > 0$ and $\mathrm{Im}\{B\} \geqq 0$. Then $|B| \leqq 1$ and $B \neq 1$. In view of Theorem 6, $\mathrm{Re}\{B\} < \frac{1}{2} = k_0(\sqrt{8}-3)$ unless B is real. Thus the inequality (13) again holds, and the inequalities (12) hold unless B is real. The preceding argument therefore applies without change unless B is real, in which case Γ is the negative real axis. The situation is illustrated in Fig. 3.

We have proved the following theorem.

Theorem 8. *Let $f \in S_0$ have minimum real part at ζ $(0 < \zeta < 1)$, and suppose $B = f(\zeta)$ has positive imaginary part. Then the curve Γ omitted by f lies in the lower half-plane and in the sector* (14). *Furthermore,* $\arg\{w-B\}$ *decreases as w moves along Γ from zero to infinity. At the origin Γ is tangent to the half-line* (8), *while it is asymptotic to the half-line* (9) *at infinity.*

Corollary. *The angle between the omitted arc Γ and confocal hyperbolas (with foci at* 0 *and* 1) *decreases to zero as the point of intersection moves along Γ from the origin to infinity.*

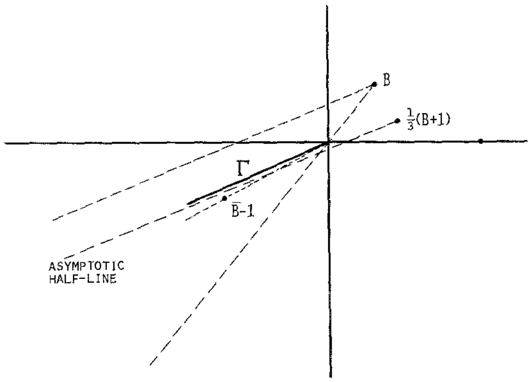

Fig. 3. *The omitted arc Γ, with* $\mathrm{Re}\{B\}>0$

Proof of Corollary. In view of the differential equation (7), the monotonicity of this angle of intersection is equivalent to the monotonicity of $\arg\{w-B\}$.

Brown [4] has used similar methods to describe the arc omitted by the extremal function for a related problem in the class S.

§8. Calculation of B

We now turn to the actual calculation of $B=f(\zeta)$. The first step is to use the differential equation (7) for the omitted arc Γ to derive a differential equation for the extremal function f. This together with Theorem 8 will allow us to determine a curve containing B.

Let Γ be parametrized by $w=f(e^{it})$, and apply (7) to conclude that

$$R(z)=\frac{B(1-B)\,z^2[f'(z)]^2}{f(z)\,[f(z)-1]\,[f(z)-B]}\geqq 0, \qquad |z|=1. \tag{15}$$

This function R has a simple zero at 0 and a simple pole at ζ. By the Schwarz reflection principle, it can be continued analytically to $|z|\geqq 1$ according to the formula $R(z)=\overline{R(1/\bar{z})}$. The extended function has a simple zero at ∞ and a simple pole at $1/\bar{\zeta}$. At the points $e^{i\alpha}$ and $e^{i\beta}$ which correspond to the finite and infinite tips of the slit, f has a double zero and a double pole, respectively, so that R is analytic and nonvanishing at these points. Thus R is a rational function of the form

$$R(z)=\frac{A\,z}{(z-\zeta)(1-\bar{\zeta}\,z)}, \tag{16}$$

where A is a constant. Since $R(z)>0$ on $|z|=1$, it follows that $A>0$.

Equating the expressions (15) and (16) for R, and setting $w=f(z)$, we find

$$\frac{B(B-1)}{w(w-1)(w-B)}\,dw^2=\frac{A}{z(\zeta-z)(1-\bar{\zeta}\,z)}\,dz^2, \tag{17}$$

or

$$\sqrt{B(B-1)} \int_1^w \frac{dw}{\sqrt{w(w-1)(w-B)}} = \sqrt{A} \int_0^z \frac{dz}{\sqrt{z(\zeta-z)(1-\zeta z)}}.$$

The relation (17) shows, in particular, that f maps the linear segment with endpoints 0 and ζ onto an arc C which is a trajectory of the quadratic differential (7). This arc C joins 1 to B, and

$$\sqrt{B(B-1)} \int_C \frac{dw}{\sqrt{w(w-1)(w-B)}} > 0. \tag{18}$$

The integrand in (18) has singularities at 0, 1, B, and ∞. According to Theorem 8, however, the omitted arc Γ (which joins 0 to ∞) lies either along the negative real axis or in the lower half-plane, under the normalizing assumption that $\operatorname{Im}\{B\} \geqq 0$. Thus on the Riemann sphere punctured at $0, 1, B$, and ∞, the arc C is homotopic to the linear segment with endpoints 1 and B. It follows that

$$J(B) = \sqrt{B(B-1)} \int_1^B \frac{dw}{\sqrt{w(w-1)(w-B)}}$$

is real and positive, where the integral is taken over the linear segment. Parametrizing this segment by

$$w = 1 + (B-1)\,t, \qquad 0 \leqq t \leqq 1,$$

we obtain

$$J(B) = \int_0^1 \left\{ \frac{B(1-B)}{1-t(1-B)} \right\}^{1/2} \frac{dt}{\sqrt{t(1-t)}}. \tag{20}$$

It is clear that $J(B) > 0$ for $0 < B < 1$. We shall now show that $J(B)$ is real on the positive imaginary axis only for $B = i$. This will show that $B = i$ is actually the value of an extremal function for some ζ, thus completing the proof of Theorem 7.

Theorem 9. *The integral $J(B)$ given in* (20) *has the properties $J(i) > 0$,*

$$\operatorname{Im}\{J(i\,b)\} > 0 \quad \textit{for } 0 < b < 1,$$

and

$$\operatorname{Im}\{J(i\,b)\} < 0 \quad \textit{for } 1 < b < \infty.$$

Proof. A simple calculation gives

$$J(i) = \sqrt{2} \int_0^1 G(t) \frac{dt}{\sqrt{t(1-t)}},$$

where

$$G(t) = \left\{ \frac{1 + (1-2t)\,i}{1 + (1-2t)^2} \right\}^{1/2}.$$

Making the substitution $t=1-s$ in the interval $\frac{1}{2}\leqq t\leqq 1$, we obtain

$$J(i)=\sqrt{2}\int_0^{1/2}[G(t)+\overline{G(t)}]\frac{dt}{\sqrt{t(1-t)}}>0.$$

A similar calculation gives

$$\operatorname{Im}\{J(ib)\}=\sqrt{b}\int_0^{1/2}\operatorname{Im}\{[H_1(t,b)]^{1/2}-[H_2(t,b)]^{1/2}\}\frac{dt}{\sqrt{t(1-t)}}$$

for $0<b<\infty$, where

$$H_1(t,b)=\frac{b+i[1-(1+b^2)t]}{(1-2t)+(1+b^2)t^2}$$

and

$$H_2(t,b)=\frac{b+i[b^2-(1+b^2)t]}{b^2(1-2t)+(1+b^2)t^2}.$$

For convenience, let

$$H_j^{1/2}=(x_j+iy_j)^{1/2}=\xi_j+i\eta_j,\qquad j=1,2.$$

Then $\operatorname{Im}\{J(ib)\}$ is the integral of $(\eta_1-\eta_2)$ with respect to the positive measure $\{b/t(1-t)\}^{1/2}dt$. We shall prove the inequalities asserted in the theorem by showing that $\eta_1>\eta_2$ throughout the interval $0\leqq t\leqq\frac{1}{2}$ if $0<b<1$, while $\eta_1<\eta_2$ for $0\leqq t\leqq\frac{1}{2}$ if $1<b<\infty$.

Suppose first that $0<b<1$. Then x_1, y_1, and x_2 are all positive in $0\leqq t\leqq\frac{1}{2}$, while y_2 changes sign at $t_0=b^2/(1+b^2)$. Thus $\eta_1>0$ in $0\leqq t\leqq\frac{1}{2}$, while $\eta_2>0$ in $0\leqq t<t_0$ and $\eta_2<0$ in $t_0<t\leqq\frac{1}{2}$. In particular, the inequality $\eta_1>\eta_2$ is trivial in $t_0\leqq t\leqq\frac{1}{2}$. In order to prove it in $0<t<t_0$, we observe that $0<x_1<x_2$ and $0<y_2<y_1$ there. Since $0<b<1$, the inequality $x_1<x_2$ is obvious. A straightforward calculation shows that the inequality $y_2<y_1$ is equivalent to

$$(1-b^2)(1+b^2)t(1-t)>0,$$

which clearly holds in $0<t<1$ if $b^2<1$. Next observe that

$$x_j=\xi_j^2-\eta_j^2\quad\text{and}\quad y_j=2\xi_j\eta_j,$$

so that

$$x_j=\frac{y_j^2}{4\eta_j^2}-\eta_j^2,\qquad j=1,2.$$

Since $0<y_2<y_1$ and $0<x_1<x_2$ in $0<t<t_0$, one sees by a simple graphical argument that $\eta_1>\eta_2$ there.

The case $1<b<\infty$ is treated by similar considerations. Now x_1, x_2, and y_2 are all positive in $0\leqq t\leqq\frac{1}{2}$, while y_1 changes sign at $t_0'=1/(1+b^2)$. The inequality

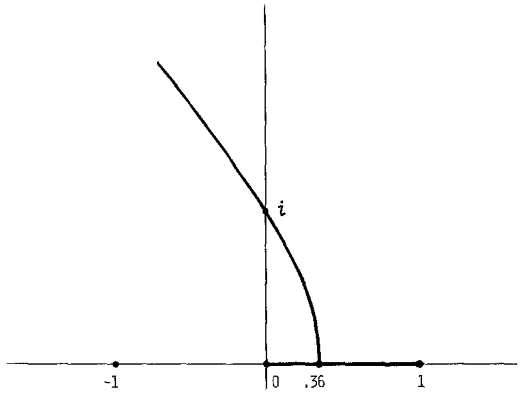

Fig. 4. *Solution set of* $\mathrm{Im}\{J(B)\}=0$, $\mathrm{Im}\{B\}\geqq 0$.

$\eta_1<\eta_2$ is trivial in $t_0'\leqq t<\frac{1}{2}$; while for $0<t<t_0'$ it follows in analogous fashion from the inequalities $0<x_2<x_1$ and $0<y_1<y_2$. Since $b>1$, the inequality $x_2<x_1$ is obvious, while the previous calculation shows that the inequality $y_1<y_2$ is equivalent to

$$(1-b^2)(1+b^2)\,t(1-t)<0.$$

This completes the proof of Theorem 9.

A numerical calculation indicates that $\mathrm{Im}\{J(B)\}$ vanishes on the curve shown in Fig. 4. In particular, these numerical results indicate that B must cover the real segment $0.36\leqq B<1$. Since $k_0(-\frac{1}{4})=0.36$, this evidence suggests the conjecture that the Koebe function $k_0(-z)$ is extremal for $0<\zeta\leqq\frac{1}{4}$, but not for $\zeta>\frac{1}{4}$. In other words, it is probable that the constant $3-\sqrt{8}=0.171\ldots$ of Theorem 6 can be increased to $\frac{1}{4}$, but not to the radius of convexity $2-\sqrt{3}=0.267\ldots$, which we have established as an upper bound.

In closing, we remark that the differential equation (17) actually provides enough relations to determine (in principle) both B and the positive real number A as functions of ζ. We have already used the imaginary part of the equation

$$J(B)=\sqrt{A}\int_0^\zeta \frac{dz}{\sqrt{z(\zeta-z)(1-\zeta z)}},$$

where the integration is performed over the linear segment $0\leqq z\leqq\zeta$. The substitution $u=(z/\zeta)^{1/2}$ transform this equation to

$$J(B)=2\sqrt{A}\,K(\zeta), \tag{21}$$

where

$$K(k)=\int_0^1 \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}}, \qquad 0<k<1,$$

is a complete elliptic integral of the first kind. In addition, we have

$$I(B)=\sqrt{B(B-1)}\int_0^1 \frac{dw}{\sqrt{w(w-1)(w-B)}}=\sqrt{A}\int_{e^{i\alpha}}^0 \frac{dz}{\sqrt{z(\zeta-z)(1-\zeta z)}},$$

where the right-hand integral is over a radial segment and the left-hand integral is over the corresponding path. The right-hand integral can be expressed as the sum of two integrals, the first over the unit circle from $e^{i\alpha}$ to -1 and the second over the radial segment from -1 to 0. The first of these integrals is real and the second imaginary. Thus we have

$$\operatorname{Im}\{I(B)\}=\sqrt{A}\int_0^1 \frac{dx}{\sqrt{x(\zeta+x)(1+\zeta x)}}. \tag{22}$$

The integral $I(B)$ in (22) can now be taken over the linear segment, since it differs from this by an integer multiple of $J(B)$, which is real.

The Eqs. (21) and (22) provide three real relations for the complex number B and the real number A.

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