Math. Z. 186, 407-417 (1984)

Mathematische Zeitschrift
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Comparison of Two Capacities in $\mathbb{C}^n$

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1. Introduction

In the past few years, several capacitary set functions have been introduced in connection with the study of analytic and plurisubharmonic functions of several variables ([1-5, 9-13]). There are many different such functions and the relationship between them is not at all clear. We shall consider here the *relative capacity* of Bedford and Taylor, and the capacity defined in terms of certain Tchebycheff constants as studied by Zaharjata [13] and Alexander [1]. (See Sect. 2 for the definitions.) We show that these capacities are essentially the same. Our main result, Theorem 2.1, gives the quantitative relationship between them.

It happens that the relationship between the two capacities is closely connected with a result of Josefson [8] on the equivalence of locally and globally pluripolar sets. The quantitative estimates of Theorem 2.1 allow us to give in Sect. 4 a new proof of Josefson's lemma about normalized polynomials which are very small on the sets where a given plurisubharmonic function is nearly $-\infty$. Josefson's own proof is a direct construction; our original motivation was to give a proof based on capacity notions. It turns out that the Tchebycheff polynomials themselves already do the job.

El Mir [7] has obtained an extension of Josefson's theorem. Given a function v psh on an open set he obtains a global psh function u of small growth at infinity such that u is dominated by a function $h(v)$ of v. His choice of h is essentially $h(x)=-\log|x|$. Thus he gets $u \leqq -\log|v|$ and so the value of the global function u is $-\infty$ whenever $v=-\infty$. Our methods give a short proof of this; in fact, they apply to functions h satisfying

$$\int_{-\infty}^{0} \frac{|h(x)|}{|x|^{1+1/n}} dx < \infty.$$

2. Statement of Main Result

For Ω a smoothly bounded, strongly pseudoconvex domain in $\mathbb{C}^n$ and K a compact subset of Ω, the *capacity of K relative to Ω* is defined in terms of the complex Monge-Ampere operator by

$$\operatorname{cap}(K;\Omega):=\sup\{\int_K (dd^c u)^n: 0<u<1, u\in P(\Omega)\}, \tag{2.1}$$

where $P(\Omega)$ denotes the class of all plurisubharmonic functions on Ω. The operator

$$(dd^c u)^n = (2i\partial\bar{\partial}u)^n = 4^n n! \det\left[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right]\beta_n$$

where β_n is the usual volume form on $\mathbb{C}^n$, and the properties of the capacitary function $\operatorname{cap}(K,\Omega)$ are discussed extensively in [3]. We refer the reader to that paper for more details. When the open set Ω is understood, we will write simply $\operatorname{cap}(K)$ for $\operatorname{cap}(K;\Omega)$ and call it the relative capacity of K.

The global capacity function we consider can be given in several essentially equivalent ways. It is defined in a manner usual when dealing with "Tchebycheff constants". For K a subset of $\mathbb{C}^n$, let

$$\|f\|_K = \sup\{|f(z)|: z\in K\}$$

denote the supremum norm. Let $\mathscr{P}_d$ denote the class of all polynomials on $\mathbb{C}^n$ of degree $\leqq d$, normalized by requiring that the maximum of the polynomial on the unit ball is at least 1. That is, a polynomial P_d of degree $\leqq d$ belongs to $\mathscr{P}_d$ if and only if

$$\|P_d\|_B \geqq 1, \tag{2.2}$$

where $B=\{z\in\mathbb{C}^n: |z|<1\}$. Then define the Tchebycheff constants for K by

$$M_d(K)=\inf\{\|P_d\|_K: P_d\in\mathscr{P}_d\} \tag{2.3}$$

and the capacity of K by

$$T(K)=\inf_{d\geqq 0}[M_d(K)]^{1/d}=\lim_{d\to\infty}[M_d(K)]^{1/d}. \tag{2.4}$$

Various other definitions of $T(K)$ can be given by changing the normalization (2.2) in defining the class $\mathscr{P}_d$. For example, the *projective capacity* treated by the first author in [1] is equivalent to replacing the normalization (2.2) by

$$\frac{1}{C}\int_{\mathbb{C}^n}\frac{\log|P_d(z)|\,d\lambda(z)}{(1+|z|^2)^{n+1}}\geqq d \tag{2.5}$$

where $C=\int_{\mathbb{C}^n}\frac{\log|z_1|\,d\lambda(z)}{(1+|z|^2)^{n+1}}$ is a dimensional constant ($d\lambda$ denotes Lebesgue measure). Other normalizations may be given by requiring that the sum of the absolute values of the coefficients of the polynomials P_d is at least 1, or the maximum coefficient is at least 1 (with respect to the usual basis of monomials). These all give capacities which are bounded above and below by a constant multiple of $T(K)$. Thus, it makes little difference which one is used. An account of many different capacities and their relationships is given in

[12]. We have used the particular normalization (2.2) because of its characterization in terms of an extremal function, given by Siciak (Theorem 9] of [12]). This is stated below as Theorem 3.2.

Note that although we call $T(K)$ the global capacity function, it is in some sense relative to the unit ball because of the choice of normalization in the definition of $\mathscr{P}_d$. In fact, if K contains the unit ball then $M_d(K)=1$ for all $d \geqq 0$ as follows by considering the constant polynomial $p_d \equiv 1$ in $\mathscr{P}_d$. The following is our main result.

Theorem 2.1 (Comparison Theorem). *Let K be a compact subset of the unit ball in $\mathbb{C}^n$. Then*

$$T(K) \leqq \exp\left[-\left(\frac{c_n}{\operatorname{cap}(K;B)}\right)^{1/n}\right] \tag{2.6}$$

where c_n is a constant given in (3.12). *For each $r<1$, there is a constant $A=A(r)$ such that for all compact sets $K \subset \{|z|<r\}$*

$$T(K) \geqq \exp\left(\frac{-A}{\operatorname{cap}(K;B)}\right). \tag{2.7}$$

Remark 1. Both the set functions $\operatorname{cap}(K,\Omega)$ and $T(K)$ are known to be "generalized capacities". Hence, the estimates of the theorem also hold for all capacitible sets – in particular all Borel sets.

Remark 2. The inequalities are sharp, at least as far as the exponents on $\operatorname{cap}(K;B)$ are concerned. For if $K=\{z:|z|\leqq\varepsilon\}$, then $T(K)=\varepsilon$ and $\operatorname{cap}(K;B) = c_n\left[\frac{-1}{\log\varepsilon}\right]^n$. Hence, equality holds in (2.6). On the other hand, if K is a small polydisc, $K=\{(z_1,\dots,z_n)\colon |z_1|\leqq\delta,\ |z_j|\leqq 1/2,\ j=2,\dots,n\}$ and $\delta<1/2$, then $T(K)\leqq\delta$, while $\operatorname{cap}(K;B)\geqq\text{const.}\left(\log\frac{1}{\delta}\right)^{-1}$. (To see the last inequality put

$$u(z)=\sum_{k=2}^{n}\frac{\log^+\left(\frac{|z_k|}{1/2}\right)}{\log 2}+\frac{\log^+\frac{|z_1|}{\delta}}{\log\frac{1}{\delta}}-n.$$

Note that $u<0$ on B and $u<-1$ on K. Hence

$$\operatorname{cap}(K;B)\geqq\int_B (dd^c u)^n \geqq \frac{\text{const}}{\log\frac{1}{\delta}}.)$$

Thus the exponent in (2.7) cannot be improved.

3. Proof of the Comparison Theorem

The proof of the comparison theorem follows from some simple quantitative relationships between extremal plurisubharmonic functions related to the capacities. For the relative capacity, the extremal function is

$$U_K^*(z) = U_K^*(z, \Omega) = \limsup_{\zeta \to z} U_K(\zeta), \tag{3.1}$$

the upper semicontinuous regularization of the envelope

$$U_K(z) = \sup\{v(z) \colon v \in P(\Omega),\ v \leqq -1 \text{ on } K,\ v < 0 \text{ on } \Omega\}. \tag{3.2}$$

The main properties of U_K^* are

$$U_K^* \in P(\Omega), \qquad -1 \leqq U_K^* \leqq 0, \qquad \lim_{z \to \partial\Omega} U_K^*(z) = 0. \tag{3.3}$$

$$(dd^c U_K^*)^n = 0 \qquad \text{on } \Omega \setminus K \tag{3.4}$$

$$U_K^* = -1 \quad \text{on } K, \text{except on a set of (relative) capacity zero} \tag{3.5}$$

$$\operatorname{cap}(K, \Omega) = \int_\Omega (dd^c U^*)^n = \int_K (dd^c U_K^*)^n. \tag{3.6}$$

Proofs of these facts are given in Prop. 5.3 of [3].

The other extremal function is the one introduced by Siciak [11]

$$u_K^*(z) = \limsup_{\zeta \to z} u_K(\zeta) \tag{3.7}$$

where

$$\begin{aligned} u_K(z) = \sup\{v(z) \colon v \in P(\mathbb{C}^n),\ v(z) \leqq 0 \text{ for } z \in K, \\ v(z) \leqq \log|z| + O(1),\ |z| \to \infty\}. \end{aligned} \tag{3.8}$$

Either $u_K^* \equiv +\infty$ or else u_K^* has the following properties:

$$u_K^* \in P(\Omega), \qquad u_K^*(z) = \log|z| + O(1), \qquad |z| \to \infty \tag{3.9}$$

$$u_K^*(z) = 0 \quad \text{for } z \in K, \text{except on a set of (relative) capacity zero.} \tag{3.10}$$

$$(dd^c u_K^*)^n \qquad \text{is a positive measure supported on } K. \tag{3.11}$$

The proof of (3.9) can be found in [11], while (3.10) and (3.11) follow directly from Proposition 9.3 and Corollary 9.4 of [3]. A further property of u_K^* is that

$$\int_{\mathbb{C}^n} (dd^c u_K^*)^n = c_n = \int_{\mathbb{C}^n} (dd^c \log^+|z|)^n; \tag{3.12}$$

the mass of the measure $(dd^c u_K^*)^n$ is independent of K. The proof of this fact requires a fundamental lemma from [3] (Theorem 4.1).

Lemma 3.1. *Let Ω be a bounded open set in $\mathbb{C}^n$ and $u, v \in P(\Omega) \cap L^\infty(\Omega)$ with $\liminf_{z \to \partial\Omega}(u(z) - v(z)) \geqq 0$. Then*

$$\int_{\{u<v\}} (dd^c v)^n \leqq \int_{\{u<v\}} (dd^c u)^n.$$

If we apply the lemma with $v(z) = u_K^*(z)$, $u(z) = (1+\varepsilon)\log^+|z| - A(\varepsilon)$ where $\varepsilon > 0$, $A(\varepsilon)$ is so large that $u(z) < 0$ on K, and Ω is a large ball, then we conclude

$$\int (dd^c u_K^*)^n \leqq (1+\varepsilon)^n \int_{\mathbb{C}^n} (dd^c \log^+ |z|)^n = c_n (1+\varepsilon)^n$$

which proves the $\leqq$ part of (3.12). Similarly, we can apply the lemma again, with $u(z)=u_K^*(z)$, $v(z)=(1-\varepsilon)\log^+|z|+A(\varepsilon)$ to obtain the opposite inequality of (3.12).

We also need to relate the capacity $T(K)$ to the extremal function u_K^*. This link is provided by the following theorem of Siciak.

Theorem 3.2 (Theorem 9.1, [12]).

$$T(K)=\exp(-a)$$

where

$$a=\sup\{u_K^*(z) \colon |z|\leqq 1\}.$$

We now have all the ingredients needed for the proof of (2.6) of the comparison theorem.

Proof of (2.6). Define

$$v(z)=\frac{u_K^*(z)-a}{a}.$$

Then $v\in P(B)$, $v<0$, and $v=-1$ on K, except on a set of (relative) capacity zero. Set $u=(1+\varepsilon)U_K^*$, $\Omega=B$, and apply the lemma to obtain

$$\frac{1}{a^n}\int_{\{u<v\}} (dd^c u_K^*)^n \leqq (1+\varepsilon)^n \int_{\{u<v\}} (dd^c U_K^*)^n.$$

However, $K\subset\{u<v\}\cup E$, where E is a set of relative capacity zero and therefore which supports no mass for either of the measures $(dd^c u_K^*)^n$ and $(dd^c U_K^*)$. So from (3.4), (3.6) and (3.12) we find

$$\operatorname{cap}(K;B)\geqq\frac{c_n}{a^n}. \tag{3.13}$$

Because of Siciak's theorem, this is equivalent to (2.6).

For the proof of the other part, (2.7), of the comparison theorem, two additional facts are needed. The first is an estimate for the mass of the measure $(dd^c u)^n$ which is sharper than the original estimate of Chern, Levine, and Nirenberg. (See e.g. [6], Sect. 2 and [3], Theorem 2.4.)

Lemma 3.3. *Let* $\omega\equiv\{|z-z_0|<\rho\}\Subset\Omega\equiv\{|z-z_0|<R\}$ *and* $u\in P(\Omega)\cap L^\infty(\Omega)$ *with* $u<0$. *There is a constant* $C=C(\rho,R,n)$, *independent of* u *such that*

$$\int_\omega (dd^c u)^n \leqq C(-u(z_0))\sup_{z\in\Omega}|u(z)|^{n-1}.$$

Remark. For the proof of this, it is convenient to give a generalization of the definition of the currents $(dd^c u)^k$ for $u\in P(\Omega)\cap L^\infty(\Omega)$. One can allow a positive closed (s,s) current ψ as a factor; such a ψ need not come from bounded psh functions. A Chern-Levine-Nirenberg type estimate involving the mass of ψ also applies in this case.

Lemma 3.4. *Let $v_1, v_2, \ldots, v_k \in P(\Omega) \cap L^\infty(\Omega)$ and let ψ be a closed positive (s,s) current on Ω with $s+k \leqq n$. Then there is a well defined closed positive $(k+s, k+s)$ current $\phi = \psi \wedge dd^c v_1 \wedge \ldots \wedge dd^c v_k$ on Ω. If $K \Subset \omega \Subset \Omega$ there is a constant $C = C(K, \omega, \Omega)$ (independent of ϕ) such that*

$$|\int_K \phi \wedge \beta_{n-(s+k)}| \leqq C \int_\omega \psi \wedge \beta_{n-s} \cdot \prod_{j=1}^{k} \|v_j\|_\omega. \tag{3.14}$$

Proof of Lemma 3.4. Define ϕ by induction on k on a $C_0^\infty(\Omega)$ $(n-(s+k), n-(s+k))$ form χ by

$$\int \phi \wedge \chi = \int v_k \phi_1 \wedge dd^c \chi$$

where $\phi_1 = \psi \wedge dd^c v_1 \wedge \ldots \wedge dd^c v_{k-1}$ is given by induction. The verification that ϕ is a positive closed $(s+k, s+k)$ current is just as in [4], Proposition 2.9.

The estimate (3.14) is obtained by induction on k. For $k=1$, let $0 \leqq \sigma \in C_0^\infty(\omega)$, $\sigma \equiv 1$ on K. Then

$$\begin{aligned}|\int_K \psi \wedge dd^c v_1 \wedge \beta_{n-(k+1)}| &\leqq |\int \psi \wedge dd^c v_1 \wedge \sigma \beta_{n-(k+1)}| \\ &= |\int v_1 \wedge \psi\, dd^c \sigma \wedge \beta_{n-(k+1)}| \\ &\leqq C \|v_1\|_\omega \int_\omega \psi \wedge \beta_{n-k}.\end{aligned}$$

Now the general case follows from this.

$$|\int_K \psi \wedge dd^c v_1 \wedge \ldots \wedge dd^c v_k \wedge \beta_{n-(s+k)}| = |\int_K \tilde{\phi} \wedge dd^c v_k \wedge \beta_{n-(s+k)}|$$

where $\tilde{\phi} = \psi \wedge dd^c v_1 \wedge \ldots \wedge dd^c v_{k-1}$. By the $k=1$ case this last integral is dominated by

$$\begin{aligned}&C \|v_k\|_{\omega_1} \int_{\omega_1} \tilde{\phi} \wedge \beta_{n-(s+k-1)} && (\text{where } K \subseteq \omega \Subset \omega_1 \Subset \Omega) \\ &\leqq C \|v_k\|_{\omega_1} \int_\omega \psi \wedge \beta_{n-s} \prod_{j=1}^{k-1} \|v_j\|_\omega && (\text{by induction hypothesis}).\end{aligned}$$

This gives (3.14).

Proof of Lemma 3.3. We can assume $z_0 = 0$. In Lemma 3.4 we take $\psi = dd^c u$ and $k = n-1$ with $v_k = u$ for $1 \leqq k \leqq n-1$. Applying (3.14) we will be finished once we show that $\int_\omega dd^c u \wedge \beta_{n-1} \leqq C(-u(0))$. To see this first apply Jensen's formula:

$$u(0) + N(r) = \frac{1}{\sigma_{2n-1}} \int_{|\alpha|=1} u(r\alpha)\, d\sigma(\alpha)$$

where $d\sigma$ is surface area measure on the unit sphere, $\sigma_{2n-1} = \int_{|\alpha|=1} d\sigma$, and where

$$N(r) = \int_0^r \frac{n(t)}{t^{2n-1}}\, dt$$

and

$$n(t)=\frac{1}{\sigma_{2n-1}}\int_{|z|\leq t}(\Delta u)(z)\,d\lambda(z)=a_n\int_{|z|\leq t}dd^c u\wedge\beta_{n-1}$$

is the total mass of the Laplacian in the ball of radius t and a_n is a dimensional constant. Since u is plurisubharmonic, $n(t)/t^{2n-2}$ is increasing, so for $\lambda>1$,

$$N(\lambda r)\geqq\int_r^{\lambda r}\frac{n(t)}{t^{2n-1}}\,dt\geqq\frac{n(r)}{r^{2n-2}}\log\lambda.$$

Also, since $u<0$, we have $N(r)<-u(0)$. Hence

$$\int_\omega dd^c u\wedge\beta_{n-1}\leqq C\,n(\rho)\leqq C(-u(0)).$$

The other ingredient we need is an observation of how the relative capacity $\operatorname{cap}(K;\Omega)$ depends on Ω. It is obvious from the definition that $K\subset\Omega_1\subset\Omega_2$ implies $\operatorname{cap}(K,\Omega_1)\geqq\operatorname{cap}(K,\Omega_2)$. However, we need an estimate in the other direction.

Lemma 3.5. *Let Ω_2 be a strongly pseudoconvex domain in $\mathbb{C}^n$. Let $\Omega_1\subset\Omega_2$ be a strongly pseudoconvex domain which is (plurisubharmonically) convex with respect to Ω_2; i.e., $z\in\Omega_2\backslash\Omega_1$ implies that for every $\Omega_3\Subset\Omega_1$ there exists $u\in P(\Omega_2)$ with $u(z)>\sup\{u(\zeta)\colon\zeta\in\Omega_3\}$. Let $\omega\Subset\Omega_1$. Then there exists a constant C, depending only on ω,Ω_1,Ω_2 and the dimension n, such that for all compact $K\subset\omega$,*

$$\operatorname{cap}(K,\Omega_1)\leqq C\operatorname{cap}(K,\Omega_2). \tag{3.15}$$

In the special case $\omega=\{|z|<\rho\}$, $\Omega_1=\{|z|<1\}$, $\Omega_2=\{|z|<R\}$, we can take $C=\left(\log\frac{1}{\rho}\right)^n\left(\log\frac{R}{\rho}\right)^{-n}$.

Proof. Let ρ be a plurisubharmonic defining function for Ω_2. It is no loss of generality to assume that for some $\delta>0$,

$$\omega\subset\{\rho<-1\}\subset\{\rho<-1+\delta\}\subset\Omega_1\Subset\Omega_2.$$

That is, $K\subset\omega\Rightarrow\rho\leqq-1$ while $\rho\geqq-1+\delta$ on $\partial\Omega_1$.

Let U_j^*, $j=1,2$ denote the extremal functions U_K^* for the compact set $K\subset\omega$ with respect to the open sets Ω_j, $j=1,2$. Then

$$U_2^*\geqq\rho\geqq-1+\delta\quad\text{on }\partial\Omega_1$$

so

$$\frac{1}{\delta}(U_2^*+1)\geqq U_1^*+1\quad\text{on }\partial\Omega_1.$$

Apply the inequality of Lemma 3.1, with $u=\frac{(1+\varepsilon)}{\delta}(U_2^*+1)$, $v=U_1^*+1$, $\Omega=\Omega_1$, and let $\varepsilon\to0$, to obtain

$$\operatorname{cap}(K,\Omega_1)=\int(dd^c U_1^*)^n\leqq\left(\frac{1}{\delta}\right)^n\int(dd^c U_2^*)^n$$
$$=\delta^{-n}\operatorname{cap}(K,\Omega_2).$$

In the special case of the balls, we can take

$$\rho(z) = -1 + [\log^+(|z|/\rho)]\,[\log(R/\rho)]^{-1}$$

to obtain

$$\delta = [\log 1/\rho]\,[\log R/\rho]^{-1}.$$

We can now prove the other half, (2.7), of the comparison theorem.

Proof of (2.7). Let $K \subset \{|z| < r\}$, $r < 1$, K compact, and $a = \max\{u_K^*(z)\colon |z| \leqq 1\} = u_K^*(z_0)$, $|z_0| = 1$. If $a = +\infty$, then K has capacity zero for both $\operatorname{cap}(K; \Omega)$ and $T(K)$ so there is nothing to prove. Thus, we assume $a < +\infty$. Then $u_K^*(z) - a \leqq 0$ on $|z| \leqq 1$ and so

$$u_K^*(z) - a \leqq \log^+|z|, \qquad z \in \mathbb{C}^n$$

(see e.g. [11, § 2]).

Let U_K^* be the extremal function for K relative to the ball $|z| < 3$. Then

$$U_K^*(z) \geqq \frac{u_K^*(z) - a - \log 3}{a + \log 3}$$

from the definition of U_K^*. If we now set $z = z_0$, we see

$$U_K^*(z_0) \geqq -\log 3/(a + \log 3).$$

Therefore, by Lemma 3.3, applied with $u = U_K^*$,

$$\omega = \{|z - z_0| < 1 + r\}, \qquad \Omega = \{|z - z_0| < 2\}$$

we have

$$\operatorname{cap}(K) = \int (dd^c U_K^*)^n \leqq C \log 3/(a + \log 3)$$

where the capacity is relative to $|z| < 3$. Then, by Lemma 3.5,

$$\operatorname{cap}(K; |z| < 1) \leqq C'(\log 3)/(a + \log 3) \leqq C''/a.$$

This inequality is equivalent to (2.7), so the proof is complete.

4. New Proofs of Josefson's Lemma and El Mir's Theorem

The essential part of Josefson's proof [8] that locally pluripolar sets are globally pluripolar is a construction of polynomials which are normalized and small on the set where a given analytic function is small. We show in this section how the quantitative estimate relating $T(K)$ and $\operatorname{cap}(K)$ implies Josefson's lemma and in fact shows that the Tchebycheff polynomials themselves already have the desired properties.

Proposition 4.1. *Let u be plurisubharmonic for $|z| < 1$ with $u(z) < 0$, $u(0) \geqq -1$. Let K be any compact subset of*

$$\{z \in \mathbb{C}^n\colon |z| \leqq r,\ u(z) < -A\} \qquad (r < 1,\ A > 1).$$

Then there exists a sequence of polynomials $\{p_d(z)\}$ for infinitely many integers d such that

(i) $p_d(z)$ *has degree* $\leqq d$,

(ii) $|p_d(z)| \leqq |z|^d$ *if* $|z| \geqq 1$,

(iii) $\sup_{|z|=1} |p_d(z)| = 1$,

(iv) $|p_d(z)| \leqq \exp(-CA^{1/n} \cdot d)$ *for all z in K*

where $C = C(r)$ is a constant depending only on $r < 1$ and n.

Proof. Let U_K^* be the relative extremal function for K. Then $U_K^* \geqq u/A$, so $U_K^*(0) \geqq -1/A$. Thus by (3.6) and Lemma 3.3 there is a constant $C = C(r)$ such that

$$\operatorname{cap}(K; |z| < 1) \leqq C/A. \tag{4.1}$$

By (2.7) of the Comparison Theorem we have

$$\frac{C}{\log\left(\frac{1}{T(K)}\right)} \leqq [\operatorname{cap}(K; |z| < 1]^{1/n}. \tag{4.2}$$

Hence $T(K) \leqq \exp(-CA^{1/n})$. Choose polynomials $p_d \in \mathscr{P}_d$ such that $\|p_d\|_B = 1$ and $\|p_d\|_K = M_d(K)$ for each $d \geqq 0$. Then since $\inf \|p_d\|_K^{1/d} = T(K) \leqq \exp(-CA^{1/n})$, we have $\|p_d\|_K < \exp(-\frac{1}{2}CA^{1/n}d)$ for infinitely many d's; this gives (iv). Finally, (ii) follows because

$$\frac{1}{d}\log|p_d| \leqq u_{B^*} \equiv \log^+|z|$$

for $|z| \geqq 1$.

Consider a convex, increasing function $h(x)$ defined for $-\infty < x < +\infty$ such that $h(0) = 0$ and

$$\int_{-\infty}^{0} \frac{|h(x)|\,dx}{|x|^{1+1/n}} < +\infty. \tag{4.3}$$

For example, if $0 < \alpha < 1/n$, then

$$h(x) = \begin{cases} -\alpha^{-1}[(1-x)^\alpha - 1] & x < 0 \\ x & x \geqq 0 \end{cases}$$

is such a function. Another example, the one used by El Mir, is

$$h(x) = \begin{cases} -\log(1-x) & x < 0 \\ x & x \geqq 0. \end{cases}$$

Theorem 4.2. *Let h be as above. Let v be psh on $\{|z| < 1\}$ with $v < 0$ and $v(0) = -1$. Then there exists a function u psh on $\mathbb{C}^n$ such that*

(i) $u = O(\log|z|)$ *as* $z \to \infty$,

(ii) $u(z) \leqq h(v(z))$ *for* $|z| < \frac{1}{2}$.

Remark. On p. 74 of [7], it is stated that the theorem is true with $h(x)=-[-x]^{1/n}$. However, it does not appear that the line of argument used in [7] will yield this result. It seems to be an interesting problem to decide if the "$1/n$" is really necessary. Could the theorem hold with $h(x)=-(-x)^{1-\varepsilon}$? An example of El Mir shows that $\varepsilon>0$ is needed; i.e., one cannot always find u such that $u(z)\leqq v(z)$ for $|z|<\frac{1}{2}$.

Proof. Let $\mathcal{O}_A$ denote the open set $\{z: |z|<\frac{1}{2},\ v(z)<A\}$ and let u_A^* denote the extremal function of $\mathcal{O}_A$. Thus, $u_A^*(z)\leqq \log|z|+O(1)$, $|z|\to\infty$, and $u_A^*(z)=0$ for $z\in\mathcal{O}_A$, because $\mathcal{O}_A$ is open. Set $\alpha(A)=\sup\{u_A^*(z): |z|=1\}$ and $v_A(z)=u_A^*-\alpha(A)$. We claim that

$$\alpha(A)\geqq \text{const.}\, A^{1/n} \tag{4.4}$$

$$v_A(z)\leqq \log^+|z| \tag{4.5}$$

$$v_A(z)=-\alpha(A), \qquad z\in\mathcal{O}_A \tag{4.6}$$

$$\int_{|z|=2} v_A(z)\,d\sigma(z)\geqq \text{const.} \tag{4.7}$$

Assuming that (4.4)-(4.7) hold, it is easy to see that the function

$$u(z)=\int_{-\infty}^{0} v_A(z)\,h'(A)\,A^{-1/n}\,dA$$

has the desired properties. For, if $|z|<\frac{1}{2}$ and $v(z)<C$, then

$$\begin{aligned}
u(z)&\leqq \int_C^0 v_A(z)\,h'(A)\,A^{-1/n}\,dA\\
&=\int_C^0 \alpha(A)\,h'(A)\,A^{-1/n}\,dA\\
&\leqq -\text{const.}\int_C^0 h'(A)\,dA\\
&=-\text{const.}\,[h(0)-h(C)]\\
&=\text{const.}\,h(C).
\end{aligned}$$

Because this holds whenever $v(z)<C$ and h is continuous, it follows that $u(z)\leqq \text{const.}\,h(v(z))$. Replacing $u(z)$ by const. $\cdot\, u(z)$ gives assertion (ii) of the theorem. Assertion (i) follows from (4.5), because an integration by parts shows that convergence of $\int_{-\infty}^{0} h'(A)\,A^{-1/n}\,dA$ is equivalent to (4.3).

It remains to prove (4.4)-(4.7). The inequality (4.4) follows from (4.1) and (4.2) applied to an exhaustion of $\mathcal{O}_A$ by compact subsets. Inequality (4.5) is clear because $v_A(z)\leqq 0$ for $|z|<1$, and (4.6) holds because $u_A^*(z)=0$ for $z\in\mathcal{O}_A$. Finally, (4.7) follows from (4.5) and the Poisson-Jensen formula on the ball $|z|\leqq 2$, because $v_A(z_0)=0$ for some z_0 with $|z_0|=1$. This completes the proof.

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Received October 19, 1983