Math. Z. 187, 97–103 (1984)

Mathematische Zeitschrift

Dilation Theory and Systems of Simultaneous Equations in the Predual of an Operator Algebra. II

Hari Bercovici[1], Bernard Chevreau[2], Ciprian Foias[3], and Carl Pearcy[4]

[1] Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA
[2] U.E.R. de Mathematique et d'Informatique, Universite de Bordeaux I, France
[3] Department of Mathematics, Indiana University, Bloomington, Indiana 47405, USA
[4] Department of Mathematics, The University of Michigan, 347 West Engineering Building, Ann Arbor, Michigan 48109, USA

1. This note is a continuation of our earlier paper [3], in which we developed a dilation theory for a certain class of contraction operators acting on a separable, infinite dimensional, complex Hilbert space $\mathscr{H}$. The notation and terminology in what follows is taken from [3]. For the convenience of the reader we recall a few pertinent definitions. The algebra of bounded linear operators on $\mathscr{H}$ is denoted by $\mathscr{L}(\mathscr{H})$. If $T\in\mathscr{L}(\mathscr{H})$, the ultraweakly closed algebra generated by T and $1_{\mathscr{H}}$ is denoted by $\mathscr{A}_T$; we recall that $\mathscr{A}_T$ can be identified with the dual space of the quotient space $Q_T=(\tau c)/{}^{\perp}\mathscr{A}_T$, where (τc) denotes the ideal of trace-class operators in $\mathscr{L}(\mathscr{H})$ and ${}^{\perp}\mathscr{A}_T$ is the pre-annihilator of $\mathscr{A}_T$ in (τc), under the pairing

$$\langle A, [L]\rangle = \operatorname{tr}(AL), \qquad A\in\mathscr{A}_T,\ [L]\in Q_T.$$

The open unit ball in $\mathbb{C}$ is denoted by $\mathbb{D}$, and we write $\mathbb{T}=\partial\mathbb{D}$. The class $\mathbb{A}(\mathscr{H})\subset\mathscr{L}(\mathscr{H})$ consists of all those absolutely continuous contractions T (i.e., all those contractions T whose unitary part is absolutely continuous or acts on the space (0)) such that the Sz.-Nagy-Foias functional calculus $\Phi_T\colon H^\infty(\mathbb{T})\to\mathscr{A}_T$ is an isometry. If $T\in\mathbb{A}(\mathscr{H})$ then Φ_T is the adjoint of an isometry ϕ_T of Q_T onto the predual $L^1(\mathbb{T})/H_0^1(\mathbb{T})$ of $H^\infty(\mathbb{T})$ (cf. [3, 5]), and via the pair $\{\phi_T,\Phi_T\}$, the pair of spaces $\{L^1(\mathbb{T})/H_0^1(\mathbb{T}), H^\infty(\mathbb{T})\}$ can be identified with the pair $\{Q_T,\mathscr{A}_T\}$.

If $x, y\in\mathscr{H}$, we write $x\otimes y$ for the rank-one operator defined, as usual, by $(x\otimes y)(u)=(u,y)x$, $u\in\mathscr{H}$. Of course, $x\otimes y\in(\tau c)$, and if some $T\in\mathscr{L}(\mathscr{H})$ is given, we write $[x\otimes y]_{Q_T}$ (or simply $[x\otimes y]$ when no confusion can result) for the image of $x\otimes y$ in Q_T. If n is any cardinal number satisfying $1\leqq n\leqq\aleph_0$, we denote by $\mathbb{A}_n(\mathscr{H})$ the set of all those T in $\mathbb{A}(\mathscr{H})$ for which every system of simultaneous equations

$$[x_i\otimes y_j]=[L_{ij}], \qquad 0\leqq i,\ j<n$$

(where the $[L_{ij}]$ are arbitrary elements from Q_T) has a solution $\{x_i\}_{0 \leqq i < n}$, $\{y_j\}_{0 \leqq j < n}$. When no confusion will result, we write simply $\mathbb{A}_n$ for $\mathbb{A}_n(\mathscr{H})$. In [3] we began the structure theory of the classes $\mathbb{A}_n$, and, in particular, the dilation theory of the class $\mathbb{A}_{\aleph_0}$. A primary motivation for the introduction of these classes in [3] was as follows. Let $(BCP) = (BCP)(\mathscr{H})$ denote the set of all those completely nonunitary contractions T in $\mathscr{L}(\mathscr{H})$ for which the intersection $\sigma_e(T) \cap \mathbb{D}$ of the essential spectrum of T with $\mathbb{D}$ is sufficiently large that almost every point of $\mathbb{T}$ is a non-tangential limit point of $\sigma_e(T) \cap \mathbb{D}$ (such sets are said to be dominating for $\mathbb{T}$). It was shown in [4] (and also in [7]) that $(BCP) \subset \mathbb{A}_{\aleph_0}$, so all of the results obtained in [3] for operators in $\mathbb{A}_{\aleph_0}$ apply, in particular, to (BCP)-operators. (In fact, in [4], an increasing family $\{(BCP)_\theta\}_{0 \leqq \theta \leqq 1}$ of classes of contractions is introduced, with $(BCP) = (BCP)_0$, and it was shown there that $\bigcup_{0 \leqq \theta < 1} (BCP)_\theta \subset \mathbb{A}_{\aleph_0}$.)

In [2] it was shown that all (BCP)-operators are reflexive, and the main purpose of this note is to show that all operators in the larger class $\mathbb{A}_{\aleph_0}$ are reflexive (Theorem 1.7). This is worthwhile because we show in the third paper [1] of this sequence that many familiar operators belong to $\mathbb{A}_{\aleph_0}$ and thus are reflexive. In particular, we will show in [1] on the basis of Theorem 1.7 that every weighted unilateral shift W that is a contraction whose spectrum satisfies $\sigma(W) \supset \mathbb{T}$ is reflexive, thus generalizing considerably the results on reflexivity of [8].

We write $\operatorname{Lat}(T)$ for the lattice of invariant subspaces of an operator T, and if $\mathscr{M}, \mathscr{N} \in \operatorname{Lat}(T)$ with $\mathscr{M} \supset \mathscr{N}$, so $\mathscr{M} \ominus \mathscr{N}$ is a semi-invariant subspace of T, we write $T_{\mathscr{M} \ominus \mathscr{N}}$ for the compression of T to this semi-invariant subspace. We also write $P_{\mathscr{M}}$ for the (orthogonal) projection whose range is a subspace $\mathscr{M}$. Our principal tool is the following result of Robel [7, Proposition 6.1].

Proposition 1.1. *Suppose $T \in (BCP)(\mathscr{H})$, $y \in \mathscr{H}$, and $\varepsilon > 0$. Then there exists a subspace $\mathscr{M} \subset \mathscr{H}$ such that $\mathscr{M} \in \operatorname{Lat}(T)$, $T|\mathscr{M} \in (BCP)(\mathscr{M})$, $T_{\mathscr{H} \ominus \mathscr{M}} \in (BCP)(\mathscr{H} \ominus \mathscr{M})$, and $\|P_{\mathscr{M}} y\| < \varepsilon$.*

We will also need the following easy lemma.

Lemma 1.2. *Suppose T is a completely nonunitary contraction in $\mathscr{L}(\mathscr{H})$ and $\{\lambda_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{D}$ that is dominating for $\mathbb{T}$. Suppose also that $\mathscr{M} \in \operatorname{Lat}(T)$ and $T|\mathscr{M}$ is a normal diagonal operator with the property that each λ_n is an eigenvalue of $T|\mathscr{M}$ of infinite multiplicity. Then $T \in (BCP)$.*

Proof. The hypothesis ensures that each λ_n belongs to $\sigma_{le}(T|\mathscr{M})$, and since $\sigma_{le}(T|\mathscr{M}) \subset \sigma_{le}(T)$, we conclude that $\sigma_e(T) \cap \mathbb{D}$ is dominating for $\mathbb{T}$.

The following result is an easy consequence of Proposition 1.1 and Lemma 1.2.

Proposition 1.3. *Suppose $T \in \mathbb{A}_{\aleph_0}(\mathscr{H})$, $\{u_1, \ldots, u_n\}$ is any finite subset of $\mathscr{H}$, and $\varepsilon > 0$. Then there exists $\mathscr{M} \in \operatorname{Lat}(T)$ such that*

(i) *both $T|\mathscr{M}$ and $T_{\mathscr{H} \ominus \mathscr{M}}$ are (BCP)-operators, and*

(ii) *$\|P_{\mathscr{M}} u_i\| < \varepsilon$ for $i = 1, \ldots, n$.*

Proof. Let $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{D}$ be dominating for $\mathbb{T}$, and let N be a normal diagonal operator of uniform infinite multiplicity whose eigenvalues constitute the sequence $\{\lambda_n\}_{n=1}^{\infty}$. By Proposition 4.2 of [3] there exist invariant subspaces $\mathcal{M}_0 \supset \mathcal{K}$ for T such that $T_{\mathcal{M}_0 \ominus \mathcal{K}}$ is unitarily equivalent to N. Thus N^* is the restriction to an invariant subspace of $(T|\mathcal{M}_0)^*$, and it follows from Lemma 1.2 that $(T|\mathcal{M}_0)^*$ (along with $T|\mathcal{M}_0$) belongs to $(BCP)(\mathcal{M}_0)$. Let y_1 be the orthogonal projection of u_1 onto $\mathcal{M}_0$. By Proposition 1.1 there exists $\mathcal{M}_1 \in \operatorname{Lat}(T|\mathcal{M}_0)$ such that $(T|\mathcal{M}_0)|\mathcal{M}_1 = T|\mathcal{M}_1$ is a (BCP)-operator and $\|P_{\mathcal{M}_1} y_1\| < \varepsilon$. Note that $\mathcal{M}_1 \in \operatorname{Lat}(T)$ and that $\|P_{\mathcal{M}_1} u_1\| = \|P_{\mathcal{M}_1} y_1\|$. By an obvious finite induction argument we can find an invariant subspace $\mathcal{M}_n \subset \mathcal{M}_1$ for T such that $T|\mathcal{M}_n$ is a (BCP)-operator and such that $\|P_{\mathcal{M}_n} u_i\| < \varepsilon$, $i=1, \ldots, n$. Since $T|\mathcal{M}_n \in \mathbb{A}_{\aleph_0}(\mathcal{M}_n)$, we may apply Proposition 4.2 of [3] to $T|\mathcal{M}_n$ and the operator $N \oplus N$ to conclude the existence of a decomposition

$$\mathcal{M}_n = \mathcal{N}_1 \oplus \mathcal{N}_2 \oplus \mathcal{N}_3 \oplus \mathcal{N}_4, \quad \text{where } \mathcal{N}_1 \text{ and } \mathcal{N}_1 \oplus \mathcal{N}_2 \oplus \mathcal{N}_3$$

belong to $\operatorname{Lat}(T|\mathcal{M}_n)$, and where $(T|\mathcal{M}_n)_{\mathcal{N}_2 \oplus \mathcal{N}_3}$ is the operator $N \oplus N$ acting on $\mathcal{N}_2 \oplus \mathcal{N}_3$ in the obvious way. We set $\mathcal{M} = \mathcal{N}_1 \oplus \mathcal{N}_2$. Clearly $\mathcal{M} \in \operatorname{Lat}(T)$, and that $T|\mathcal{M} \in (BCP)(\mathcal{M})$ follows as before. Furthermore the restriction of $T_{\mathcal{H} \ominus \mathcal{M}}$ to the invariant subspace $\mathcal{N}_3$ is the operator N, so, once again by Lemma 1.2, $T_{\mathcal{H} \ominus \mathcal{M}} \in (BCP)(\mathcal{H} \ominus \mathcal{M})$. Finally, since $\mathcal{M} \subset \mathcal{M}_n$, it is obvious that $\|P_{\mathcal{M}} u_i\| < \varepsilon$, $i = 1, \ldots, n$, so the proof is complete.

The next corollary now follows from Proposition 1.3 by the same argument that Robel used to prove [7, Propositions 6.2 and 6.3] from [7, Proposition 6.1].

Corollary 1.4. *Suppose $T \in \mathbb{A}_{\aleph_0}(\mathcal{H})$. Then $\mathcal{H}$ admits a decomposition $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{M}_n$ such that the operator matrix (T_{ij}) for T relative to this decomposition is in upper triangular form and satisfies $T_{nn} \in (BCP)(\mathcal{M}_n)$, $0 \leqq n < \infty$. Furthermore $\mathcal{H}$ admits another decomposition $\mathcal{H} = \bigoplus_{n=-\infty}^{\infty} \mathcal{N}_n$ such that the operator matrix $(\tilde{T}_{ij})$ for T relative to this decomposition is in upper triangular form and satisfies $\tilde{T}_{nn} \in (BCP)(\mathcal{N}_n)$, $-\infty < n < \infty$.*

The following theorem shows that, for operators in $\mathbb{A}_{\aleph_0}$, finite systems of simultaneous equations can be solved with reasonable estimates on the distance from the initial data to the solution.

Theorem 1.5. *Suppose $T \in \mathbb{A}_{\aleph_0}(\mathcal{H})$, $\{[L_{ij}]\}_{1 \leqq i,j \leqq n}$ is a finite set of elements of Q_T, $\{z_1, \ldots, z_m\}$ is an arbitrary finite set of vectors from $\mathcal{H}$, and $\varepsilon > 0$. Suppose also that $\{x_1^0, \ldots, x_n^0\}$ and $\{y_1^0, \ldots, y_n^0\}$ are sequences from $\mathcal{H}$ and $\delta > 0$ is such that $\|[L_{ij}] - [x_i^0 \otimes y_j^0]\| < \delta$ for $1 \leqq i, j \leqq n$. Then there exist sequences $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ of vectors from $\mathcal{H}$ such that*

$$[L_{ij}] = [x_i \otimes y_j], \quad 1 \leqq i, j \leqq n, \tag{1}$$

$$\|x_i^0 - x_i\| < n\delta^{1/2}, \quad \|y_i^0 - y_i\| < n\delta^{1/2}, \tag{2}$$

and

$$\|[(x_i^0-x_i)\otimes z_k]\|<\varepsilon, \quad \|[z_k\otimes(x_i^0-x_i)]\|<\varepsilon,$$
$$\|[(y_i^0-y_i)\otimes z_k]\|<\varepsilon, \quad \|[z_k\otimes(y_i^0-y_i)]\|<\varepsilon, \tag{3}$$
$$1\leqq i\leqq n,\ 1\leqq k\leqq m.$$

Proof. Let $d_{ij}=\|[L_{ij}]-[x_i^0\otimes y_j^0]\|$, $1\leqq i, j\leqq n$, and let τ be a positive number such that

$$\tau<n(\delta^{1/2}-\max_{i,j}(d_{ij})^{1/2}). \tag{4}$$

Let $M>0$ be an upper bound for $\|x_i^0\|$, $\|y_j^0\|$, and $\|z_k\|$ for $1\leqq i, j\leqq n$ and $1\leqq k\leqq m$. We choose a positive number η such that

$$\eta<\min\{\tau/2,\ \varepsilon/3M,\ \varepsilon/3n\delta^{1/2}\} \tag{5}$$

and such that

$$0\leqq t, t' \quad \text{and} \quad |t'-t|<3M\eta \quad \text{imply } |\sqrt{t'}-\sqrt{t}|<\tau/2n. \tag{6}$$

(The reason for this choice of η will appear later. We choose it now to make it clear that η does not depend upon the choice of the upcoming vectors x_i and y_j.) It follows from Proposition 1.3 that there exists $\mathcal{M}\in\operatorname{Lat}(T)$ such that $T|\mathcal{M}$ and $S=T_{\mathcal{H}\ominus\mathcal{M}}$ are both (BCP)-operators and such that the norm of the (orthogonal) projection onto $\mathcal{M}$ of each of the $2n+m$ vectors $\{x_1^0,\dots,x_n^0\}$, $\{y_1^0,\dots,y_n^0\}$, and $\{z_1,\dots,z_m\}$ is less than η. We write $x_i'=P_{\mathcal{H}\ominus\mathcal{M}}x_i^0$, $1\leqq i\leqq n$, and define similarly y_j', $1\leqq j\leqq n$, and z_k', $1\leqq k\leqq m$. (The idea of the proof of this theorem should now be clear. We will transfer the equation solving problem to the semi-invariant subspace $\mathcal{H}\ominus\mathcal{M}$, using the fact that $S=T_{\mathcal{H}\ominus\mathcal{M}}$ is a (BCP)-operator to solve equations there with "good" bounds, and the smallness of the η we have chosen will then give us the estimates we desire.)

For $1\leqq i, j\leqq n$, let $[M_{ij}]\in Q_S$ be defined by $[M_{ij}]=\phi_S^{-1}\phi_T([L_{ij}])$, and note that the $[M_{ij}]$ are uniquely determined by the relations

$$\langle S^p,[M_{ij}]\rangle=\langle\lambda^p,\phi_S([M_{ij}])\rangle=\langle T^p,[L_{ij}]\rangle, \quad p=0,1,2,\dots. \tag{7}$$

In particular, since the $[L_{ij}]$ are arbitrary elements of Q, for $u, v\in\mathcal{H}\ominus\mathcal{M}$, we have

$$[u\otimes v]_{Q_S}=\phi_S^{-1}\phi_T([u\otimes v]_{Q_T}) \tag{8}$$

by virtue of (7), since

$$\langle S^p,[u\otimes v]_{Q_S}\rangle=(S^p u,v)=(T^p u,v)=\langle T^p,[u\otimes v]_{Q_T}\rangle, \quad 0\leqq p<\infty.$$

Let $\alpha=M\eta+\max_{i,j}d_{ij}'$, where $d_{ij}'=\|[M_{ij}]-[x_i'\otimes y_j']\|_{Q_S}$. It now follows from Corollary 6.13 and Remark 6.14 of [3] (applied with $\theta=0$ to the operator S) that there exist sequences $\{x_1,\dots,x_n\}$ and $\{y_1,\dots,y_n\}$ of vectors in $\mathcal{H}\ominus\mathcal{M}$ such that

$$[M_{ij}]=[x_i\otimes y_j]_{Q_S}, \quad 1\leqq i,\ j\leqq n, \tag{9}$$

$$\|x_i'-x_i\|<n\alpha^{1/2}, \quad \|y_i'-y_i\|<n\alpha^{1/2}, \quad 1\leqq i\leqq n, \tag{10}$$

and

$$\begin{aligned}&\|[(x_i'-x_i)\otimes z_k']\|_{Q_S}<\varepsilon/3, \quad \|[z_k'\otimes(x_i'-x_i)]\|_{Q_S}<\varepsilon/3,\\ &\|[(y_i'-y_i)\otimes z_k']\|_{Q_S}<\varepsilon/3, \quad \|[z_k'\otimes(y_i'-y_i)]\|_{Q_S}<\varepsilon/3, \quad 1\leqq i\leqq n,\ 1\leqq k\leqq m.\end{aligned} \tag{11}$$

By applying $\phi_T\phi_S^{-1}$ to (9) and using (8), we see that (1) is satisfied. We will now prove (2) for the x_i's, recalling that ϕ_S and ϕ_T are isometries. We have from (5) and (10) that

$$\|x_i^0-x_i\|\leqq\|x_i^0-x_i'\|+\|x_i'-x_i\|<(\tau/2)+n\alpha^{1/2}, \quad 1\leqq i\leqq n. \tag{12}$$

Furthermore, from the inequalities

$$\begin{aligned}d_{ij}'&=\|[M_{ij}]-[x_i'\otimes y_j']\|_{Q_S}=\|[L_{ij}]-[x_i'\otimes y_j']\|_{Q_T}\\ &\leqq\|[L_{ij}]-[x_i^0\otimes y_j^0]\|_{Q_T}+\|[x_i^0\otimes y_j^0]-[x_i'\otimes y_j']\|_{Q_T},\end{aligned}$$

we obtain

$$\begin{aligned}d_{ij}'&\leqq d_{ij}+\|[x_i^0\otimes y_j^0]-[x_i'\otimes y_j']\|_{Q_T}\\ &\leqq d_{ij}+\|[x_i^0\otimes(y_j^0-y_j')]\|_{Q_T}+\|[(x_i^0-x_i')\otimes y_j']\|_{Q_T}<d_{ij}+2M\eta.\end{aligned}$$

Therefore

$$\alpha=M\eta+\max_{i,j}d_{ij}'<(\max_{i,j}d_{ij})+3M\eta,$$

and from (6) we obtain

$$\alpha^{1/2}<\max_{i,j}(d_{ij}^{1/2})+\tau/2n. \tag{13}$$

Hence from (12), (13), and (4) we conclude that

$$\|x_i^0-x_i\|<\tau/2+n(\max_{i,j}d_{i,j}^{1/2})+\tau/2<n\delta^{1/2}, \quad 1\leqq i\leqq n, \tag{14}$$

as desired. Of course this argument works equally well to prove that $\|y_i^0-y_i\|<n\delta^{1/2}$, $1\leqq i\leqq n$. To conclude the proof of the theorem we content ourselves with proving the first inequality in (3). For $1\leqq i\leqq n$ and $1\leqq k\leqq m$ we have

$$\begin{aligned}\|[(x_i-x_i^0)\otimes z_k]\|_{Q_T}&\leqq\|[(x_i-x_i')\otimes z_k']\|_{Q_T}+\|[(x_i-x_i')\otimes(z_k-z_k')]\|_{Q_T}\\ &\quad+\|[(x_i'-x_i^0)\otimes z_k]\|_{Q_T},\end{aligned}$$

and using (11), (14), (5) and the fact that ϕ_S and ϕ_T are isometries, we obtain

$$\|[(x_i-x_i')\otimes z_k']\|_{Q_T}=\|[(x_i-x_i')\otimes z_k']\|_{Q_S}<\varepsilon/3,$$

$$\|[(x_i-x_i')\otimes(z_k-z_k')]\|_{Q_T}\leqq\|x_i-x_i'\|\cdot\|z_k-z_k'\|\leqq n\alpha^{1/2}\eta<n\delta^{1/2}\eta<\varepsilon/3,$$

and

$$\|[(x_i^0 - x_i') \otimes z_k]\|_{Q_T} \leqq \|x_i^0 - x_i'\| \cdot \|z_k\| \leqq \eta \cdot M < \varepsilon/3.$$

Thus $\|[(x_i^0 - x_i) \otimes z_k]\|_{Q_T} < \varepsilon$ as desired, and the proof is complete.

The special case of Theorem 1.5 when $n=1$ shows that [2, Proposition 1] is valid for all operators in $\mathbb{A}_{\aleph_0}$, and since the proof of [2, Corollary 1] only depends on [2, Proposition 1] we have the following.

Corollary 1.6. *Suppose $T \in \mathbb{A}_{\aleph_0}(\mathscr{H})$, and denote by $\mathscr{W}_T$ the smallest subalgebra of $\mathscr{L}(\mathscr{H})$ that contains T and $1_{\mathscr{H}}$ and is closed in the weak operator topology. Then $\mathscr{W}_T = \mathscr{A}_T$ and the weak operator and ultraweak operator topologies coincide on $\mathscr{A}_T$.*

It follows from this corollary and a result from [1] that every weighted unilateral shift operator W that is a contraction such that $\sigma(W) \supset \mathbb{T}$ satisfies $\mathscr{W}_W = \mathscr{A}_W$. This partly answers Question 5 of [8].

Theorem 1.5 also shows that [2, Proposition 2] is valid for all operators in $\mathbb{A}_{\aleph_0}$, and since the proof of the reflexivity of (BCP)-operators used only [2, Proposition 2], we also have the following corollary, which generalizes Theorems 3, 4, and 5 of [2].

Theorem 1.7. *Every operator in $\mathbb{A}_{\aleph_0}(\mathscr{H})$ is reflexive. In particular, all of the operators in the classes $(BCP)_\theta$, $0 \leqq \theta < 1$, defined in [4] are reflexive.*

As mentioned earlier, the utility of Theorem 1.7 will be greatly enhanced by the appearance of [1], because of the large number of operators that turn out to belong to $\mathbb{A}_{\aleph_0}$. For the moment we deduce the following corollary of Corollary 1.6 and Theorem 1.7.

Corollary 1.8. *Suppose $T \in C_{00}$ and also $T \in \bigcap_{n=1}^{\infty} \mathbb{A}_n(\mathscr{H})$. Then T is reflexive, the algebras $\mathscr{W}_T$ and $\mathscr{A}_T$ coincide, and the weak operator and ultraweak topologies agree on $\mathscr{A}_T$.*

Proof. Exner showed in [6] that $\left(\bigcap_{n=1}^{\infty} \mathbb{A}_n\right) \cap C_{00} \subset \mathbb{A}_{\aleph_0}$.

This corollary raises the interesting question whether operators in a fixed class $\mathbb{A}_n$ $(n < \aleph_0)$ and not in C_{00} have these same properties.

We also note that the upper bounds on $\|x_i^0 - x_i\|$ and $\|y_i^0 - y_i\|$ given by (2) in Theorem 1.5 for all operators in $\mathbb{A}_{\aleph_0}$ are better than those given in [4, Corollary 6.11] for $(BCP)_\theta$-operators, so Theorem 1.5 generalizes [4, Corollary 6.11].

We close this note with a further consequence of Theorem 1.5. If $n \in \mathbb{N}$, we denote by $\tilde{\mathscr{H}}_n$ the direct sum of n copies of the Hilbert space $\mathscr{H}$.

Corollary 1.9. *Suppose $T \in \mathbb{A}_{\aleph_0}$, $n \in \mathbb{N}$, and $\{[L_{ij}]\}_{i,j=1}^{n}$ is a doubly indexed sequence of elements in Q_T. Then the set of vectors $(x_1, \ldots, x_n)$ in $\tilde{\mathscr{H}}_n$ for which there exists a vector $(y_1, \ldots, y_n)$ in $\tilde{\mathscr{H}}_n$ satisfying (1) is dense in $\tilde{\mathscr{H}}_n$.*

Proof. Let $\tilde{x}_0 = (x_1^0, \ldots, x_n^0)$ be an arbitrary vector in $\tilde{\mathscr{H}}_n$, let τ be a positive number, and use as initial data in Theorem 1.5 the vectors $(\tau x_1^0, \ldots, \tau x_n^0)$ and

$(0, \ldots, 0)$ in $\tilde{\mathscr{H}}_n$. Then, according to that theorem, there exists a solution $\tilde{x}_\tau = (x_1^\tau, \ldots, x_n^\tau)$, $\tilde{y}_\tau = (y_1^\tau, \ldots, y_n^\tau)$ of (1) such that

$$\|x_i^\tau - \tau x_i^0\| < n\delta^{1/2}, \qquad \|y_i^\tau - 0\| < n\delta^{1/2}, \quad 1 \leqq i \leqq n, \tag{15}$$

where δ is any fixed positive number that exceeds $\max_{i,j} \|[L_{ij}]\|$. Thus, since for every $\tau > 0$, the pair $(1/\tau)\tilde{x}_\tau, \tau\tilde{y}_\tau$ is also a solution of (1), and since $\|(1/\tau)\tilde{x}_\tau - \tilde{x}_0\| \to 0$ by (15), the result follows. In fact, to obtain $\|(1/\tau)\tilde{x}_\tau - \tilde{x}_0\| < \varepsilon$, it suffices to take $\tau = n^2\delta^{1/2}/\varepsilon^{1/2}$, in which case the vector $\tau\tilde{y}_\tau$ satisfies $\|\tau\tilde{y}_\tau\| < n^4\delta/\varepsilon$.

References

1. Apostol, C., Bercovici, H., Foias, C., Pearcy, C.: Dilation theory, invariant subspaces, and dual algebras, Preprint.
2. Bercovici, H., Foias, C., Langsam, J., Pearcy, C.: (*BCP*)-operators are reflexive. Michigan Math. J. **29**, 371–379 (1982)
3. Bercovici, H., Foias, C., Pearcy, C.: Dilation theory and systems of simultaneous equations in the predual of an operator algebra. I., Michigan Math. J. **30**, 335–354 (1983)
4. Bercovici, H., Foias, C., Pearcy, C.: Factoring trace-class operator-valued functions with applications to the class $\mathbb{A}_{\aleph_0}$, to appear in J. Operator Theory
5. Brown, S., Chevreau, B., Pearcy, C.: Contractions with rich spectrum have invariant subspaces. J. Operator Theory **1**, 123–136 (1979)
6. Exner, G.: Systems of equations in the predual of an operator algebra, the classes $\mathbb{A}_n$, and related operators. Thesis, University of Michigan 1983
7. Robel, G.: On the structure of (*BCP*)-operators and related algebras. I. To appear in J. Operator Theory
8. Shields, A.: Weighted shift operators and analytic function theory, Topics in Operator Theory, pp. 49–128. Providence: Amer. Math. Soc. 1974
9. Sz.-Nagy, B., Foias, C.: Harmonic analysis of operators on Hilbert space. Budapest: Akademiai Kiado 1970

Received December 21, 1983